Calculus I

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Contents

1 Functions: Domain, Range, and Operations

Key Concepts

- **Limits:** How does the output change?
- **Derivative:** How sensitive is the output (velocity)?
- **Integration:** Can we recover outputs just by knowing the derivative?

Set Notation

- $[a, b]$: Closed interval $\{x \mid a \leq x \leq b\}$.
- (a, b): Open interval $\{x \mid a < x < b\}$.
- $X \in S$: X is an element of set S.
- $\mathbb{R}$: Real numbers, $\mathbb{R} = (-\infty, \infty) = \{x \mid -\infty < x < \infty\}$.

Function Types

- **Piecewise Function:**
$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$
- **Constant Function:** For $b \in \mathbb{R}$, $f(x) = b$.
- **Linear Function:** $y = f(x) = mx + b$, slope is rise over run.
- **Polynomials:** $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$; e.g., $f(x) = x^4 + 3x - 2$.
- **Rational Function:** $f(x) = \frac{P(x)}{Q(x)}$, domain $D = \{x \mid Q(x) \neq 0\}$.
- **Power Functions:** $f(x) = x^a$, where $a \in \mathbb{R}$; e.g., $f(x) = x^{1/3}, x^{-1/2}$.
- **Trigonometric Functions:**
$$\sin(\theta), \cos(\theta), \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \sec(\theta) = \frac{1}{\cos(\theta)}, \csc(\theta) = \frac{1}{\sin(\theta)}.$$

2 Transformations, Building Functions, and Inverses

New Functions from Old

Shifting Graphs

- $f(x) + c$ shifts the graph up or down by c.
- $f(x - c)$ shifts the graph to the right by c.
- $f(x + c)$ shifts the graph to the left by c.

Stretching and Shrinking Graphs

- $y = cf(x)$ stretches the graph vertically by a factor of c.
- If $c > 1$, the graph stretches.
- If $0 < c < 1$, the graph shrinks.
- $y = f(cx)$ compresses the graph horizontally by a factor of c.
- If $c > 1$, the graph becomes narrower (sharper).
- If $0 < c < 1$, the graph becomes wider (shallower).

Reflections

- $y = -f(x)$ reflects the graph across the x-axis.
- $y = f(-x)$ reflects the graph across the y-axis.

Periodic Functions

A function $f(x)$ is periodic if there exists a constant $c > 0$ such that:

$$f(x + c) = f(x).$$

Odd and Even Functions

- Odd functions satisfy $f(-x) = -f(x)$.
- Even functions satisfy $f(-x) = f(x)$.

Symmetry of Odd and Even Functions

- Odd functions have rotational symmetry.
- Even functions have reflection symmetry.

Composition of Functions

If $f(x) = x^2$ and $g(x) = x + 3$, then:

$$f \circ g(x) = f(g(x)) = (x + 3)^2.$$

If $f(x) = x^2$ and $g(x) = \sqrt{x}$, then:

$$(f \circ g)(x) = f(g(x)) = (\sqrt{x})^2 = x.$$

And:

$$(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|.$$

Note that $f \circ g \neq g \circ f$ in general.

Domain of Composition

For $f \circ g$, we can only plug in values of $g(x)$ into f if the range of $g(x)$ is within the domain of $f(x)$.

One-to-One Functions

A function $f(x)$ is one-to-one if:

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

In other words, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Examples of One-to-One Functions

- Non-constant linear functions $f(x) = mx + b$ (where $m \neq 0$) are one-to-one.
- For $f(x) = x^2$ with domain $[0, \infty)$, the function is one-to-one.

Inverse Functions

If f is one-to-one, then its inverse function f^{-1} is defined as follows:

$$f^{-1}(y) = x \quad \text{such that} \quad f(x) = y$$

- For $f(x) = x^2$ with domain and range $[0, \infty)$, the inverse is:

$$f^{-1}(y) = \sqrt{y}.$$

- For $f(x) = x^3 + 2$, the inverse is:

$$f^{-1}(y) = (y - 2)^{1/3}.$$

Periodic Functions are not One-to-One

Periodic functions are never one-to-one because they repeat their values.

Inverse Trigonometric Functions

- $\sin^{-1}(y) = \arcsin(y)$ — inverse of sine.
- $\cos^{-1}(y) = \arccos(y)$ — inverse of cosine.
- $\tan^{-1}(y) = \arctan(y)$ — inverse of tangent.

The graphs of these functions transform the domain into the range.

Exponential Functions

Exponential functions have the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.
- If $b = 1$, $f(x) = 1^x = 1$ for all x.
- If $b > 1$, the graph of $f(x) = b^x$ increases as x increases.
- If $b < 1$, the graph of $f(x) = b^x$ decreases as x increases.

The exponential function has several properties:

$$b^x \cdot b^y = b^{x+y}.$$

$$b^x / b^y = b^{x-y}.$$

$$b^0 = 1 \quad \text{for all} \quad b > 0.$$

Logarithms

The logarithm is the inverse of the exponential function. If $f(x) = b^x$, then:

$$f^{-1}(x) = \log_b(x).$$

Logarithmic properties include:

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

$$\log_b(x/y) = \log_b(x) - \log_b(y).$$

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}.$$

Exponential and Logarithmic Functions with $b > 1$

- The graph of $f(x) = e^x$ (where e is Euler's number) touches the line $y = x + 1$ at the origin with slope 1.
- The inverse of $f(x) = b^x$ is $f^{-1}(x) = \log_b(x)$.

The composition of an exponential function and its inverse yields:

$$f(f^{-1}(x)) = x.$$

3 Tangent Lines, Secant Lines, and Instantaneous Velocity

Parallel Lines

Two lines are parallel if their slopes are equal. In mathematical terms, for two lines with equations $y = m_1x + b_1$ and $y = m_2x + b_2$, the lines are parallel if:

$$m_1 = m_2,$$

where m_1 and m_2 are the slopes of the respective lines.

Tangent Lines

A tangent line is a line that touches a curve at exactly one point and does not cross it. The tangent line to a function $f(x)$ at the point $p = (x_0, f(x_0))$ can be written as:

$$y - f(x_0) = f'(x_0)(x - x_0),$$

where $f'(x_0)$ is the derivative of $f(x)$ at the point x_0.

Examples

- Let $f(x) = e^x$ and $p = (0, 1)$, the tangent line at $x = 0$ is:

$$t(x) = x + 1.$$

- For a linear function $l(x) = mx + b$ with $p = (0, b)$, the tangent line is just the function itself. Tangents to linear functions are themselves.

Slope of a Line

The slope of a line between two points $p_1 = (x_1, f(x_1))$ and $p_2 = (x_2, f(x_2))$ is given by the formula for the slope of a secant line:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example: Finding the Slope of a Secant Line

Let $l(x) = 3x + 1$ and the point $p = (0, 1)$. If another point $q = (1, 4)$ is chosen, the slope of the secant line is:

$$m = \frac{4 - 1}{1 - 0} = 3.$$

Secant Lines Approaching Tangent Line

For $f(x) = x^2$ and the point $p = (1, 1)$, the secant line through p and a nearby point $q = (2, 4)$ has slope:

$$m_{\text{sec}} = \frac{4 - 1}{2 - 1} = 3.$$

However, as the point q gets closer to p, the slope of the secant line approaches the slope of the tangent line at p.

Table of Secant Slopes for $f(x) = x^2$

x	Slope of Secant Line
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001
0.5	1.5
0.9	1.9
0.99	1.99

Limit of the Slope

As the point q approaches p, the slope of the secant line approaches the derivative of $f(x)$ at p. For $f(x) = x^2$, the derivative at $x = 1$ is:

$$f'(1) = \lim_{q \to p} m_{\text{sec}} = 2.$$

Equation of the Tangent Line

The equation of the tangent line at $p = (1, 1)$ for $f(x) = x^2$ is:

$$t(x) = 2x - 1.$$

This is found by taking the limit of the secant slope and applying the point-slope form of the line equation.

Derivatives as Limits

The derivative of a function $f(x)$ at $x = a$ is given by the following limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This can be thought of as the limit of the slope of secant lines as the second point approaches the first.

Instantaneous Velocity

Instantaneous velocity can be found by taking the derivative of the position function. For a ball dropped from a height, the position function might be:

$$s(t) = 4.9t^2.$$

The instantaneous velocity at time t is the derivative of $s(t)$:

$$v(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

For $t = 5$ seconds, the velocity is:

$$v(5) = 9.8 \times 5 = 49\,\text{m/s}.$$

Average Velocity

The average velocity between two times t_1 and t_2 is given by:

$$\text{Average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

For example, between $t = 5$ and $t = 6$:

$$\text{Average velocity} = \frac{4.9(6^2) - 4.9(5^2)}{6 - 5} = 53.9\,\text{m/s}.$$

Conclusion

Understand limits and you will understand calculus.

4 Derivatives and Interpretations

Definition of Derivative

The derivative is the slope of the tangent line at a given point. The formula for the derivative of a function, f, at a point, a, is given by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 1:

Let $f(x) = x^2$. Compute the derivative at $x = 1$:

$$f'(1) = 2.$$

$$\frac{(a+h)^2 - a^2}{h} = h + 2a.$$

For small h, the term $h + 2a$ approaches $2a$, so $f'(x) = 2x$.

Example 2:

Let $f(x) = \frac{3}{x}$. The derivative is computed as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{1}{h}\left[\frac{3}{x+h} - \frac{3}{x}\right] = \frac{1}{h}\left[\frac{3x - 3(x+h)}{x(x+h)}\right]$$

$$= \frac{-3h}{hx(x+h)} = -\frac{3}{x^2} \quad (\text{as } h \to 0).$$

Thus, $f'(x) = -\frac{3}{x^2}$.

Normal Line

The normal line to a curve at a point has a slope that is the negative reciprocal of the tangent line's slope (one can remember that the slope of the tangent times the slope of the normal is always equal to -1).
For instance, if the slope of the tangent line is $-\frac{1}{3}$, the slope of the normal line is 3. The equation of the normal line would be:

$$y = 3x - 8.$$

Derivative Interpretation in Economics

Let $c = f(x)$ represent the cost of producing x meters of fabric.

- **Q1:** What does $f'(x)$ represent?
  A1: The rate of change of the production cost with respect to the number of meters produced.
- **Q2:** What are the units of the derivative?
  A2: The units of $f'(x)$ are dollars per meter.
- **Q3:** If $f'(1000) = 9$, what does this mean?
  A3: After producing 1000 meters, the cost of the 1001st meter is expected to be \$9 more than the 1000th meter.
- **Q4:** Which should be higher: $f'(50)$ or $f'(500)$?
  A4: Bulk production is cheaper, so $f'(50) > f'(500)$.

Non-Differentiability

Intuitively, a function is not differentiable at a point if it has a discontinuity or a sharp corner.

Example:

Let $f(x) = |x|$:

$$f(x) = \begin{cases} -x & x < 0, \\ x & x \geq 0. \end{cases}$$

The derivative is:

$$f'(x) = \begin{cases} -1 & x < 0, \\ 1 & x > 0. \end{cases}$$

The derivative is not defined at $x = 0$ due to a discontinuity in slope.

Other Examples of Differentiability Issues:

- $f(x) = \tan(x)$ has vertical asymptotes.
- $f(x) = x^{1/3}$ and $f(x) = \sqrt{x}$ have points where the derivative is undefined or discontinuous.

Graph of the Derivative

The derivative is the slope of the tangent line at each point. The graph of a derivative function provides insights into the behavior of the original function's slope at every point.

5 Limits

Definition of Limit

A function $f(x)$ is defined near a point a if for any x close to a, $f(x)$ is defined. The limit of $f(x)$ as $x \to a$ is L, written:

$$\lim_{x \to a} f(x) = L.$$

if we can make $f(x)$ as close to L as we like for all x close to a, but not necessarily at $x = a$.

Example:

Let $g(x) = \frac{1}{x+1}$ for $x \neq 1$ and $g(1) = 2$. The limit as $x \to 1$ is:

$$\lim_{x \to 1} g(x) = \frac{1}{2}.$$

Limits That Do Not Exist

Limits may fail to exist in cases of infinite discontinuities or jump discontinuities.

Example:

For $f(x) = \frac{1}{x^2}$, the limit as $x \to 0$ does not exist because the function tends towards infinity.

Heaviside Step Function:

Let:

$$h(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

The left-hand and right-hand limits as $x \to 0$ differ:

$$\lim_{x \to 0^-} h(x) = 0, \quad \lim_{x \to 0^+} h(x) = 1.$$

Thus, the limit does not exist at $x = 0$.

Infinite Limits

If $f(x)$ is defined near a, we say $\lim_{x \to a} f(x) = +\infty$ if $f(x)$ grows arbitrarily large as x approaches a.

