Calculus on Manifolds

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Contents

1 Introduction

The aim of this course is to introduce a suitable generalisation of Euclidean space, namely the manifold, on which one can make sense of various concepts introduced in calculus. As a guiding example, consider the function which outputs the temperature at each point on the surface of the earth. What would it mean for such a function to be differentiable? Could such a function be integrated?

1.1 Course information

The content of the course is an amalgamation of results from various notes and books I used to learn about manifolds. Below I list some references I am at least somewhat familiar with; I recommend that you find a few (not necessarily from this list) that you enjoy reading and do not restrict yourself to a specific text in understanding each concept:

- [Bow17] is an updated version of a course I took, which heavily inspired this course (these notes are no longer accessible online, I put a pdf on the Canvas page).
- [Lee12] is a comprehensive introductory text, lots of good exercise problems.
- [Lot21] is a geometrically inclined set of notes, which I have frequently drawn upon.
- [Mil67] is a classical text, includes an appendix classifying 1-dimensional manifolds.
- [Spi18] is the university recommended text, very concise.

The prerequisites for the course are a strong background in multivariable calculus (in particular smooth maps and the inverse function theorem) and linear algebra (vector spaces, bases, dimension, subspaces, linear maps etc.). Although topology is not a formal prerequisite (we will introduce all the notions we need), I would recommend an introductory course be taken alongside.

The four homework sets will consist of unproven results from the notes as well as specific problems to deepen your understanding of concepts or provide a new perspective on them. The first quiz will be a review of background fundamentals (linear algebra, multivariable calculus, topology) and some basic notions from the first few classes. The second and third quiz consist of new problems based on the concepts covered in class. The quizzes, and solutions to the homework, are available upon request.

1.2 What is (and is not) a manifold?

The concepts of calculus make sense locally, by which we mean one only needs to know how a function behaves near a given point. We can thus reasonably expect to make sense of those same concepts in more general spaces where one only asks for a Euclidean structure locally, allowing for the global picture to potentially look very different from Euclidean space. A slightly more precise formulation of what it means to be locally Euclidean is given by the notion of coordinates; for example, near any point on the sphere one needs only two numbers to smoothly parametrise the space.

Loosely, a manifold will be a space formed by patching together small regions in which the object has a Euclidean structure. This is in strong analogy with the way in which maps of the earth are created through the use of charts and atlases (terms we will soon see in a mathematical context). Before diving into the formal definition of a manifold, we first familiarise ourselves with some examples (and non-examples) of manifolds with the idea in mind that a manifold should be an object composed of regions that can be smoothly deformed to look like Euclidean space.

1.2.1 Examples of manifolds in Euclidean space

We first look at some examples of manifolds that arise as subsets of Euclidean space:

- For each non-negative integer, n, the **n-dimensional Euclidean space**, $\mathbb{R}^n$, is an n-dimensional manifold, hereafter an n-manifold, as is any open subset, $U \subset \mathbb{R}^n$. If $n = 0$ then $\mathbb{R}^0$ is a point, unions of points are 0-manifolds.

- Any m-dimensional subspace (or affine plane) of $\mathbb{R}^n$ is an m-manifold. Necessarily $m \leq n$, $m = 1$ corresponds to lines, and $m = 2$ corresponds to planes.

- The circle, $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$, is a 1-manifold (explicit coordinates are given by $\theta \mapsto (\cos(\theta), \sin(\theta))$ and the sphere, $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$, is a 2-manifold (coordinates can be given by stereographic projection). More generally, the n-sphere, $S^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\} \subset \mathbb{R}^{n+1}$, is an n-manifold.

- The (donut) torus, given by $\{((2 + \cos(\theta)) \cos(\phi), (2 + \cos(\theta)) \sin(\phi), \sin(\theta)) \,|\, \theta, \phi \in \mathbb{R}\} \subset \mathbb{R}^2$, is a 2-manifold. The (standard) torus, $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$, is also a 2-manifold. These two object, while different as sets, are the 'same' manifold in an appropriate sense (they are diffeomorphic).

- The Möbius band, $B = \{(\cos(\theta), \sin(\theta), r \cos(\frac{\theta}{2}), r \sin(\frac{\theta}{2}) \in \mathbb{R}^4 \,|\, r, \theta \in \mathbb{R}\} \subset \mathbb{R}^4$, is a 2-manifold. Notice that projection to the first two coordinates gives a copy of the circle, S^1, whose preimage is a straight line. As θ varies over an interval of length 2π this line rotates a half turn (so two loops round the circle in the projection arrive at the same point).

1.2.2 Examples of abstract manifolds

Not all manifolds sit inside of Euclidean space in a natural way. For example, in physics one thinks of our universe as a 4-manifold (three dimensions of space and one of time), but we do not seem to exist as a subset of some bigger ambient space. We now look at examples of abstract manifolds (abstract in the sense that they do not lie naturally in a copy of $\mathbb{R}^n$):

- Space-time itself as a 4-manifold does not seem to sit inside of some copy of $\mathbb{R}^n$, but is locally Euclidean.

- Consider the matrix groups, $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$, formed of $n \times n$ matrices with non-zero and unit determinant respectively. These form manifolds of dimension n^2 and $n^2 - 1$ respectively. While these groups can be viewed as subsets of $\mathbb{R}^{n^2}$ (for instance by stacking columns of the matrices), this perspective does not tell us much, if anything, about their group structure.

- The (Pac-Man) torus, formed by identifying opposite sides of a square (with the same orientation) is a 2-manifold. This does not sit inside of $\mathbb{R}^3$, like the donut torus, without having to 'bend/twist' the square.

- The Klein bottle, formed by identifying two sides of a square with the same orientation and identifying the other two sides with opposite orientation, is a 2-manifold. There is no way to make this sit in $\mathbb{R}^3$ without it self-intersecting.

- The projective space, $\mathbb{R}P^n$, is formed by either taking the n-sphere, S^n, and identifying antipodal points or by identifying colinear points in $\mathbb{R}^{n+1}$. This is an n-manifold and may be thought of as the space of lines in $\mathbb{R}^{n+1}$.

The final three examples of abstract manifolds above are formed as quotient spaces of Euclidean space. As it turns out, and as we will see later on, every manifold can realised as a subset of $\mathbb{R}^n$ but this is not always a natural way to view them, and may not preserve any additional structres on the manfiold (such as a notion of distance). We will thus be interested in definitions and concepts that are intrinsic to the manifold itself, and do not make reference to an ambient space.

1.2.3 Examples that are not manifolds

Let us finally discuss some examples of objects that we want to preclude as manifolds:

- Consider a figure of eight (or just the union of the coordinate axes in $\mathbb{R}^2$). While this is a 1-manifold away from the intersection point, at this intersection point there is no way to assign local coordinates there. Similarly, the cone, $C = \{(x_1, x_2, x_3) \,|\, x_1^2 + x_2^2 - x_3^2 = 0\}$ is not a manifold.

- Consider the square or cube. while the faces of the shapes locally look like $\mathbb{R}^2$, the corners of the object are not smooth. These are examples of topological but not smooth manifolds (which we will refer to simply as manifolds in this course).

- Fractal like objects such as the Koch snowflake, Peano space filling curves, and Alexander's horned sphere (homeomorphic but not diffeomorphic to the sphere) are locally homeomorphic to Euclidean space but not smooth in an appropriate sense.

The second and third bullet points above describe objects that are topologically manifolds in the sense that they locally can be deformed to look like Euclidean space, but this deformation cannot be done in a smooth manner; and hence they are not (smooth) manifolds.

2 Manifold preliminaries

We want to identify an appropriate mathematical notion that allows us to piece together sets that look like Euclidean space into a global object. This will be done by first introducing topological spaces, sometimes referred to 'rubber sheet geometry', which posses the desired property of the abstract examples described above in that they are not required to sit inside of some ambient space. Once we have introduced topological spaces, we will then need to make sense of smooth maps for these objects in analogy with the usual notion in Euclidean space.