Example:

For $f(x) = \frac{1}{x}$, we have:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty, \quad \lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

Logarithmic Example:

Let $g(x) = \ln(|x|)$. The limit as $x \to 0$ is:

$$\lim_{x \to 0} \ln(|x|) = -\infty.$$

Oscillating Function

Let $f(x) = \cos\left(\frac{\pi}{x}\right)$. This function oscillates between -1 and 1 as x approaches 0, and hence the limit does not exist.

$$f\left(\frac{1}{2k}\right) = 1, \quad f\left(\frac{1}{2k+1}\right) = -1.$$

6 Limit Laws

Example:

$$\lim_{x \to 0}(x^2 \cos\left(\frac{\pi}{x}\right)).$$

We suppose both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

Even if the limits don't exist individually, the sum may exist.
For example:

$$\lim_{x \to a} \cos^2\left(\frac{\pi}{x}\right) \text{ does not exist.}$$

$$\lim_{x\to a} \sin^2\left(\frac{\pi}{x}\right) \text{ does not exist.}$$

However, $\sin^2\left(\frac{\pi}{x}\right) + \cos^2\left(\frac{\pi}{x}\right) = 1$, so:

$$\lim_{x\to 0}\left(\sin^2\left(\frac{\pi}{x}\right) + \cos^2\left(\frac{\pi}{x}\right)\right) = 1.$$

$$\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

If $g(x) = C$, then:

$$\lim_{x\to a}(C\cdot f(x)) = C\cdot \lim_{x\to a} f(x).$$

If $\lim_{x\to a} g(x) \neq 0$, then:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

Let n be a positive integer:

$$\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n.$$

Also,

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

If $\lim_{x\to a} f(x) > 0$, then n must be even.

Example:

$$\lim_{x\to 5}(2x - 3x + 4).$$

Use the rules to justify this in your homework.

Example:

$$\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)} = \frac{-1}{11}.$$

Question: What about $\lim_{x\to 1}\frac{x^2-1}{x-1}$?

The function $\frac{x^2-1}{x-1}$ is not defined at $x = 1$. However, using Rule 5, if $f(x) = g(x)$ for all $x \neq a$, then:

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) \text{ (if the limit exists).}$$

Example:

$$\lim_{x\to -1}\frac{x+1}{x+1} = 1.$$

All rules work for one-sided limits if they exist.
If $f(x) \leq g(x)$ for all x near a, then:

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \text{ (if limits exist).}$$

Squeeze and Sandwich Theorems

If $h(x) \leq f(x) \leq g(x)$ when x is near a, and $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$, then:

$$\lim_{x\to a} f(x) = L.$$

Example:

$$\lim_{x\to 0} x^2 \cos\left(\frac{\pi}{x}\right).$$

Note that $-1 \leq \cos\left(\frac{\pi}{x}\right) \leq 1$. Multiply both sides by x^2, so:

$$-x^2 \leq x^2 \cos\left(\frac{\pi}{x}\right) \leq x^2.$$

Since $\lim_{x\to 0} \pm x^2 = 0$, by the Sandwich Theorem:

$$\lim_{x\to 0} x^2 \cos\left(\frac{\pi}{x}\right) = 0.$$

When Do Left and Right Limits Differ?

These are called discontinuity points.

Types of Discontinuity:

- Jump discontinuity: e.g. Heaviside.
- Vertical asymptote: e.g. $\tan(x)$ at $x = \frac{\pi}{2}$.
- Removable discontinuity: e.g. If $f(x)$ is not defined at a, but $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ then we could define $f(a) = L$.
- Other types of discontinuity: e.g. $\cos(\frac{\pi}{x})$ at $x = 0$ (defining it to be 0 at $x = 0$ say).

7 Continuity

Definition:

A function f is continuous (cts) at a point a if:

$$\lim_{x\to a} f(x) = f(a).$$

This tells us three things:
1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

If any of these conditions fail, the function is discontinuous at a.
A function is continuous from the right/left at a point a if:

$$\lim_{x\to a^\pm} f(x) = f(a).$$

We say f is continuous on the interval (a, b) if f is continuous at every point $c \in (a, b)$. A similar definition holds for $[a, b]$, $[a, b)$, and $(a, b]$.

Examples:

$$f(x) = \sqrt{1 - x^2} \quad \text{on } [-1, 1].$$

$$g(x) = \sqrt{x} \quad \text{on } [0, \infty).$$

$$h(x) = \tan(x) \quad \text{is continuous away from } (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}.$$

Which Functions Are Continuous?

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Exponentials

Building Continuous Functions:

If f and g are continuous at a point a, and c is some constant, then $f \pm g$, $f \cdot g$, $c \cdot f$, $\frac{f}{g}$ (if $g(a) \neq 0$) are all continuous at a.

Example:

Where is $f(x) = \frac{\ln(x)+\arctan(x)}{x^2-1}$ continuous? It is continuous on $(0, 1)$ and $(1, \infty)$.

Fact:

If f is continuous at b, and $\lim_{x \to a} g(x) = b$, then:

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

Example:

$$\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) = \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right).$$

Fact:

If g is continuous at a, and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Example:

- $\sin(x^2)$ is continuous everywhere.
- $\ln(1 + \cos(x))$ is continuous if $x \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$.

Intermediate Value Theorem (IVT):

Suppose f is continuous on $[a, b]$, and let N be some number between $f(a)$ and $f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = N$.
In layman's terms: if f is continuous, then every value between the endpoint values is reached.

Example:

Let's solve $4x^3 = 6x^2 - 3x + 2$.
Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. Solving this equation is the same as solving $f(x) = 0$.
We find that $f(1) < 0$ and $f(2) > 0$. Now, since $f(x)$ is continuous, the IVT tells us that there exists $c \in (1, 2)$ such that $f(c) = 0$.

8 Limits at Infinity

Definition:

We say $\lim_{x\to\infty} f(x) = L$ if $f(x)$ can be made as close to L as we like by making x large.

Examples:

$$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}.$$

Remark: All limit laws apply to infinite limits.

Examples:

$$\lim_{x\to-\infty} \frac{1}{x} = 0, \quad \lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}.$$

$$\lim_{x\to\infty} \left(\sqrt{x^2 - x} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2 - x} - x\right) \cdot \frac{\sqrt{x^2 - x} + x}{\sqrt{x^2 - x} + x} = -\frac{1}{2}.$$

Limits to Infinity

We say that:

$$\lim_{x\to\pm\infty} f(x) = L.$$

if $f(x)$ can be made as close to L as we like by taking x very large or very small.
Limit Laws:

- Limit laws hold for limits as $x \to \pm\infty$.
- If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist, then:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}, \text{ if } g(x) \neq 0.$$

9 Definition of the Derivative

For a function f, the derivative of f at a is defined as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

We then think of f' as a function of x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Domain of f':

$$\{x \mid f'(x) \text{ exists}\}.$$

Examples:

- $f(x) = x^2$, $f'(x) = 2x$.
- $g(x) = \frac{3}{x}$, $g'(x) = -\frac{3}{x^2}$.
- $L(x) = d$, $L'(x) = 0$.
  $f(x) = \sqrt{x}$:

  $$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

  Rationalizing the numerator:

  $$f'(x) = \frac{1}{2\sqrt{x}}.$$

Notation

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(f(x)) = df(x).$$

If $y = f(x)$, then:

$$y' = f'(x) = \frac{dy}{dx}.$$

Higher Derivatives

If f' is differentiable, then:

$$f''(x) = \frac{d^2 f}{dx^2}, \quad y'' = \frac{d^2 y}{dx^2}.$$

Example:

- If $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$.

The n-th derivative is denoted $f^{(n)}(x)$.

Physical Interpretation

If f represents position as a function of time:

- f' is velocity.
- f'' is acceleration.
- f''' is called the jerk (apparently).

Differentiability Implies Continuity

If f is differentiable at a, then f is continuous at a.
Proof: Since:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

the existence of the limit implies:

$$\lim_{x \to a} f(x) = f(a).$$

Basic Derivative Rules

- $(f \pm g)'(a) = f'(a) \pm g'(a)$.
- $(cf)'(a) = cf'(a)$, where c is a constant.

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for any } n \in \mathbb{R}.$$

Examples:

- $\frac{d}{dx}(x^2) = 2x$.
- $\frac{d}{dx}(x^{-3}) = -3x^{-4}$.
- $\frac{d}{dx}(1/x) = -1/x^2$.

Exponential Functions

For $f(x) = b^x$, $b > 0$:

$$f'(x) = \ln(b)b^x.$$

If $f(x) = e^x$, then:

$$f'(x) = e^x.$$

Key Property: The derivative of b^x is proportional to itself.

10 Product Rule and Quotient Rule

The product rule states:

$$\frac{d}{dx}[fg] = gf' + fg'.$$

Derivation:

- If $f(t)$ and $g(t)$ are position functions (in meters, m) as a function of time t (in seconds, s), then:

$$f'(t)g'(t) \quad \text{has units of } m^2/s^2,$$
$$f(t)g(t) \quad \text{has units of } m^2,$$
$$\frac{d}{dt}[f(t)g(t)] \quad \text{has units of } m^2/s.$$

- The total change is given by:

$$\Delta(fg) = g\Delta f + f\Delta g + \Delta g\Delta f.$$

  Dividing by Δt:

$$\frac{\Delta(fg)}{\Delta t} = g\frac{\Delta f}{\Delta t} + f\frac{\Delta g}{\Delta t} + \frac{\Delta g\Delta f}{\Delta t}.$$

- Taking the limit as $\Delta t \to 0$:

$$(fg)' = \lim_{\Delta t \to 0} \frac{\Delta(fg)}{\Delta t} = gf' + fg'.$$

Example:

$$\begin{aligned} f(x) &= x, \quad g(x) = e^x, \\ h(x) &= f(x)g(x) = xe^x, \\ h'(x) &= (xe^x)' = 1 \cdot e^x + x \cdot e^x = e^x(1+x). \end{aligned}$$

Quotient Rule

The quotient rule states:

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}.$$

Derivation:

- Consider $h(x) = \frac{f(x)}{g(x)}$:

$$h = \frac{f}{g}, \quad \Delta h = \frac{f + \Delta f}{g + \Delta g} - \frac{f}{g}.$$

  Simplifying:

$$\Delta h = \frac{g(f + \Delta f) - f(g + \Delta g)}{g(g + \Delta g)}.$$

  Dividing by Δt:

$$\frac{\Delta h}{\Delta t} = \frac{g\frac{\Delta f}{\Delta t} - f\frac{\Delta g}{\Delta t}}{g^2} + \text{higher-order terms.}$$

  Taking the limit as $\Delta t \to 0$:

$$h'(x) = \frac{gf' - fg'}{g^2}.$$

Example:

$$h(x) = \frac{e^x}{1 + x^2}, \quad h'(x) = \frac{(1 + x^2)e^x - e^x \cdot 2x}{(1 + x^2)^2} = \frac{e^x(1 - x^2)}{(1 + x^2)^2}.$$

Alternative Simplification

The quotient rule can sometimes be avoided:

$$f(x) = \frac{3x^2 + 2\sqrt{x}}{x} = 3x + 2x^{-1/2}.$$

Taking the derivative:

$$f'(x) = 3 - x^{-3/2}.$$

11 Trig Derivatives

We **guess**:

$$\frac{d}{dx}[\sin(x)] = \cos(x).$$

Derivation:

$$f(x) = \sin(x), \quad \frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin(x)}{h}.$$

Using the sum identity:

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h),$$

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}.$$

Taking the limit:

$$\lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}.$$

Using:

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1, \quad \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0,$$

we find:

$$\frac{d}{dx}[\sin(x)] = \cos(x).$$

Key Results

- $\frac{d}{dx}[\sin(x)] = \cos(x)$,
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$,
- $\frac{d}{dx}[\tan(x)] = \sec^2(x)$.

Trig Derivative Details

To derive $\frac{d}{dx}[\sin(x)] = \cos(x)$:

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = \frac{\cos(\theta) - 1}{\theta} \cdot \frac{\cos(\theta) + 1}{\cos(\theta) + 1} = \frac{\cos^2(\theta) - 1}{\theta(\cos(\theta) + 1)}.$$

Simplifying:

$$\frac{-\sin^2(\theta)}{\theta(\cos(\theta) + 1)} = \frac{\sin(\theta)}{\theta} \cdot \frac{-\sin(\theta)}{\cos(\theta) + 1}.$$

As $\theta \to 0$, the limit is 0, so:

$$\frac{d}{dx}[\sin(x)] = \cos(x).$$

Key derivatives:

$$\frac{d}{dx}[\sin(x)] = \cos(x), \quad \frac{d}{dx}[\cos(x)] = -\sin(x), \quad \frac{d}{dx}[\tan(x)] = \sec^2(x).$$

Verification of $\frac{d}{dx}[\tan(x)] = \sec^2(x)$:

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}.$$

Using $\sin^2(x) + \cos^2(x) = 1$:

$$\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

12 Chain Rule

If $y = f(u)$, $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example:

$$f(x) = \sqrt{x^2 + 1}.$$

Let $y = f(u) = \sqrt{u} = u^{1/2}$ and $u = x^2 + 1$. Then:

$$f'(x) = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}.$$

General formula:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

Examples:

- $\frac{d}{dx}[\sin(x^2 + 1)] = \cos(x^2 + 1) \cdot 2x$,
- $\frac{d}{dx}[\cos(x)^3] = 3\cos^2(x) \cdot (-\sin(x)) = -3\cos^2(x)\sin(x)$,
- $\frac{d}{dx}[\cos(x^3)] = -\sin(x^3) \cdot 3x^2 = -3x^2\sin(x^3)$.

Power Rule + Chain Rule

If n is a real number and $y = u^n$, $u = g(x)$, the chain rule gives:

$$\frac{dy}{dx} = nu^{n-1} \cdot \frac{du}{dx} = ng(x)^{n-1} \cdot g'(x).$$

Example:

$$\frac{d}{dt}\left[\left(\frac{t-2}{2t+1}\right)^9\right] = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{d}{dt}\left(\frac{t-2}{2t+1}\right).$$

Simplify:

$$\frac{d}{dt}\left(\frac{t-2}{2t+1}\right) = \frac{(2t+1) - (t-2) \cdot 2}{(2t+1)^2} = \frac{t+2}{(2t+1)^2}.$$

Thus:

$$\frac{d}{dt}\left[\left(\frac{t-2}{2t+1}\right)^9\right] = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{t+2}{(2t+1)^2}.$$

Another example:

$$\frac{d}{dz}[\sin(\cos(\tan(z)))] = \cos(\cos(\tan(z))) \cdot \frac{d}{dz}[\cos(\tan(z))].$$

$$\frac{d}{dz}[\cos(\tan(z))] = -\sin(\tan(z)) \cdot \sec^2(z),$$

so:

$$\frac{d}{dz}[\sin(\cos(\tan(z)))] = \cos(\cos(\tan(z))) \cdot (-\sin(\tan(z)) \cdot \sec^2(z)).$$

Exponential Functions Revisited

We saw that:

$$\frac{d}{dx}[e^x] = e^x.$$

For a general base $b > 0$, if $f(x) = b^x$, then:

$$f'(x) = \ln(b) \cdot b^x.$$

Derivation:

$$b^x = (e^{\ln(b)})^x = e^{x\ln(b)}.$$

Using the chain rule:

$$\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x\ln(b)}] = e^{x\ln(b)} \cdot \ln(b) = b^x \ln(b).$$

Special case:

$$\frac{d}{dx}[e^x] = e^x \ln(e) = e^x \cdot 1 = e^x.$$

13 Implicit Differentiation

Examples

1. $x^2 + y^2 = 1$: This is an implicit equation. It cannot be written as a single explicit function.
2. $x^3 + y^3 = 6xy$: This could be written as three explicit functions.

Example: Solve for $\frac{dy}{dx}$

$$x^2 + y^2 = 1.$$

Differentiating implicitly:

$$2x + 2y\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

At the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$:

$$\frac{dy}{dx} = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1.$$

The equation of the tangent line is:

$$y = -x + \sqrt{2}.$$

Example: $\sin(x+y)y^2\cos(x)$**, find** y'**.**

Using the chain rule and product rule:

$$\cos(x+y)(1+y') = 2y'\cos(x) + y^2(-\sin(x)).$$

Rearranging:

$$y'[-\cos(x+y) + 2y\cos(x)] = \cos(x+y) + y^2\sin(x),$$

$$y' = \frac{\cos(x+y) + y^2\sin(x)}{2y\cos(x) - \cos(x+y)}.$$

14 Inverse Derivatives

If $y = f^{-1}(x)$, this means $x = f(y)$. Explicit vs implicit example:

- $y = \arcsin(x) \implies x = \sin(y)$.

Implicitly differentiating:

$$1 = \cos(y)\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}.$$

For $y \in (-\pi/2, \pi/2)$, $\cos(y) \geq 0$. Using $\cos^2(y) + \sin^2(y) = 1$, we derive:

$$\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}.$$

Thus:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}.$$

Similarly, we can show:

$$\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}.$$

For $y = \arctan(x)$, where $x = \tan(y)$:

$$1 = \sec^2(y)\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{1+x^2}.$$

Example: $f(x) = x\arctan(\sqrt{x})$**, find** $f'(x)$**.**

Using the product rule:

$$f'(x) = 1\cdot\arctan(\sqrt{x}) + x\cdot\frac{d}{dx}\arctan(\sqrt{x}),$$

$$f'(x) = \arctan(\sqrt{x}) + x\left(\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\right),$$

$$f'(x) = \arctan(\sqrt{x}) + \frac{\sqrt{x}}{2(1+x)}.$$

15 Derivatives of Logarithms

For $y = b^x$, where $b > 0$:

$$\frac{d}{dx}(b^x) = b^x \ln(b).$$

For $y = \log_b(x)$, we have $b^y = x$. Differentiating implicitly:

$$b^y \ln(b)\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{x\ln(b)}.$$

Thus:

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)}.$$

If $b = e$, then $\ln(e) = 1$:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

Example: $\frac{d}{dx}(\ln(\sin(x)))$

$$\frac{d}{dx}(\ln(\sin(x))) = \frac{1}{\sin(x)} \cdot \cos(x) = \cot(x).$$

For $\frac{d}{dx}(\log_{10}(2 + \sin(x)))$:

$$\frac{d}{dx} = \frac{1}{\ln(10)(2 + \sin(x))} \cdot \cos(x) = \frac{\cos(x)}{\ln(10)(2 + \sin(x))}.$$

General Formula for Inverse Derivatives

If $f^{-1}(x)$ exists and f is differentiable then:

$$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}.$$

Example: $y = \ln(x)$

Let $f(y) = e^y = x$. Then:

$$f'(y) = e^y = x \implies \frac{dy}{dx} = \frac{1}{x}.$$

16 Maxima and Minima

Definitions

- A function $f(c)$ is a *global maximum* if $f(c) \geq f(x)$ for all x.
- A function $f(c)$ is a *global minimum* if $f(c) \leq f(x)$ for all x.
- c must be in the domain of f.
- Sometimes we replace *global* with *absolute*.
- The global maximum and minimum values are called the *extreme values* of f.
- Infinity is not a valid value for global extreme values.

Local Maximum and Minimum

- A function $f(c)$ is a *local maximum* if $f(c) \geq f(x)$ for all x near c.
- A function $f(c)$ is a *local minimum* if $f(c) \leq f(x)$ for all x near c.
- A global maximum or minimum is also a local maximum or minimum if f is defined near c.

Examples

Example 1: $f(x) = 1 - x^2$ The global maximum is at $f(0) = 1$, but there is no global minimum.
Example 2: $g(x) = 1 - x^2$ on $[-1, 1]$ The global and local maximum is at $f(0) = 1$. The global minimum is at $f(1) = 0$, but it is not a local minimum.
Example 3: $h(x) = 1 - x^2$ on $(-1, 1)$ The global and local maximum is at $f(0) = 1$, but there is no global minimum because $f(-1)$ and $f(1)$ are not defined.
Example 4: $f(x) = x^3$ This function has neither a global nor a local maximum or minimum.
Example 5: $g(x) = \cos(x)$ The global and local maximum is at $f(0) = 1$, and the global and local minimum is at $f(\pi) = -1$.

Extreme Value Theorem

If f is continuous on $[a, b]$, then there are points $c, d \in [a, b]$ such that $f(c)$ is a global maximum and $f(d)$ is a global minimum. Note: This theorem does not tell us how to find these points or whether there are local maximum/minimum points.

Fermat's Theorem

If $f'(c)$ exists and $f(c)$ is a local maximum or minimum, then $f'(c) = 0$. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$, and $f'(0) = 0$. However, this does not imply that $f(0)$ is a local maximum or minimum.

Critical Points

A critical point of a function f is a point c where $f'(c) = 0$ or where the derivative does not exist.

Example:

For $f(x) = 6x^5 + 33x^4 - 30x^3 + 100$, we find:

$$f'(x) = 30x^4 + 152x^3 - 90x^2 = 6x^2(5x^2 + 22x - 15).$$

Thus, $f'(x) = 0$ if $6x^2 = 0$ or $5x^2 + 22x - 15 = 0$. Therefore, the critical points are $x = 0$, $x = \frac{3}{5}$, and $x = -5$.

Closed Interval Method

If f is continuous on $[a, b]$, we can find the global maximum and minimum as follows:

1. Find the critical values of f.

2. Evaluate $f(a)$ and $f(b)$.

3. The largest value from step 1 and step 2 is the global maximum, and the smallest value is the global minimum.

Example: Find extrema of $f(x) = 3x^{\frac{1}{3}} - x$ on $[-8, 1]$.

Step 1: $f'(x) = x^{-\frac{2}{3}} - 1$ $f'(x) = 0$ at $x = 1$ and $x = -1$, and $f'(0)$ is not defined. Step 2: Evaluate:

$$f(1) = 2, \quad f(-1) = -2, \quad f(0) = 0.$$

Step 3: Evaluate at endpoints:

$$f(-8) = 2, \quad f(1) = 2.$$

Thus, the global maximum is 2, and the global minimum is $f(-1) = -2$.

17 Rolle's Theorem and the Mean Value Theorem

Recall:

1. **Extreme Value Theorem:** If f is continuous on a closed interval $[a, b]$, then f has both a global maximum and a global minimum on $[a, b]$.

2. **Fermat's Theorem:** If $f'(c)$ exists and $f(c)$ is a local maximum or minimum, then $f'(c) = 0$.

3. The **Closed Interval Method** works by combining the two theorems above.

Example 1: Find the extrema of $g(t) = 2t^3 + 3t^2 - 12t + 4$ on the interval $[-4, 2]$.

First, we find the derivative:

$$g'(t) = 6t^2 + 6t - 12 = 6(t - 1)(t + 2).$$

The critical points are where $g'(t) = 0$, which gives $t = 1$ and $t = -2$. Now, evaluate $g(t)$ at these critical points and endpoints:

$$g(1) = -3, \quad g(-2) = 24, \quad g(-4) = -28, \quad g(2) = 8.$$

We conclude that the global maximum is at $g(-2) = 24$ and the global minimum is at $g(-4) = -28$.

Rolle's Theorem

Rolle's Theorem: Suppose that:

1. f is continuous on the closed interval $[a, b]$,

2. f is differentiable on the open interval (a, b),

3. $f(a) = f(b)$.

Then, there exists a point $c \in (a, b)$ such that $f'(c) = 0$.
Remark: All hypotheses of Rolle's Theorem are necessary for the theorem to apply.

Example 2: Show that $x^3 + x - 1 = 0$ has exactly one solution on $[0, 1]$.

Let $f(x) = x^3 + x - 1$. f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$. We have:

$$f(0) = -1, \quad f(1) = 1.$$

By the Intermediate Value Theorem (IVT), there exists $a \in (0, 1)$ such that $f(a) = 0$.
Suppose that there were another root, say $b \in (0, 1)$, where $f(b) = 0$ and $a < b$. Since $f(x) = x^3+x-1$ is continuous and differentiable on $[a, b]$, by Rolle's Theorem, there exists a point $c \in (a, b)$ such that:

$$f'(c) = 0.$$

But $f'(x) = 3x^2 + 1$. Since $3x^2 + 1 > 0$, we have $f'(x) > 0$ for all x. Therefore, there is only one root in the interval $[0, 1]$.

What if $f(a) \neq f(b)$?

Mean Value Theorem

Mean Value Theorem (MVT): Suppose that:

1. f is continuous on the closed interval $[a, b]$,
2. f is differentiable on the open interval (a, b),

Then, there exists a point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This means that there is at least one point where the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$.

Example 3: If I drive 130 miles in 2 hours on roads with a speed limit of 60 mph, did I speed?

The average speed is:

$$\frac{130}{2} = 65 \text{ mph}.$$

Since the speed limit is 60 mph, and the average speed is greater than 60 mph, by the Mean Value Theorem, there must be a point $c \in (0, 2)$ such that the instantaneous speed $f'(c) = 65$ mph. Hence, I did indeed speed.