2.1 A primer in topology

In Euclidean space, $\mathbb{R}^n$, we declared a set, $U \subset \mathbb{R}^n$, to be open if for every point, $x \in U$, there was some ball around this point that lay inside of the set; precisely if there existed some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$ (where $B_\varepsilon(x) = \{y \in \mathbb{R}^n \,|\, |y - x| < \varepsilon\}$). With this notion we were able to characterise the continuity of functions, $f : \mathbb{R}^m \to \mathbb{R}^n$, by requiring that the pre-image of any open set was open; i.e. that $f^{-1}(U) = \{x \in \mathbb{R}^n \,|\, f(x) \in U\}$ was open whenever $U \subset \mathbb{R}^n$ was open. On the grounds that any good mathematical notion is worth generalising, let us try to do so.

2.1.1 Topological spaces and some properties

We want to make the idea of open sets and their characterisation of continuity more general.

Definition 1. *Let X be a (non-empty) set. A* ***topology****, τ, is a collection of subsets of X, called* ***open sets****, such that*

- $\emptyset, X \in \tau$,
- *Any union of open sets is an open set,*
- *Any finite intersection of open sets is an open set.*

We then call the pair (X, τ) a ***topological space****.*

Remark 1. *As in $\mathbb{R}^n$, we call a set, $C \subset X$,* ***closed*** *if $C = X \setminus U$ for some open set U.*

Let's see some examples:

Example 1. *On any (non-empty) set X we can consider the* ***trivial topology*** *by letting $\tau = \{\emptyset, X\}$.*

Example 2. *On any (non-empty) set X we can consider the* ***discrete topology*** *by letting τ contain every subset of X (i.e. declare every subset of X to be open).*

Example 3. *On any (non-empty) set X we can consider the* ***co-finite topology*** *by defining the open sets to be $\emptyset, X$ and sets, $U \subset X$ whose complement is finite (i.e. $|X \setminus U| < \infty$).*

Example 4. *For a topological space, (X, τ), we can induce the* ***subspace topology*** *on any subset, $S \subset X$, by declaring a set to open in S if it is the intersection of S with an open set in X.*

Example 5. *For a topological space, (X, τ), and an equivalence relation, $\sim$, on X we can define the* ***quotient space*** *$X/\sim$ consisting of all equivalence classes of X with this relation, which we denote for $x \in X$ by $[x] = \{y \in X \,|\, x \sim y\}$. We then get a projection map $\pi : X \to X/\sim$ taking a point $x \in X$ to its equivalence class $[x] \in X/\sim$. The topology on X induces the* ***quotient topology*** *on $X/\sim$ by declaring $U \subset X/\sim$ open if $\pi^{-1}(U)$ is open in X.*

Example 6. *The usual notion of open sets in $\mathbb{R}^n$ gives it the* ***standard topology****. In this course we will always consider $\mathbb{R}^n$ with the standard topology.*

As the first three examples above show, topological spaces can be fairly weird. Since we want to restrict our attention to spaces that locally look like Euclidean space, we will consider further restrictions on our topological spaces.

The first property we consider allows us to separate distinct points by disjoint open sets (ensuring they are 'housed off'):

Definition 2. *A topological space, (X, τ), is* ***Hausdorff*** *if for any distinct points, $x \neq y$, in X there exist open sets, $U, V \in \tau$, with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$.*

Remark 2. $\mathbb{R}^n$ *is Hausdorff.*

Remark 3. *The trivial and co-finite topology on any set are not Hausdorff (in the latter case one can show that any two open sets have non-empty intersection).*

Remark 4. *In a Hausdorff topological space, any convergent sequence has a unique limit. By a convergent sequence here we mean that $x_n \to x$ in (X, τ) if for any $x \in U \in \tau$ there is some $N \geq 1$ such that $x_n \in U$ for all $n \geq N$ (i.e. the sequence is eventually inside of every open set containing the limit point).*