Applications of the Mean Value Theorem

Fact 1: If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is constant on (a, b). Why? If c and d are any two points in (a, b), then the Mean Value Theorem gives:

$$\frac{f(c) - f(d)}{c - d} = f'(y) = 0 \quad \Rightarrow \quad f(c) = f(d).$$

Since c and d were chosen arbitrarily, $f(x)$ must be constant on (a, b).
Fact 2: If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists a constant c such that $f(x) = g(x) = c$ for all $x \in (a, b)$. Derivatives determine the shape of the graphs. If two functions have the same derivative, their graphs must be identical, differing only by a constant.

18 Increasing/Decreasing Test

Recall, if $f'(x) = g'(x)$, then $f(x) = g(x) + C$ for some constant C.
A function f is increasing on (a, b) if $f(x) < f(y)$ for $a < x < y < b$.
A function f is decreasing on (a, b) if $f(x) > f(y)$ for $a < x < y < b$.

- (a) If $f'(x) > 0$ on (a, b), then f is increasing on (a, b).
- (b) If $f'(x) \leq 0$ on (a, b), then f is decreasing on (a, b).

Example:

Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. To determine if f is increasing or decreasing:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1).$$

Setting $f'(x) = 0$ gives $x = 0, 2, -1$. We now consider the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$, and $(2, \infty)$. Make a sign chart for all intervals.

19 First Derivative Test

Suppose c is a critical point of f:

- (a) If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a local maximum.
- (b) If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a local minimum.
- (c) If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local max nor a local min.

First Derivative Test Conclusion:

- $f(-1)$ is a local minimum.
- $f(0)$ is a local maximum.
- $f(2)$ is a local minimum.

Example:

For $g(x) = |x|$, the critical point is $x = 0$.
We have:

$$g'(x) = -1 \quad \text{on } (-\infty, 0),$$
$$g'(x) = 1 \quad \text{on } (0, \infty).$$

The first derivative test tells us that $g(0)$ is a local minimum.

20 Second Derivative and the Concavity Test

- Convex up = CU, Concave down = CD.
- (a) If $f''(x) > 0$ on (a, b), then f is concave up on (a, b).
- (b) If $f''(x) < 0$ on (a, b), then f is concave down on (a, b).

Example:

For $f(x) = x^3$, we have:

$$f'(x) = 3x^2, \quad f''(x) = 6x,$$
$$f''(x) > 0 \quad \text{on } (0, \infty),$$
$$f''(x) < 0 \quad \text{on } (-\infty, 0),$$
$$f''(x) = 0 \quad \text{at } x = 0.$$

Thus, $x = 0$ is an inflection point.

21 Inflection Test

A point c is an inflection point if f is continuous at c and the concavity changes from concave up to concave down or vice versa.
c is an inflection point if $f''(x)$ changes sign at c.
Example: $x = 0$ is an inflection point for $f(x) = x^3$.

22 Second Derivative Test

Suppose $f''(x)$ is continuous at c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Remarks:

The second derivative test is easier to apply than the first derivative test (just check if $f'(x) = 0$ and $f''(x) > 0$ or $f''(x) < 0$), but it has a more restrictive condition (requires $f''(x)$ to be continuous).

Examples:

For $g(x) = x^2$, we have $g'(x) = 2x$, $g'(0) = 0$, and $g''(x) = 2 > 0$. So $x = 0$ is a local minimum.
For $g(x) = x^3$, we have $g'(x) = 3x^2$, $g'(0) = 0$, and $g''(x) = 6x$. Since $f''(x)$ changes sign at $x = 0$, the inflection test tells us that $x = 0$ is an inflection point.

Example:

Consider $f(x) = x^4 - 4x^3$. We have:

$$f'(x) = 4x^2(x - 3),$$
$$f'(x) = 0 \text{ if } x = 0, 3,$$
$$f''(x) = 12x(x - 2),$$
$$f''(x) = 0 \text{ if } x = 0, 2.$$

Applying the second derivative test:

$$f'(0) = 0, \quad f''(0) = 0 \quad \text{(no information!)}.$$

$$f'(3) = 0, \quad f''(3) = 36 > 0 \quad \text{(local minimum at } x = 3\text{)}.$$

First derivative test:

$$f'(x) < 0 \text{ on } (-\infty, 3).$$

Since $f'(x) < 0$ on both $(-\infty, 0)$ and $(0, 3)$, $f(0)$ is not a local max or min.
The sign chart is then:

- Interval $(-\infty, 0)$: Concave up (CU).
- Interval $(0, 2)$: Concave down (CD).
- Interval $(2, \infty)$: Concave up (CU).

Curve sketching example:

Use the first and second derivative tests and limits to sketch the graph of $f(x) = e^{1/x}$.
The domain is $(-\infty, 0) \cup (0, \infty)$.
As $x \to 0^+$, $f(x) \to +\infty$.
As $x \to 0^-$, $f(x) \to -\infty$.
The derivative is:

$$f'(x) = e^{1/x} \cdot \left(-\frac{1}{x^2}\right).$$

Thus, $f'(x) < 0$ for all x in the domain, so there are no local maxima or minima.
What about $f''(x)$?

$$f''(x) = e^{1/x} \cdot \frac{2x+1}{x^4}.$$

So:

$$f''(x) < 0 \quad \text{if } x < -\frac{1}{2} \quad \text{(CD)},$$

$$f''(x) > 0 \quad \text{if } x > -\frac{1}{2} \quad \text{(CU)}.$$

Thus, $x = -\frac{1}{2}$ is an inflection point.

23 Rates of Change in Science

In physics, $s(t)$ is the position of a particle measured horizontally.
Let $s(t) = t^3 - 6t^2 + 9t$.

Questions:

1. When is the particle at rest?
We find the velocity $s'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$, so the particle is at rest at $t = 1$ and $t = 3$.
2. What is the initial speed?
We compute $s'(0) = 9\,\text{m/s}$.
Note: Speed is the absolute value of the velocity, $|s'(t)|$.
3. When is the particle moving forward?
The particle moves forward when $s'(t) > 0$. This occurs for $0 \le t < 1$ and $t > 3$.
The particle moves backward for $1 < t < 3$.
Thus, the particle moves forward for 1 second, then backward for 2 seconds, and then forward again for the rest of time.

24 Exponential Growth/Decay

If a quantity grows/decays at a rate proportional to the amount of the quantity, then the quantity is modeled exponentially:

$$\frac{dp}{dt} = k \cdot p(t), \quad k = \text{growth/decay rate.}$$

The solution is $p(t) = ce^{kt}$, where $p(0)$ is the initial amount of the quantity.

Example:

The Earth's population since 1950 (in millions) is modeled by:

$$p(t) = 2560e^{0.017t},$$

where 0.017 is the growth rate $\approx 1.7\%$.

Example:

Recall that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

If you have \$D at an interest rate r, and if it compounds once a year, you get $D + rD$. If it compounds n times a year, you get:

$$\left(1+\frac{r}{n}\right)^n D.$$

As $n \to \infty$, this approaches:

$$e^r D.$$

25 Logarithmic Differentiation (not examinable)

Example

Let $y = \frac{x^3\sqrt{x^2+1}}{(2x-1)^2}$. Taking the natural logarithm of both sides,

$$\ln(y) = \ln\left(\frac{x^3\sqrt{x^2+1}}{(2x-1)^2}\right).$$

Using logarithmic properties $\ln(ab) = \ln(a) + \ln(b)$, $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$, and $\ln(a^b) = b\ln(a)$, we get

$$\ln(y) = \ln(x^3) + \ln(\sqrt{x^2+1}) - \ln((2x-1)^2).$$

This simplifies to

$$\ln(y) = 3\ln(x) + \frac{1}{2}\ln(x^2+1) - 2\ln(2x-1).$$

Now, differentiating both sides with respect to x,

$$\frac{y'}{y} = \frac{3}{x} + \frac{x}{x^2+1} - \frac{4}{2x-1}.$$

Thus,

$$y' = y\left(\frac{3}{x} + \frac{x}{x^2+1} - \frac{4}{2x-1}\right) = \frac{x^3\sqrt{x^2+1}}{(2x-1)^2}\left(\frac{3}{x} + \frac{x}{x^2+1} - \frac{4}{2x-1}\right).$$

General Rule

For functions of the form $y = f(x)^{g(x)}$, we have

$$\ln(y) = g\ln(f).$$

Differentiating,

$$\frac{y'}{y} = g'\ln(f) + g\frac{f'}{f},$$

or equivalently,

$$y' = y\left(g'\ln(f) + g\frac{f'}{f}\right) = f^g\left(g'\ln(f) + g\frac{f'}{f}\right).$$

Example

Let $y = x^{\sqrt{x}}$ with $f(x) = x$ and $g(x) = \sqrt{x}$. Then $f' = 1$ and $g' = \frac{1}{2\sqrt{x}}$. So,

$$y' = x^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln(x) + \frac{\sqrt{x}}{x}\right),$$

or

$$y' = x^{\sqrt{x}}\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right).$$

26 Related Rates

Example: Balloon Inflation

Suppose air is pumped into a balloon, causing both the volume and radius to increase. Given $\frac{dV}{dt} = 100\,\text{cm}^3/\text{s}$, find $\frac{dr}{dt}$ when $r = 25\,\text{cm}$.

$$V = \frac{4}{3}\pi r^3,$$

Differentiating both sides with respect to t,

$$\frac{dV}{dt} = \frac{4}{3}\pi\frac{d}{dt}(r^3) = 4\pi r^2\frac{dr}{dt}.$$

Solving for $\frac{dr}{dt}$,

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\cdot\frac{dV}{dt} = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi}\,\text{cm/s}.$$

Strategy for Related Rate Problems

1. Read the question and draw a picture.
2. Define quantities with names (e.g., V, r, t in this example).
3. Express the given information in terms of these quantities (e.g., $\frac{dV}{dt} = 100$, $\frac{dr}{dt} = ?$).
4. Relate the quantities using an equation (e.g., $V = \frac{4}{3}\pi r^3$).
5. Use the chain rule and substitution to find the desired rate.

Example: Ladder Against a Wall

A ladder 10 m long leans against a flat wall. How fast is the top sliding down if the bottom moves horizontally at 1 m/s and the height of the top is 8 m?
1. Draw a picture of a right triangle.
2. Let x be the horizontal distance and y be the vertical distance from the corner.
3. Given $\frac{dx}{dt} = 1\,\text{m/s}$, find $\frac{dy}{dt}$ when $y = 8$.
4. By the Pythagorean theorem, $x^2 + y^2 = 10^2$. - When $y = 8$, $x^2 + 64 = 100 \Rightarrow x = 6$.
5. Differentiating, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. - Substituting $x = 6$, $y = 8$, and $\frac{dx}{dt} = 1$,

$$12 + 16\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{3}{4}\,\text{m/s}.$$

Example: Airplane and Searchlight

A plane flying at a height of 3 km is moving horizontally at 300 km/hr towards a searchlight fixed on the ground. We want to know how fast the searchlight is turning when the plane is 5 km away from the searchlight.
1. Draw a picture, where θ is the angle between the ground and the line connecting the searchlight to the plane.
2. Define quantities: - x: the horizontal distance from the searchlight (km), - θ: the angle between the ground and the searchlight beam (radians), - t: time (hours).
3. Given $\frac{dx}{dt} = -300\,\text{km/hr}$ (negative as x decreases), we want $\frac{d\theta}{dt}$ when the distance to the searchlight is 5 km or $x = 4$ km.
4. Since $\tan(\theta) = \frac{3}{x}$, we have $x\tan(\theta) = 3$.
5. Differentiating both sides with respect to t,

$$\frac{dx}{dt}\tan(\theta) + x\sec^2(\theta)\frac{d\theta}{dt} = 0.$$

Substitute $\tan(\theta) = \frac{3}{4}$, $x = 4$, and $\frac{dx}{dt} = -300$: - Using $\sec^2(\theta) = \frac{25}{16}$,

$$-300 \cdot \frac{3}{4} + 4 \cdot \frac{25}{16}\frac{d\theta}{dt} = 0.$$

Solving for $\frac{d\theta}{dt}$,

$$\frac{d\theta}{dt} = \frac{36}{\text{radians/hr}}.$$

Example: Water Tank Draining

A water tank in the shape of an inverted cone has a base radius of 2 m and height of 4 m. Water is draining at a rate of $\frac{dV}{dt} = -2\,\text{m}^3/\text{s}$. Find the rate of change of the water level, $\frac{dh}{dt}$, when the water depth is 3 m.
1. Draw a picture of the cone with water depth h and top radius r.
2. Define variables: - h: height of the water (m), - r: radius of the water surface (m), - V: volume of water (m^3), - t: time (s).
3. Given $\frac{dh}{dt} =?$ when $h = 3$ and $\frac{dV}{dt} = -2\,\text{m}^3/\text{s}$.
4. Since $V = \frac{1}{3}\pi r^2 h$ and $\frac{r}{h} = \frac{1}{2}$, we have $r = \frac{h}{2}$.
5. Substituting for r in the volume formula:

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3.$$

Differentiating with respect to t,

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \frac{dh}{dt}.$$

Solving for $\frac{dh}{dt}$,

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{1}{4}\pi h^2} = \frac{-2}{\frac{1}{4}\pi \cdot 9} = -\frac{8}{9\pi} \text{ m/s}.$$

27 Linear Approximation (not examinable)

The idea is that tangent lines approximate functions near a point. Given $f(x)$, we can approximate $f(x)$ near $x = a$ using the tangent line $L(x)$:

$$L(x) = f(a) + f'(a)(x - a).$$

Thus, $f(x) \approx L(x)$ if x is close to a.

Example: Approximating Square Roots

To approximate $\sqrt{3.98}$ and $\sqrt{4.05}$, let $f(x) = \sqrt{x+3}$ and set $a = 1$, where $f(a) = 2$ and $f'(x) = \frac{1}{2\sqrt{x+3}}$. At $x = 1$, $f'(1) = \frac{1}{4}$, so

$$L(x) = 2 + \frac{1}{4}(x - 1) = \frac{x}{4} + \frac{7}{4}.$$

Then,

$$\sqrt{3.98} \approx \frac{0.98}{4} + \frac{7}{4}, \quad \sqrt{4.05} \approx \frac{1.05}{4} + \frac{7}{4}.$$

Example: Approximating $\sin(\theta)$ Near $\theta = 0$

Let $f(\theta) = \sin(\theta)$. Near $\theta = 0$, $f'(\theta) = \cos(\theta)$, and $f'(0) = 1$, so

$$\sin(\theta) \approx 0 + 1 \cdot (\theta - 0) = \theta \quad \text{if } \theta \approx 0.$$

28 L'Hôpital's Rule

Definition

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or both limits approach $\pm\infty$, then $\frac{f(x)}{g(x)}$ is an indeterminate form. In such cases, if f and g are differentiable and $g'(x) \neq 0$ near a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

Examples

1. **Limit of $\frac{\ln(x)}{x-1}$ as $x \to 1$:** - This is a $\frac{0}{0}$ form. Differentiating numerator and denominator:

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1.$$

2. **Limit of $\frac{e^x}{x^2}$ as $x \to \infty$:** - This is an $\frac{\infty}{\infty}$ form. Differentiating repeatedly,

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

3. **Limit of $\frac{x^2-1}{2x^2+1}$ as $x \to \infty$:** - This is also an $\frac{\infty}{\infty}$ form. Dividing numerator and denominator by x^2,

$$\lim_{x\to\infty} \frac{x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{1}{2}.$$

Notes on L'Hôpital's Rule

L'Hôpital's Rule should only be used when an expression is indeterminate (e.g., $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Other techniques, such as factoring or trigonometric identities, may be more efficient in cases that are not indeterminate form

Definition

We say that $\lim_{x\to a} f(x)g(x)$ is an indeterminate form of type $0 \cdot \infty$ if:

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty.$$

To resolve this form, we can rewrite $f \cdot g$ as:

$$\frac{f}{1/g} \quad \text{or} \quad \frac{g}{1/f}.$$

Example

$\lim_{x\to 0^+} x\ln(x)$
Firstly, $\lim_{x\to 0^+} x = 0$, and $\lim_{x\to 0^+} \ln(x) = -\infty$, so this is a $0 \cdot \infty$ form.
Rewriting:

$$x\ln(x) = \frac{\ln(x)}{1/x}.$$

Now, $\lim_{x\to 0^+} \frac{\ln(x)}{1/x}$ is an $\frac{\infty}{\infty}$ form. Using L'Hôpital's Rule:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Thus:

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

Definition

We say that $\lim_{x\to a}(f(x)-g(x))$ is an indeterminate form of type $\infty-\infty$ if:

$$\lim_{x\to a} f(x)=\infty \quad \text{and} \quad \lim_{x\to a} g(x)=\infty.$$

To resolve this form, we rewrite:

$$f-g=f\left(1-\frac{g}{f}\right) \quad \text{or} \quad f-g=g\left(\frac{f}{g}-1\right),$$

then apply L'Hôpital's Rule if necessary.

Example

$\lim_{x\to\infty} e^x - x$
Here:

$$\lim_{x\to\infty} e^x=\infty, \quad \lim_{x\to\infty} x=\infty,$$

so this is an $\infty-\infty$ form.
Rewrite:

$$e^x-x=x\left(\frac{e^x}{x}-1\right).$$

Now, $\lim_{x\to\infty}\frac{e^x}{x}=\infty$, so:

$$\lim_{x\to\infty} x\left(\frac{e^x}{x}-1\right)=\infty.$$

Thus, $\lim_{x\to\infty} e^x-x=\infty$.

Definition

We say that $\lim_{x\to a} f(x)^{g(x)}$ is an indeterminate form of type:

1. 0^0 if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$.
2. ∞^0 if $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=0$.
3. 1^∞ if $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$.

We write $y=f^g$, then $\ln(y)=g\ln(f)$, which is a $0\cdot\infty$ form in all cases.

Example

$\lim_{x\to 0^+} x^x$
This is a 0^0 form. Writing:

$$\ln(x^x)=x\ln(x).$$

We already evaluated $\lim_{x\to 0^+} x\ln(x)=0$. Thus:

$$\lim_{x\to 0^+} x^x=\lim_{x\to 0^+} e^{\ln(x^x)}=e^0=1.$$

29 Optimization Problems

General Strategy for Optimization Problems

To solve optimization problems, follow these steps:

1. **Understand the problem:** Read the question carefully and draw a diagram to visualize the situation.

2. **Identify variables:** Determine the independent and dependent variables along with their units.

3. **Express given and unknown information:** Formulate the known constraints and the quantity to be optimized.

4. **Refine relationships:** Use equations to express constraints and relationships among variables.

5. **Solve the problem:** Use calculus methods, such as the Closed Interval Method (CIM), first derivative test, or second derivative test, to find the maximum or minimum values.

Example 1: Maximizing the Area of a Rectangle

Problem: Given 100 ft of fencing, what is the largest area that can be enclosed with a rectangular fence?
Solution:

1. **Understand the problem:** Let the width of the rectangle be x ft and the height be y ft. The area is given by $A = xy$.

2. **Identify variables:** x and y (in ft) are independent variables, and A (in square feet) is the dependent variable.

3. **Express constraints:** The perimeter constraint is:

$$2x + 2y = 100 \quad \Rightarrow \quad x + y = 50 \quad \Rightarrow \quad y = 50 - x.$$

4. **Substitute and simplify:** Substitute $y = 50 - x$ into $A = xy$:

$$A(x) = x(50 - x) = 50x - x^2.$$

5. **Find critical points:** Differentiate $A(x)$:

$$A'(x) = 50 - 2x.$$

   Set $A'(x) = 0$:

$$50 - 2x = 0 \quad \Rightarrow \quad x = 25.$$

6. **Second derivative test:** Compute $A''(x)$:

$$A''(x) = -2.$$

   Since $A''(x) < 0$, $x = 25$ is a local maximum.

7. **Conclusion:** The maximum area is achieved when $x = 25$ and $y = 25$, giving:

$$A = 25 \cdot 25 = 625 \text{ square feet.}$$

Example 2: Minimizing the Cost of a Box

Problem: A box has a base length three times its width. The top and bottom cost $10/\text{ft}^2$, and the sides cost $6/\text{ft}^2$. What is the cheapest box enclosing 50ft^3?
Solution:

1. **Understand the problem:** Let the width of the base be w ft, the height be h ft, and the length of the base be $3w$ ft. The cost C (in dollars) and the volume V (in cubic feet) are related.

2. **Identify variables:** Independent variables: w, h (in ft). Dependent variables: C (in dollars).

3. **Express constraints:** The volume constraint is:

$$V = 3w^2h = 50.$$

The cost is:

$$C = 2(10)(3w^2) + 2(6)(3wh) + 2(6)(wh) = 60w^2 + 48wh.$$

Substitute $h = \frac{50}{3w^2}$:

$$C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w}.$$

4. **Minimize cost:** Differentiate $C(w)$:

$$C'(w) = 120w - \frac{800}{w^2}.$$

Set $C'(w) = 0$:

$$120w = \frac{800}{w^2} \quad \Rightarrow \quad w^3 = \frac{800}{120} = \frac{20}{3} \quad \Rightarrow \quad w = \sqrt[3]{\frac{20}{3}}.$$

5. **Second derivative test:** Compute $C''(w)$:

$$C''(w) = 120 + \frac{1600}{w^3}.$$

Since $C''(w) > 0$, $w = \sqrt[3]{\frac{20}{3}}$ minimizes cost.

6. **Conclusion:** Substitute $w = \sqrt[3]{\frac{20}{3}}$ into $C(w)$:

$$C \approx 637 \text{ dollars.}$$

30 Antiderivatives

Physically, we aim to go from velocity to position or from rate to amount.
Definition: A function F is an antiderivative for a function f if $F'(x) = f(x)$.
Example 1: If $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative of f, as $(x^2)' = 2x$.
Example 2: If $g(x) = x^2$, then by the power rule $(x^3)' = 3x^2$, so $G(x) = \frac{1}{3}x^3$ satisfies $G'(x) = x^2 = g(x)$.
Remarks: 1. If F is an antiderivative of f, then F is differentiable and continuous. 2. Antiderivatives are not unique because we can add a constant: for example, $(x^2 + c)' = 2x$.

Recall: The Mean Value Theorem implies that if $F' = G'$, then $F = G + c$ for some constant c.
General Antiderivative: If F is a specific antiderivative, then any other antiderivative G of f is of the form $G = F + c$. The collection of functions $F(x) + c$ is called the general antiderivative of f.
Examples: 1. $x^2 + c$ are the general antiderivatives of $2x$. 2. $\frac{1}{3}x^3 + c$ are the general antiderivatives of x^2. 3. The general antiderivative of $\cos(x)$ is $\sin(x) + c$. 4. If $f(x) = \frac{1}{x}, x \neq 0$, then $\ln|x|$ is a specific antiderivative, and the general antiderivative is:

$$\begin{cases} \ln|x| + c & x > 0, \\ \ln|x| + d & x < 0. \end{cases}$$

Rules for Antiderivatives

1. If $F' = f$, then $(cF)' = cf$, so cF is an antiderivative of cf. 2. If $F' = f$ and $G' = g$, then $(F+G)' = f + g$, so $F + G$ is an antiderivative for $f + g$. 3. If $f(x) = e^x$, then $F(x) = e^x$ is an antiderivative, and the general antiderivative is $e^x + c$. 4. If $f(x) = x^n$, $n \neq -1$, then the general antiderivative is $\frac{1}{n+1}x^{n+1} + c$.
Examples:

1. Find all $g(x)$ such that $g'(x) = 4\sin(x) + 2x^5 - \frac{\sqrt{x}}{x}$.

$$g'(x) = 4\sin(x) + 2x^4 - x^{-1/2},$$

$$g(x) = -4\cos(x) + \frac{2}{5}x^5 - 2x^{1/2} + c.$$

2. Find f such that $f''(x) = 2 - x$, $f(1) = 1$, $f'(0) = 0$.

   - Guess: $f'(x) = 2x - \frac{1}{2}x^2 + c$ (general antiderivative of f'').
   - Use $f'(0) = 0$ to solve for c: $c = 0$. Thus, $f'(x) = 2x - \frac{1}{2}x^2$.
   - Antiderivative of $f'(x)$: $f(x) = x^2 - \frac{1}{6}x^3 + d$.
   - Use $f(1) = 1$ to solve for d: $1 - \frac{1}{6} + d = 1 \Rightarrow d = \frac{1}{6}$.
   - Final solution: $f(x) = x^2 - \frac{1}{6}x^3 + \frac{1}{6}$.

3. A ball is thrown upward at 10 m/s from a 100 m high cliff. Gravity is -10 m/s^2. Find how long it takes to reach the ground.

   - $a(t) = -10$ m/s^2. Velocity $v(t)$ satisfies $v'(t) = a(t)$, so $v(t) = -10t + c$.
   - Given $v(0) = 10$ m/s, solve for c: $c = 10$, so $v(t) = 10 - 10t$.
   - Height $h(t)$ satisfies $h'(t) = v(t)$, so $h(t) = 10t - 5t^2 + 100$.
   - Set $h(t) = 0$ to find when the ball hits the ground:

$$10t - 5t^2 + 100 = 0 \Rightarrow 2t^2 - 4t - 20 = 0 \Rightarrow t = 1 + \sqrt{21} \approx 5.6 \text{ s}.$$

Antiderivative of $|x|$:

- If $x > 0$, $|x| = x$, so $\frac{x^2}{2} + c$ is an antiderivative.
- If $x < 0$, $|x| = -x$, so $-\frac{x^2}{2} + d$ is an antiderivative.