The second property we will consider is easier to motivate retrospectively as it will allow for numerous useful constructions on manifolds later on (partitions of unity, Whitney embedding theorem, Riemannian metrics etc.):

Definition 3. *A topological space, (X, τ), is* ***second countable*** *if there is a countable collection of open sets, $\{U_i\}_{i=1}^{\infty} \subset \tau$, such that any open set can be written as the union of open sets from this collection; i.e. any $U \in \tau$ is equal to the (countable) union of some of the sets $\{U_i\}_{i=1}^{\infty}$. The collection $\{U_i\}_{i=1}^{\infty}$ are referred to as a* ***countable base*** *for the topology.*

Remark 5. $\mathbb{R}^n$ *is second countable. As is any subset equipped with the subspace topology. One can consider balls of rational radii centred at points with rational coordinates.*

Remark 6. *Second countable spaces are separable (assuming the axiom of choice). Separable means that the space has a countable dense subset (where dense means that the set intersects every open set). In metric spaces (e.g. $\mathbb{R}^n$) these notions are equivalent.*

Remark 7. *A space that is not second countable is given by the 'long-line', which is formed by stacking the half-open interval $[0, 1)$ end-to-end uncountably many times.*

The final property we consider is a notion of 'smallness' for topological spaces, generalising the notion of closed and bounded subsets of Euclidean space:

Definition 4. *A topological space, (X, τ), is* ***compact*** *if whenever $X = \cup_{i \in I} U_i$ for some collection $\{U_i\}_{i \in I} \subset \tau$ (called a* ***cover****), there is a finite subcollection of the $U_{i_1}, \ldots, U_{i_N} \in \{U_i\}_{i \in I}$ such that $X = U_{i_1} \cup \cdots \cup U_{i_N}$ (i.e. every cover of X has a finite subcover).*

Remark 8. *Closed and bounded subsets of $\mathbb{R}^n$ are compact (this is the Heine–Borel theorem).*

Remark 9. *In a Hausdorff space, every compact set is also closed.*

Remark 10. *$\mathbb{R}^n$, $(0, 1)$, and $[0, 1)$ are not compact (one should think compact is small, not compact is big).*

2.1.2 Topological continuity and homeomorphisms

The definition of a topological space generalises the notion of open sets, we now wish to generalise the notion of continuity to maps between these spaces.

Definition 5. *A map $f : X \to Y$ between topological spaces is* ***continuous*** *if the pre-image of every open set is open.*

Remark 11. *In $\mathbb{R}^n$ this agrees with the familiar definition of continuity described above.*

Remark 12. *Whether a given map $f : X \to Y$ is continuous depends on the topology on the spaces X and Y. For instance, if X is given the discrete topology every function is continuous (as every subset of X is open)!*

We want to understand continuous maps that preserve topological properties (e.g. those introduced above), for which we need their inverses to also be continuous:

Definition 6. *A map $f : X \to Y$ is a* ***homeomorphism*** *if it is continuous, bijective, and has continuous inverse (i.e. $f^{-1} : Y \to X$ is well defined and continuous). If there is a homeomorphism between X and Y we say that the spaces are* ***homeomorphic****.*

Remark 13. *Homeomorphisms preserve topological structure. For instance, if X and Y are homeomorphic then X is Hausdorff/second countable/compact if and only if Y is Hausdorff/second countable/compact.*

The importance of this definition lies in the fact that it allows us to determine whether two topological spaces are the 'same' if they can be obtained from one another by stretching and bending (e.g. a mug and a donut). One of the main goals of the field of topology is to determine which topological spaces are homeomorphic.