- At $x = 0$, $F(x)$ must be continuous. Thus, the antiderivative is piecewise continuous:

$$F(x) = \begin{cases} \frac{x^2}{2} + c & x > 0, \\ -\frac{x^2}{2} + d & x < 0. \end{cases}$$

We will see that every continuous function has an antiderivative.

31 Area and Distance

If $f(x) \geq 0$ and is continuous, how do we find the area below the function?

Basic Shapes for Area Calculation

- Rectangle: Area is $w \cdot h$.
- Triangle: Area is $\frac{1}{2}b \cdot h$.
- Polygon: Divide into triangles and sum their areas.

For curved regions, such as $f(x) = x^2$ on $[0, 1]$, the area is less than 1, as it fits within the unit square.

Approximation with Rectangles

To approximate the area under $f(x)$, use rectangles:
- Right-hand sum with 4 rectangles:

$$R_4 = \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(\frac{3}{4})^2 + \frac{1}{4}(1)^2 = 0.46875.$$

- Left-hand sum with 4 rectangles:

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(\frac{3}{4})^2 = 0.21875.$$

Observation:

Left-hand sum < Actual Area < Right-hand sum.

Refining the Approximation

Using more rectangles gives better estimates:

Number of Rectangles	L_n (Left-hand sum)	R_n (Right-hand sum)
10	0.285	0.385
100	0.32885	0.33835
1000	0.3328335	0.3338335

As the number of rectangles approaches infinity, both L_n and R_n converge to $\frac{1}{3}$.

General Formula for R_n

$$R_n = \frac{1}{n}\left(\frac{1}{n}^2 + \frac{2}{n}^2 + \frac{3}{n}^2 + \cdots + \frac{n}{n}^2\right).$$

The width of each rectangle is $\frac{1}{n}$, and the height is the square of the endpoint.

Using the formula for the sum of squares:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

we can write:

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{6} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right).$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} R_n = \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}.$$

Similarly, $\lim_{n\to\infty} L_n = \frac{1}{3}$.

Defining Area under $f(x)$

For $f(x) \geq 0$ and continuous on $[a, b]$: - Divide $[a, b]$ into n rectangles of width $\Delta x = \frac{b-a}{n}$. - Right-hand endpoints: $x_i = a + i \cdot \Delta x$ for $i = 1, 2, \ldots, n$. - Define area as:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i).$$

Key Notes: 1. The limits of L_n and R_n agree. 2. Choosing other sample points gives the same limit.

Riemann Sums

Using sigma notation, write:

$$R_n = \sum_{i=1}^{n} \Delta x f(x_i).$$

The area under $f(x)$ is the limit of the Riemann sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i).$$

Distance

For constant velocity, distance equals speed $\times$ time. For varying velocity $v(t)$, the distance is the area under the velocity-time graph.
By the same reasoning as for area, the distance traveled for $v(t) \geq 0$ and continuous is:

$$\text{Distance} = \int_a^b v(t)\, dt.$$

Units check out: seconds $\times$ meters/second = meters.

32 The Definite Integral

When computing areas, we looked at limits of the form:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i \cdot \Delta x$.

Definition

If a function f is defined on $[a, b]$, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i \cdot \Delta x$ for $i = 1, 2, 3, \ldots, n$, and we choose sample points $x_i^* \in [x_{i-1}, x_i]$, then the definite integral of f from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided this limit exists and is independent of the choice of x_i^*.
Key Notes: 1. $\int_a^b f(x)\,dx$ is a number and does not depend on x (a dummy variable). For example:

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr.$$

2. We do not require f to be continuous or positive, but if it is, $\int_a^b f(x)\,dx$ corresponds to the area bounded by f and the x-axis.

Theorems

1. If f is continuous on $[a, b]$ or has finitely many jump discontinuities, then $\int_a^b f(x)\,dx$ exists (e.g., for polynomials, trig functions, exponentials, logs, etc.). 2. If f is integrable on $[a, b]$, then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n,$$

where $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$.

Signed Area

If f is negative, we interpret $\int_a^b f(x)\,dx$ as the signed area:

$$\int_a^b f(x)\,dx = A_+ - A_-,$$

where: - A_+ = area between $f \geq 0$ and the x-axis, - A_- = area between $f \leq 0$ and the x-axis.

Examples

1. $\int_0^3 f(x-1)\,dx$, $f(x) = x - 1$:

$$A_+ = \frac{2 \cdot 2}{2} = 2, \quad A_- = \frac{1 \cdot 1}{2} = \frac{1}{2}.$$

So, $\int_0^3 f(x-1)\,dx = A_+ - A_- = 2 - \frac{1}{2} = \frac{3}{2}$.
2. $g(x) = \begin{cases} -2, & x < 0, \\ x^2, & x \geq 0 \end{cases}$:

$$\int_{-2}^{1} g(x)\,dx = \frac{1}{3} - 4 = -\frac{11}{4}.$$

3. $h_+(x) = \sqrt{1-x^2}$, $h_-(x) = -\sqrt{1-x^2}$:

$$\int_0^1 h_+(x)\,dx = \frac{\pi}{4}, \quad \int_{-1}^{0} h_-(x)\,dx = -\frac{\pi}{4}.$$

Recognizing Limits

On $[0, \pi]$, consider:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \sin(x_i)\right) \Delta x.$$

We identify $f(x) = x^3 + x\sin(x)$ and write:

$$\int_0^{\pi} \left(x^3 + x\sin(x)\right)\, dx.$$

Example: Non-integrable Function

For $\int_0^2 f(x^2+1)\, dx$, with $f(x) = x^2 + 1$, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, and $x_i = 0 + i\Delta x = \frac{2i}{n}$:

$$\int_0^2 f(x^2+1)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^2 + 1\right) \frac{2}{n}.$$

So:

$$R_n = \sum_{i=1}^{n} \frac{8i^2}{n^3} + \frac{2}{n}.$$

Using the sum of integers and sum of squares:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

we find:

$$R_n = \frac{2}{n} \cdot \sum_{i=1}^{n} 1 + \frac{8}{n^3} \cdot \sum_{i=1}^{n} i^2 = \frac{2n}{n} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

Simplifying:

$$R_n = 2 + \frac{8(n+1)(2n+1)}{6n^2}.$$

Taking the limit as $n \to \infty$:

$$\int_0^2 (x^2+1)\, dx = \lim_{n\to\infty} \left(2 + \frac{8(n+1)(2n+1)}{6n^2}\right) = \frac{14}{3}.$$

33 Properties of the Definite Integral

We saw that if f is integrable on $[a, b]$, then

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} R_n,$$

where

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

Let f, g be integrable functions and C a constant.

1. $\int_a^b C\,dx = C(b-a)$
2. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$
   ("C stretches f, so C stretches the approximating rectangles").
   In particular,
   $$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$
4. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
   (This can be visualized as the total area being split between $[a,b]$ and $[b,c]$).

Special Definitions

We always assumed that $a < b$. Now we define:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$$

where if $a < b$, $\Delta x = \frac{b-a}{n}$, and if $a > b$, $\Delta x = \frac{a-b}{n}$.
Additionally,

$$\int_a^a f(x)\,dx = 0.$$

From property (4):

$$\int_a^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^b f(x)\,dx - \int_a^b f(x)\,dx = 0.$$

Examples

Example 1:

$$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx.$$

$$= 4(1-0) + 3\int_0^1 x^2\,dx = 4 + 3\cdot\frac{1}{3} = 4 + 1 = 5.$$

Example 2: If $\int_0^1 f(x)\,dx = 2$ and $\int_{10}^{11} f(x)\,dx = -1$, what is $\int_0^{10}(2f(x)-1)\,dx$?

$$\int_0^{10}(2f(x)-1)\,dx = \int_0^{10} 2f(x)\,dx + \int_0^{10} -1\,dx.$$

$$= 2\int_0^{10} f(x)\,dx - 1(10-0).$$

$$= 2\int_0^{10} f(x)\,dx - 10.$$

$$= 2\left(\int_0^{11} f(x)\,dx - \int_{10}^{11} f(x)\,dx\right) - 10.$$

$$= 2(2-(-1)) - 10 = 4.$$

Example 3: Given $\int_0^\pi \sin^4(x)\,dx = \frac{3\pi}{8}$, find $\int_\pi^0 \sin^4(\theta)\,d\theta$.
The definite integral does not depend on the variable:

$$\int_0^\pi \sin^4(x)\,dx = \int_0^\pi \sin^4(\theta)\,d\theta = -\int_\pi^0 \sin^4(\theta)\,d\theta = -\frac{3\pi}{8}.$$

Comparison of Definite Integrals

1. If $f \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.
2. If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
3. If $m \leq f(x) \leq M$ on $[a,b]$ (assuming f is continuous, m and M are global extrema), then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example: Estimate $\int_0^2 e^x\,dx$. Since e^x is continuous, on $[0,2]$:

$$0 \leq e^x \leq e^2.$$

Additionally,

$$e^x \geq x+1 \text{ on } [0,2].$$

By comparison,

$$\int_0^2 e^x\,dx \geq \int_0^2 (x+1)\,dx = \int_0^2 x\,dx + \int_0^2 1\,dx = 2+2 = 4.$$

Also,

$$\int_0^2 e^x\,dx \leq e^2(2-0) = 2e^2.$$

Thus,

$$4 \leq \int_0^2 e^x\,dx \leq 2e^2.$$

34 Which Functions are not Integrable?

If f is defined on $[a,b]$, let $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1,2,3,\ldots,n$. If we choose $x_i^* \in [x_{i-1}, x_i]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x,$$

if the limit exists, is finite, and is independent of the choice of x_i^*.

We saw that if f is continuous or has finitely many jump discontinuities on $[a,b]$, then $\int_a^b f(x)\,dx$ exists (i.e., f is integrable on $[a,b]$).

If a function is not integrable, then one of the following two things could happen:

1. If $\lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x$ does not exist or is $\pm\infty$.
2. If $\lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x$ depends on different choices of x_i^*.

Example: Let $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} & \text{if } 0 < x \leq 1 \end{cases}$ on $[0, 1]$.
For $a > 0$, this function is continuous on $[a, b]$, so it is integrable.
Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$, and $x_i = 0 + i\Delta x = \frac{i}{n}$ for $i = 1, 2, 3, \ldots, n$.
Choose $x_1^* = \frac{1}{n^2} \in [x_0, x_1]$ and $x_i^* = x_i$ for $i = 2, 3, \ldots, n$.

- For $i = 1$,
$$f(x_1^*)\Delta x = \frac{1}{x_1^*} \cdot \frac{1}{n} = n^2 \cdot \frac{1}{n} = n.$$

- For $i = 2, 3, \ldots, n$,
$$f(x_i^*)\Delta x = \frac{1}{x_i^*} \cdot \frac{1}{n} = \frac{1}{i}.$$

Thus,
$$\sum_{i=1}^{n} f(x_i^*)\Delta x = n + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} > n.$$

Taking the limit as $n \to \infty$,
$$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x > \lim_{n\to\infty} n = +\infty.$$

Thus, $f(x)$ is not integrable on $[0, 1]$.

Example: Let $g(x) = \begin{cases} 1 & \text{if } x \text{ is irrational}, \\ 0 & \text{if } x = \frac{p}{q} \text{ where } p, q \text{ are integers.} \end{cases}$ on $[0, 1]$.
This function has infinitely many jump discontinuities. Let $\Delta x = \frac{1}{n}$, and $x_i = \frac{i}{n}$.
Choose two types of x_i^*:

1. Choose x_i^* to be a rational number in $[x_{i-1}, x_i]$. Then $g(x_i^*) = 0$, so
$$g(x_i^*)\Delta x = 0.$$
Hence,
$$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x = 0.$$

2. Choose x_i^* to be an irrational number in $[x_{i-1}, x_i]$. Then $g(x_i^*) = 1$, so
$$g(x_i^*)\Delta x = \frac{1}{n}.$$
Hence,
$$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x = \lim_{n\to\infty} n \cdot \frac{1}{n} = 1.$$

Since the limit depends on the choice of x_i^*, $g(x)$ is not integrable on $[0, 1]$.

Integrals as Functions

Given a function f integrable on $[a, b]$, for $x \in [a, b]$ the definite integral,

$$\int_a^x f(x)\,dx,$$

is a number. Define $F(x) = \int_a^x f(x)\,dx$, which is a function. We observe that:

- $F(b) = \int_a^b f(x)\,dx$,
- $F(a) = \int_a^a f(x)\,dx = 0$.

Example: If $f(x) = -2$ on $[-1, 1]$, then

$$F(x) = \int_{-1}^x (-2)\,dx = (-2)(x - (-1)) = -2x - 2.$$

If $g(x) = -2$ on $[-3, 1]$, then

$$G(x) = \int_{-3}^x (-2)\,dx = -2(x - (-3)) = -2x - 6.$$

Note that $F(x) = G(x) + 4$.

General Rule: If $F(x) = \int_a^x f(x)\,dx$ and $G(x) = \int_b^x f(x)\,dx$, then

$$F(x) - G(x) = \int_a^x f(x)\,dx - \int_b^x f(x)\,dx = \int_a^b f(x)\,dx,$$

which is a constant.

Antiderivative Relation

Let $F(x) = \int_a^x f(t)\,dt$. If $f \geq 0$ and is continuous, then $F'(x) = f(x)$. By drawing a picture we have

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \approx f(x).$$

Thus, $F(x)$ is an antiderivative of $f(x)$.

35 Fundamental Theorem of Calculus

For a continuous f on $[a, b]$, we define:

$$F(x) = \int_a^x f(t)\,dt \quad \text{(area up to } x\text{)}.$$

By considering:

$$\frac{F(x+h) - F(x)}{h} \approx f(x) \quad \text{if } h \neq 0,$$

we deduce $F'(x) = f(x)$. Any other antiderivative of f is $F(x) + C$, or:

$$\int_a^x f(t)\,dt + C.$$

FTC Part 1 (FTC1)

If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$.
Concisely:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Remarks: - f must be continuous here (see later).
- F is the unique antiderivative of f with $F(a) = 0$.

Tangent/velocity problem ↔ Area/distance problem

Example: If $g(x) = \int_0^x \sqrt{x+t^2}\,dt$, then $g'(x) = \sqrt{1+x^2}$.

Example (Fresnel functions): $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)\,dt$. By FTC1, $S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$.

Non-example: Let:

$$f(x) = \begin{cases} 0, & x \in [0,1), \\ 1, & x \in [1,2]. \end{cases}$$

Then:

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} 0, & x \in [0,1), \\ x-1, & x \in [1,2]. \end{cases}$$

This example shows why f must be continuous in FTC1 (as $F(x)$ is not differentiable at $x = 1$).

Example: Chain Rule + FTC1

Find $\frac{d}{dx}\int_0^{\ln(x)} \sin(t^2)\,dt$.
Let $y = \ln(x)$. Define:

$$G(x) = \int_0^{\ln(x)} \sin(t^2)\,dt = \int_0^y \sin(t^2)\,dt.$$

We want $\frac{d}{dx}G(x)$. By FTC1:

$$\frac{d}{dy}\int_0^y \sin(t^2)\,dt = \sin(y^2).$$

Using the chain rule:

$$\frac{d}{dx}G(x) = \frac{dy}{dx}\cdot\frac{d}{dy}\int_0^y \sin(t^2)\,dt = \frac{1}{x}\cdot\sin\left((\ln(x))^2\right).$$

Example: (FTC1 + Chain Rule) Find $\frac{d}{dx}\int_0^{x^4} \sec(t)\,dt$. Write $y = x^4$ so $\frac{dy}{dx} = 4x^3$ and:

$$\frac{d}{dy}\int_0^y \sec(t)\,dt = \sec(y).$$

Thus:

$$\frac{d}{dx}\int_0^{x^4} \sec(t)\,dt = \frac{dy}{dx}\cdot\frac{d}{dy}\int_0^y \sec(t)\,dt = 4x^3\sec(x^4).$$

FTC Part 2

If f is continuous on $[a,b]$, then:

$$\int_a^b f(t)\,dt = G(b) - G(a),$$

for any antiderivative G of f (i.e., any G such that $G' = f$).
Concisely:

$$\int_a^b G'(t)\,dt = G(b) - G(a).$$

Examples:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

$$\int_0^2 e^x\,dx = [e^x]_0^2 = e^2 - e^0 = e^2 - 1.$$

$$\int_3^6 \frac{1}{x}\,dx = [\ln(x)]_3^6 = \ln(6) - \ln(3) = \ln\left(\frac{6}{3}\right) = \ln(2).$$

$$\int_0^{\pi/2} \cos(x)\,dx = [\sin(x)]_0^{\pi/2} = 1 - 0 = 1.$$

Non-example:

$$\int_{-1}^3 \frac{1}{x^2}\,dx.$$

Here, $\frac{1}{x^2}$ is not continuous on $[-1,3]$. In fact, $\int_{-1}^3 \frac{1}{x^2}\,dx$ does not exist.
The FTC:

Part 1

Let f be continuous on $[a,b]$. Then

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$, differentiable on (a,b), and satisfies

$$F'(x) = f(x) \quad \text{for all } x \in (a,b).$$

In concise form:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Part 2

If f is continuous on $[a,b]$, then

$$\int_a^b f(t)\,dt = G(b) - G(a),$$

for any antiderivative G of f, where $G'(t) = f(t)$. In concise form:

$$\int_a^b G'(t)\,dt = G(b) - G(a).$$

If $G = F$ as in Part 1, then $F(a) = 0$ and

$$F(b) = \int_a^b f(t)\,dt.$$

Examples

1. Find $\frac{d}{dx}\int_a^x (t-t^2)^8\,dt$:

$$\frac{d}{dx}\int_a^x (t-t^2)^8\,dt = (x-x^2)^8.$$

2. Find $\frac{d}{dx}\int_x^0 \sqrt{1+\sec(t)}\,dt$:

$$\frac{d}{dx}\int_x^0 \sqrt{1+\sec(t)}\,dt = -\sqrt{1+\sec(x)}.$$

3. Find $\frac{d}{dx}\int_1^{e^x} \ln(t)\,dt$: Let $y = e^x$, so $\frac{dy}{dx} = e^x$. Using the chain rule:

$$\frac{d}{dx}\int_1^{e^x} \ln(t)\,dt = \frac{dy}{dx}\cdot\frac{d}{dy}\int_1^y \ln(t)\,dt.$$

   Since $\frac{d}{dy}\int_1^y \ln(t)\,dt = \ln(y)$, we have:

$$\frac{d}{dx}\int_1^{e^x} \ln(t)\,dt = e^x\cdot\ln(e^x) = xe^x.$$

4. Find $\frac{d}{dx}\int_{\sqrt{x}}^{\pi/4} \theta\tan(\theta)\,d\theta$:

$$\frac{d}{dx}\int_{\sqrt{x}}^{\pi/4} \theta\tan(\theta)\,d\theta = -\frac{d}{dx}\int_{\pi/4}^{\sqrt{x}} \theta\tan(\theta)\,d\theta.$$

   Let $y = \sqrt{x}$, so $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. Using the chain rule:

$$\frac{d}{dx}\int_{\pi/4}^{\sqrt{x}} \theta\tan(\theta)\,d\theta = \frac{dy}{dx}\cdot\frac{d}{dy}\int_{\pi/4}^{y} \theta\tan(\theta)\,d\theta.$$

   Since $\frac{d}{dy}\int_{\pi/4}^{y} \theta\tan(\theta)\,d\theta = y\tan(y)$, we have:

$$\frac{d}{dx}\int_{\sqrt{x}}^{\pi/4} \theta\tan(\theta)\,d\theta = -\frac{1}{2\sqrt{x}}\cdot\sqrt{x}\tan(\sqrt{x}) = -\frac{1}{2}\tan(\sqrt{x}).$$

Part 2: More Examples

1. Compute $\int_0^1 x^3\,dx$:

$$\int_0^1 x^3\,dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}.$$

2. Compute $\int_1^3 e^{2x}\,dx$:

$$\int_1^3 e^{2x}\,dx = \left.\frac{e^{2x}}{2}\right|_1^3 = \frac{1}{2}(e^6 - e^2).$$

3. Compute $\int_{-\pi/2}^{\pi/2} \sin(x)\,dx$:

$$\int_{-\pi/2}^{\pi/2} \sin(x)\,dx = -\cos(x)\Big|_{-\pi/2}^{\pi/2} = 0 - 0 = 0.$$

4. Compute $\int_{-\pi/2}^{\pi/2} \cos(x)\,dx$:

$$\int_{-\pi/2}^{\pi/2} \cos(x)\,dx = \sin(x)\Big|_{-\pi/2}^{\pi/2} = 1 - (-1) = 2.$$

Note:

$$\int_{-\pi/2}^{\pi/2} \cos(x)\,dx = 2\int_0^{\pi/2} \cos(x)\,dx.$$

5. General Facts:
   - If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.
   - If g is even, then $\int_{-a}^{a} g(x)\,dx = 2\int_0^a g(x)\,dx$.

6. Compute $\int_{-1}^{1} \frac{\tan(x)}{1+x^2+x^4}\,dx$: Since $\tan(x)$ is odd and $1+x^2+x^4$ is even, the integrand is odd. Hence:

$$\int_{-1}^{1} \frac{\tan(x)}{1+x^2+x^4}\,dx = 0.$$

7. Compute $\int_{-2}^{2} (x^6+1)\,dx$: Since x^6+1 is even:

$$\int_{-2}^{2} (x^6+1)\,dx = 2\int_0^2 (x^6+1)\,dx = 2\left(\frac{x^7}{7} + x\right)\Big|_0^2 = 2\left(\frac{2^7}{7} + 2\right) = \frac{284}{7}.$$

8. Compute $\int_0^2 3^x\,dx$: Using substitution:

$$\int 3^x\,dx = \frac{3^x}{\ln(3)} + C.$$

Thus:

$$\int_0^2 3^x\,dx = \left.\frac{3^x}{\ln(3)}\right|_0^2 = \frac{9}{\ln(3)} - \frac{1}{\ln(3)} = \frac{8}{\ln(3)}.$$

36 Applications of the Fundamental Theorem of Calculus

Piecewise Function Example

Let $f(x)$ be a piecewise-defined function:

$$f(x) = \begin{cases} 4x - 1 & \text{if } x \leq 1, \\ \frac{2}{x^2} & \text{if } x > 1. \end{cases}$$

Evaluate $\int_0^2 f(x)\,dx$:

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx.$$

Substitute $f(x)$:

$$\int_0^2 f(x)\,dx = \int_0^1 (4x-1)\,dx + \int_1^2 \frac{2}{x^2}\,dx.$$

Simplify each integral:

$$\int_0^1 (4x-1)\,dx = \int_0^1 4x\,dx - \int_0^1 1\,dx,$$

$$\int_1^2 \frac{2}{x^2}\,dx = \int_1^2 2x^{-2}\,dx.$$

Compute:

$$\int_0^1 4x\,dx = 2x^2\Big|_0^1 = 2(1)^2 - 2(0)^2 = 2,$$

$$\int_0^1 1\,dx = x\Big|_0^1 = 1 - 0 = 1,$$

$$\int_1^2 2x^{-2}\,dx = -\frac{2}{x}\Big|_1^2 = -\frac{2}{2} - \left(-\frac{2}{1}\right) = -1 + 2 = 1.$$

Combine the results:

$$\int_0^2 f(x)\,dx = 2 - 1 + 1 = 2.$$

Indefinite Integrals

The second part of the FTC tells us that $\int_a^b f(x)\,dx$ depends only on the antiderivative of $f(x)$ and the limits a, b.

Definition: The indefinite integral, denoted $\int f(x)\,dx$, represents the collection of all antiderivatives of $f(x)$:

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is an arbitrary constant.

Examples:

- $\int x^2\,dx = \frac{x^3}{3} + C.$
- $\int e^x\,dx = e^x + C.$

Important Distinction:

- $\int_a^b f(x)\,dx$ is a *number.*
- $\int f(x)\,dx$ is a *collection of functions.*

The FTC states:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b = F(b) - F(a),$$

where $F'(x) = f(x)$.