We now consider some examples of homeomorphic topological spaces, bearing in mind that topological properties are preserved by homeomorphisms:

Example 7. *The circle and the square are homeomorphic.*

Example 8. *The interval $(0, 1)$ is homeomorphic to $\mathbb{R}$.*

Example 9. *Each of the three tori defined previously are homeomorphic.*

Example 10. *$\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$ (this is hard to prove).*

Example 11. *The intervals $(0, 1)$ and $[0, 1]$ are not homeomorphic; one is compact and the other is not. Similarly, S^1 and $\mathbb{R}$ are not homeomorphic.*

2.1.3 Topological manifolds, charts, atlases, and transition maps

We are now ready to define a topological manifold, with the idea that a manifold should locally look like Euclidean space near every point:

Definition 7. *An* ***n-dimensional topological manifold*** *or* ***topological n-manifold*** *is a Hausdorff, second countable topological space for which every point belongs to an open set homeomorphic to an open subset of $\mathbb{R}^n$.*

Remark 14. *By virtue of the fact, mentioned above, that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$, the dimension of a manifold is uniquely determined.*

We will not discuss topological manifolds in too much detail, since we are interested in manifolds we can do calculus on, but mention here that every example discussed in the introduction is a topological manifold; in particular, we note that the examples discussed in Euclidean space are automatically Hausdorff and second countable.

Since we want to generalise notions from calculus to manifolds, and since we are familiar with these notions in Euclidean space, we first show that all topological manifolds induce natural maps between Euclidean spaces. We first observe that, as every point in a topological n-manifold belongs to an open set homeomorphic to an open subset of $\mathbb{R}^n$, we get a homeomorphism between an open subset of the manifold and $\mathbb{R}^n$; this notion gives the following definition:

Definition 8. *Given a topological n-manifold, M, we call a homeomorphism $\varphi : U \to V$ between open subsets $U \subset M$ and $V \subset \mathbb{R}^n$ a **chart** on M.*

The open sets of the manifold in the definition above are often referred to as coordinate patches. Since we can take the union over all such open sets as above, we can cover the entire manifold with charts in the following manner:

Definition 9. *A collection of charts, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$, is called an **atlas** for M if $M = \cup_{\alpha \in \mathcal{A}} U_\alpha$.*

With the above two definitions, one could equivalently define a topological n-manifold to be a Hausdorff, second countable topological space that admits an atlas of charts mapping into $\mathbb{R}^n$. The terminology for charts and atlases is made in strong analogy with the manner in which maps of the world are created; regions of the curved earth are depicted by flat two dimensional images which are pieced together to give a complete map.

We conclude this topological section by discussing natural maps that are induced by an atlas on a topological manifold. Whenever two coordinate patches in a given atlas overlap we yield a map between Euclidean spaces:

Definition 10. *Given two charts, $\varphi_\alpha : U_\alpha \to V_\alpha$ and $\varphi_\beta : U_\beta \to V_\beta$, in the atlas of a topological n-manifold, M, we call the homeomorphism*

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

*a **transition map**.*

Transition maps are homeomorphisms as they are compositions of homeomorphisms (so one does not need to check the condition given a transition map). Given an atlas on a topological manifold, the transition maps tell us how coordinate patches of the manifold relate in regions where they overlap.

2.2 Notions from multivariable calculus

We now recall some background in multivariable calculus to make sense of the differentiability of maps between Euclidean spaces. These ideas will be generalised to manifolds shortly.

Definition 11. *Given an open set $U \subset \mathbb{R}^n$ we say that a map $f : U \to \mathbb{R}^m$ is **smooth** if its derivatives of all orders exist; i.e. if $\frac{\partial^k f_i}{\partial x_j^k}(x)$ exists for each $i = 1, \ldots, m$, $j = 1, \ldots, n$, and every $k \geq 1$ whenever $x \in U$. Given a point $x \in U$ we call the linear map $D_x f : \mathbb{R}^n \to \mathbb{R}^m$ the **derivative of** f **at** x which is represented by the **Jacobian matrix**, denoted $J_f(x)$, with entries $\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{ij}$.*

Just as a homeomorphism between open sets $U, V \subset \mathbb{R}^n$ gave a way to continuously deform one set to the other, if we ask that this deformation is smooth we arrive at the following:

Definition 12. *Given open sets $U, V \subset \mathbb{R}^n$ a map $f : U \to V$ is a **diffeomorphism** if it is smooth, bijective, and has a smooth inverse (i.e. $f^{-1} : V \to U$ is well defined and smooth). If there is a diffeomorphism between U and V we say that the sets are **diffeomorphic**.*

As smooth maps are continuous, every diffeomorphism is necessarily a homeomorphism. This definition captures the idea that we deform one set to the other in a differentiable or smooth manner; for example any open interval is diffeomorphic to the real line. One effective way to locally construct diffeomorphisms is provided whenever the derivative of a smooth map is invertible:

Theorem 1. *(Euclidean Inverse Function Theorem) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth with $D_x f$ invertible at some point $x \in \mathbb{R}^n$, then there exist open sets $U, V \subset \mathbb{R}^n$ with $x \in U$ such that $f : U \to V$ is a diffeomorphism.*

One sometimes refers to the the conclusion of the inverse function theorem as saying that a smooth map is a **local diffeomorphism** whenever it has invertible derivative; the use of local here means that the point at which the derivative is invertible belongs to an open set on which the map is a diffeomorphism.

2.3 Smooth manifolds

2.3.1 The definition

Since a transition map between charts in the atlas of a topological manifold provides a homeomorphism between open subsets of Euclidean space, it is natural to ask whether these maps are smooth. This leads us to the formal definition of a smooth manifold:

Definition 13. *An n-**dimensional smooth manifold**, hereafter an n-**manifold**, is a topological n-manifold admitting an atlas with smooth transition maps.*

Let's unpack this definition carefully. We first take a topological n-manifold, M, which is a Hausdorff, second countable space for which every point belongs to an open set homeomorphic to an open subset of $\mathbb{R}^n$. The definition then stipulates that there is some atlas, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$, of charts, i.e. homeomorphisms between open sets $U_\alpha \subset M$ and $V_\alpha \in \mathbb{R}^n$, such that whenever $\alpha, \beta \in \mathcal{A}$ the transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth as a map between open subsets of $\mathbb{R}^n$. Since the transition maps told us how coordinate patches of the manifold relate in regions where they overlap, the additional requirement that they are smooth tells us that the piecing together of coordinate patches is done in a smooth manner; as opposed to just continuously for a topological manifold.

Remark 15. *Since charts are homeomorphisms, and we may swap the role of α and β in the transition maps, we see that transition maps are smooth, bijective and have smooth inverse. This implies that the transition maps are in fact diffeomorphisms between the open sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$; the inverse of $\varphi_\beta \circ \varphi_\alpha^{-1}$ is given by $\varphi_\alpha \circ \varphi_\beta^{-1}$. Observe that the smoothness of a manifold implicitly depends on the choice of atlas for the manifold; given one choice of atlas however, one can always define an equivalence class of atlases, called a **smooth structure**, for which there will exist a uniquely defined (maximal) atlas. We will not discuss this point too much further here, but simply mention that differing (non-equivalent) choices of atlases on smooth manifolds give rise to different ways of differentiating functions, i.e. different smooth structures on a given topological manifold.*

Remark 16. *One can define other classes of manifold, each of which has their own corresponding theory, by replacing the word smooth in the definition of a manifold with the assumption that they are 'X', provided being X is a condition closed under composition and taking inverses. By taking X = homeomorphic one recovers the definition of a topological manifold. One could also relax the smoothness assumption to C^k for some finite integer $k \geq 1$, bi-Lipschitz, conformal, real analytic, or complex analytic (if n is even). This last condition gives rise to the notion of a **complex manifold**. In this course however we will only consider the class of smooth manifolds.*

Let's check that some of the examples we previously discussed are indeed manifolds:

Example 12. *We mentioned that $\mathbb{R}^n$ is Hausdorff and second countable. Take the atlas $\{I_n : \mathbb{R}^n \to \mathbb{R}^n\}$ consisting of a single chart given by the identity on $\mathbb{R}^n$; the transition maps are then the identity also, which is a diffeomorphism. Thus $\mathbb{R}^n$ is an n-manifold; indeed the same argument works to show that any open $U \subset \mathbb{R}^n$ is also an n-manifold. This immediately shows that $GL(n, \mathbb{R})$ is an n^2-manifold as it arises as an open subset of $\mathbb{R}^{n^2}$ (as the determinant map is continuous).*