Table of Indefinite Integrals

$$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx,$$
$$\int kf(x)\,dx = k \int f(x)\,dx,$$
$$\int k\,dx = kx + C,$$
$$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{if } n \neq -1, \\ \ln|x| + C & \text{if } n = -1, \end{cases}$$
$$\int e^x\,dx = e^x + C,$$
$$\int b^x\,dx = \frac{b^x}{\ln(b)} + C,$$
$$\int \sin(x)\,dx = -\cos(x) + C,$$
$$\int \cos(x)\,dx = \sin(x) + C,$$
$$\int \sec^2(x)\,dx = \tan(x) + C,$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C,$$
$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C.$$

The Net Change Theorem

The second part of the FTC states:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

This describes the net change of $F(x)$ over the interval $[a, b]$. Physically, this is a very natural idea.
Example: If $V(t)$ represents the volume of water in a tank (in m^3) and $V'(t)$ represents the rate of inflow minus outflow (in m^3/s), then:

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1).$$

Example: Let $S(t)$ represent the position of a particle. Its velocity is $S'(t)$, and the net change in position (displacement) is:

$$\int_{t_1}^{t_2} S'(t)\,dt = S(t_2) - S(t_1).$$

If $S'(t) = t^2 + t - 6$, find the displacement as a function:

$$\int S'(t)\,dt = \int (t^2 + t - 6)\,dt = \frac{t^3}{3} + \frac{t^2}{2} - 6t + C.$$

Evaluate from $t = 0$ to $t = 3$:

$$\int_0^3 S'(t)\,dt = \left[\frac{t^3}{3} + \frac{t^2}{2} - 6t + C\right]_0^3 = \left(9 + \frac{9}{2} - 18 + C\right) - C = -\frac{9}{2}.$$

Question: What is the total distance travelled? *Answer:* Total distance is:

$$\int |S'(t)|\,dt, \quad \text{with displacement } \leq \text{total distance}.$$

37 Integration by Substitution

Example 1

Find $\int e^{2x}\,dx$. Guess $u = e^{2x}$, then $u' = (e^{2x})' = 2e^{2x}$. Correct the guess:

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C.$$

Differential Definition

If $u = f(x)$, then the differential is $du = f'(x)dx$.

Examples

- If $u = e^{2x}$, then $du = 2e^{2x}\,dx$.
- If $u = \cos(x)$, then $du = -\sin(x)\,dx$.
- If $u = x$, then $du = dx$.
- If $u = e^{3x} + x$, then $du = (3e^{3x} + 1)\,dx$.

Simplifying Integrals with du

Example: $\int 2x\sqrt{1 + x^2}\,dx$ Let $u = 1 + x^2$, so $du = 2x\,dx$. Thus

$$\int 2x\sqrt{1 + x^2}\,dx = \int \sqrt{u}\,du = \frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{3}(1 + x^2)^{3/2} + C.$$

Check by differentiating:

$$\frac{d}{dx}\left(\frac{2}{3}(1 + x^2)^{3/2} + C\right) = \frac{2}{3} \cdot \frac{3}{2} \cdot (2x)(1 + x^2)^{1/2} = 2x\sqrt{1 + x^2}.$$

Chain Rule Recap:

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))g'(x).$$

Thus,

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

General Rule: If $u = g(x)$, then $du = g'(x)dx$.

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Additional Examples

1. $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$ Let $u = \cos(x)$, so $du = -\sin(x)\,dx$.

$$\int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-1}{u}\,du = -\ln|u| + C = -\ln|\cos(x)| + C.$$

2. $\int \frac{\ln(x)^2}{x}\,dx$ Let $u = \ln(x)$, so $du = \frac{1}{x}\,dx$.

$$\int \frac{\ln(x)^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\ln(x)^3}{3} + C.$$

3. $\int x^3\cos(x^4+2)\,dx$ Let $u = x^4 + 2$, so $du = 4x^3\,dx$, hence $\frac{1}{4}\,du = x^3\,dx$.

$$\int x^3\cos(x^4+2)\,dx = \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(u) + C = \frac{1}{4}\sin(x^4+2) + C.$$

4. $\int x^5\sqrt{1+x^2}\,dx$ Let $u = 1 + x^2$, so $du = 2x\,dx$.

$$\int x^5\sqrt{1+x^2}\,dx = \frac{1}{2}\int \sqrt{u}\cdot x^4\,du.$$

Substitute $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\frac{1}{2}\int \sqrt{u}(u-1)^2\,du = \frac{1}{2}\int u^{1/2}(u^2 - 2u + 1)\,du.$$

Expand and integrate:

$$\frac{1}{2}\left(\frac{u^{7/2}}{7/2} - 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C = \frac{(1+x^2)^{7/2}}{7} - \frac{4}{5}(1+x^2)^{5/2} + \frac{2}{3}(1+x^2)^{3/2} + C.$$

38 Definite Integrals via Substitution

Example 1

Evaluate $\int_0^4 \sqrt{2x+1}\,dx$. Let $u = 2x + 1$, so $du = 2\,dx$, hence $\frac{1}{2}\,du = dx$.
Method 1:

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{3}(2(4)+1)^{3/2} - \frac{1}{3}(2(0)+1)^{3/2} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}.$$

Method 2: Change the limits of integration: When $x = 0$, $u = 1$; when $x = 4$, $u = 9$.

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_1^9 u^{1/2}\,du = \frac{1}{3}\left[u^{3/2}\right]_1^9 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}.$$

Additional Examples

1. $\int_1^e \frac{\ln(x)}{x}\,dx$: Let $u=\ln(x)$, so $du=\frac{1}{x}\,dx$. Change limits: when $x=1$, $u=0$; when $x=e$, $u=1$.

$$\int_1^e \frac{\ln(x)}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}.$$

2. $\int_0^4 x\sqrt{9+x^2}\,dx$: Let $u=9+x^2$, so $du=2x\,dx$. Change limits: when $x=0$, $u=9$; when $x=4$, $u=25$.

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{2}\int_9^{25} u^{1/2}\,du = \frac{1}{3}\left[u^{3/2}\right]_9^{25} = \frac{98}{3}.$$

39 Area Between Curves

The area between two curves is given by:

$$\text{Area of } S = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x)-g(x))\,dx.$$

Example 1: Area between $y=e^x$ and $y=e^{-x}$ from $x=0$ to $x=1$.

$$\text{Area of } S = \int_0^1 \left(e^x - e^{-x}\right)\,dx = \left[e^x + e^{-x}\right]_0^1 = e + \frac{1}{e}.$$

Example 2: Area enclosed by $y=x^2$ and $y=2x-x^2$.

First, set $x^2 = 2x - x^2$ to find $b=1$. So, the area of S is:

$$\text{Area of } S = \int_0^1 \left((2x-x^2)-x^2\right)\,dx = \int_0^1 (2x-2x^2)\,dx = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = \frac{1}{3}.$$

General Formula

If $f(x)\geq g(x)$, the area of S is:

$$\text{Area of } S = \int_a^b (f(x)-g(x))\,dx.$$

If $f(x)$ is not greater than $g(x)$

The area is given by:

$$\text{Area of } S = \int_a^b |f(x)-g(x)|\,dx = \int_a^c (g(x)-f(x))\,dx + \int_c^b (f(x)-g(x))\,dx.$$

Example 3: Area between $y=\frac{8}{x}$ and $y=2x$ from $x=1$ to $x=4$.

The area of S is:

$$\text{Area of } S = \int_1^4 \left|\frac{8}{x}-2x\right|\,dx = \int_1^2 \left(\frac{8}{x}-2x\right)\,dx + \int_2^4 \left(2x-\frac{8}{x}\right)\,dx.$$

Solving for c, set $\frac{8}{x}=2x$, which gives $x=2$. The integral is:

$$= \left[8\ln(x)-x^2\right]_1^2 + \left[x^2-8\ln(x)\right]_2^4 = 9.$$

Example 4: Area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

First, reflect the equations in the line $y = x$:

$$y = x - 1 \quad \Rightarrow \quad x = y + 1,$$

$$y^2 = 2x + 6 \quad \Rightarrow \quad x = \frac{1}{2}y^2 - 3.$$

The area of S is:

$$\text{Area of } S = \int_{-2}^{4} \left((y+1) - \left(\frac{1}{2}y^2 - 3 \right) \right) \, dy = \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4 \right) \, dy.$$

Evaluating this integral:

$$= \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y \right]_{-2}^{4} = 18.$$

40 Integration by Parts

Using the product rule:

$$\frac{d}{dx} \left(f(x)g(x) \right) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides with respect to x:

$$f(x)g(x) = \int \frac{d}{dx} \left(f(x)g(x) \right) \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx.$$

Integration by Parts (IBP) Formula

$$\int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx.$$

This is often described as "passing the derivative from f to g, up to some error (given by $f(x)g(x)$)." Recall that if $u = f(x)$, $v = g(x)$, then $du = f'(x)dx$ and $dv = g'(x)dx$, so IBP becomes:

$$\int v \, du = uv - \int u \, dv.$$

Example 1: $\int (3t + 5) \cos\left(\frac{t}{4}\right) \, dt$.

Let $u = (3t + 5)$, $dv = \cos\left(\frac{t}{4}\right) \, dt$, $du = 3 \, dt$, and $v = 4 \sin\left(\frac{t}{4}\right)$. Applying IBP:

$$\int (3t + 5) \cos\left(\frac{t}{4}\right) \, dt = 4(3t+5) \sin\left(\frac{t}{4}\right) - \int 4 \sin\left(\frac{t}{4}\right) (3 \, dt)$$

$$= 4(3t+5) \sin\left(\frac{t}{4}\right) - 12 \int \sin\left(\frac{t}{4}\right) \, dt$$

$$= 4(3t+5) \sin\left(\frac{t}{4}\right) + 48 \cos\left(\frac{t}{4}\right) + C.$$

Definite Integration by Parts

By the Fundamental Theorem of Calculus:

$$\int_a^b u\,dv = \left[uv - \int v\,du\right]_a^b = [uv]_a^b - \int_a^b v\,du.$$

Example 2: $\int_0^{\frac{\pi}{2}} x\sin(x)\,dx$.

Let $u = x$, $du = dx$, $dv = \sin(x)\,dx$, and $v = -\cos(x)$. Applying IBP:

$$\int_0^{\frac{\pi}{2}} x\sin(x)\,dx = \left[-x\cos(x) + \int \cos(x)\,dx\right]_0^{\frac{\pi}{2}}$$

$$= [-x\cos(x)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(x)\,dx = (0-0) + (\sin(x))_0^{\frac{\pi}{2}} = 1.$$

Example 3: $\int x^5\sqrt{x^3+1}\,dx$.

We first rewrite the integral as:

$$\int x^5\sqrt{x^3+1}\,dx = \int x^3\left(x^2\sqrt{x^3+1}\right)\,dx.$$

Let $u = x^3$, $dv = x^2\sqrt{x^3+1}\,dx$, $du = 3x^2\,dx$, and $v = \frac{2}{9}(x^3+1)^{3/2}$. Applying IBP:

$$\int x^5\sqrt{x^3+1}\,dx = x^3\left(\frac{2}{9}(x^3+1)^{3/2}\right) - \frac{2}{3}\int x^2(x^3+1)^{3/2}\,dx.$$

Continuing the evaluation:

$$\frac{2}{3}\int x^2(x^3+1)^{3/2}\,dx = \frac{2}{15}(x^3+1)^{5/2} + C.$$

Thus:

$$\int x^5\sqrt{x^3+1}\,dx = \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{4}{45}(x^3+1)^{5/2} + C.$$

Example 4: $\int \theta^2\sin(\theta)\,d\theta$.

Let $u = \theta^2$, $du = 2\theta\,d\theta$, $dv = \sin(\theta)\,d\theta$, and $v = -\cos(\theta)$. By IBP:

$$\int \theta^2\sin(\theta)\,d\theta = -\theta^2\cos(\theta) + 2\int \theta\cos(\theta)\,d\theta.$$

Now, use IBP again on $\int \theta\cos(\theta)\,d\theta$:

$$\int \theta\cos(\theta)\,d\theta = \theta\sin(\theta) - \int \sin(\theta)\,d\theta.$$

Thus:

$$\int \theta^2\sin(\theta)\,d\theta = -\theta^2\cos(\theta) + 2\theta\sin(\theta) + 2\cos(\theta) + C.$$

Example 5: $\int \arctan(x)\,dx$.

Let $u = \arctan(x)$, $du = \frac{1}{x^2+1}\,dx$, and $dv = dx$, $v = x$. By IBP:

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{x^2+1}\,dx.$$

Use a substitution $U = x^2 + 1$, so $dU = 2x\,dx$, and the integral becomes:

$$\frac{1}{2}\int \frac{1}{U}\,dU = \frac{1}{2}\ln|U| = \frac{1}{2}\ln|x^2+1|.$$

Thus:

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(x^2+1) + C.$$

Example 6: Self-Similar IBP

Let $I = \int e^\theta \cos(\theta)\,d\theta$. Using IBP:

$$I = e^\theta \sin(\theta) - \int e^\theta \sin(\theta)\,d\theta.$$

Now apply IBP again on $\int e^\theta \sin(\theta)\,d\theta$:

$$I = e^\theta \sin(\theta) - e^\theta \cos(\theta) + I.$$

Thus:

$$2I = e^\theta \sin(\theta) - e^\theta \cos(\theta).$$

So:

$$I = \frac{e^\theta \sin(\theta) - e^\theta \cos(\theta)}{2}.$$