Example 13. *If M is an n-manifold and $U \subset M$ is open, then U is also an n-manifold by restricting charts in the atlas of M to U; precisely if $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$ is an atlas for M, then $\{\varphi_\alpha|_U : U_\alpha \cap U \to V_\alpha \cap \varphi_\alpha(U)\}_{\alpha \in \mathcal{A}}$ is an atlas for U.*

Example 14. *We will check that the n-sphere, $S^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ is an n-manifold. Firsly, we note that $S^n \subset \mathbb{R}^{n+1}$ with the subspace topology is Hausdorff and second countable. Let us denote the 'north' and 'south' poles of S^n by the points $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ respectively. Define open sets $U_N = S^n \setminus \{N\}$ and $U_S = S^n \setminus \{S\}$, so that $S^n = U_N \cup U_S$, and consider the **stereographic projection** maps, $\varphi_{N/S} : U_{N/S} \to \mathbb{R}^n$ defined for $x \in U_{N/S}$ by setting*

$$\varphi_{N/S}(x) = \frac{1}{1 \mp x_{n+1}}(x_1, \ldots, x_n).$$

These maps are continuous on their domains and have continuous inverses defined for $y \in \mathbb{R}^n$ by setting

$$\varphi_{N/S}^{-1}(y) = \frac{1}{1 + |y|^2}(2y_1, \ldots, 2y_n, \pm(|y|^2 - 1));$$

hence the maps are homeomorphisms and $\{\varphi_{N/S} : U_{N/S} \to \mathbb{R}^n\}$ defines an atlas for S^n; thus it is a topological n-manifold. Noting that $U_N \cap U_S = \mathbb{R}^n \setminus \{0\}$ and $\varphi_{N/S}(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$ one can compute that, for $y \in \mathbb{R}^n \setminus \{0\}$, a transition map is given by

$$\varphi_S \circ \varphi_N^{-1}(y) = \frac{y}{|y|^2};$$

which is its own inverse. The transition maps are thus seen to be smooth on $\varphi_{N/S}(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$ and hence S^n is an n-manifold.

Example 15. *One can check that if M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold. This can be used to show, with the previous example, that the standard n-torus, $T^n = S^1 \times \cdots \times S^1$ (product of n circles), is an n-manifold.*

2.3.2 Smooth maps and diffeomorphisms

With the formal definition of a manifold now established, we want to start to make sense of notions from calculus on these objects. The topological properties of a manifold mean that the notion of continuity for maps on and between manifolds is clear, but we need to see how to define a notion of smoothness. Smoothness of the manifold itself was defined by requiring that the transition maps were smooth, and since these were maps between Euclidean spaces this notion was clear. We can do the same for a map from a manifold into some Euclidean space as follows:

Definition 14. *Given an n-manifold, M, with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ a map $f : M \to \mathbb{R}^m$ is* ***smooth*** *if the map*

$$f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}^m$$

is smooth for all $\alpha \in \mathcal{A}$.

Remark 17. *Since the map $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}^m$ is a map between Euclidiean spaces, the smoothness of this map is required in the sense defined previously; i.e. if the derivatives of all orders exist at every point of V_α. By considering $f = \varphi_\alpha$ in the definition we see that every chart of an n-manifold is smooth in the above sense as $\varphi_\alpha \circ \varphi_\alpha^{-1} = I_n$ is the identity map (which is smooth). Observe that the smoothness of a map implicitly depends on the choice of atlas on the manifold; indeed this can be used to define another equivalence relation between atlases as discussed previously.*

We can very quickly generalise this notion to define smooth maps between manifolds:

Definition 15. *Given an m-manifold, M, with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ and an n-manifold, N, with an atlas $\{\phi_\beta : U_\beta \to V_\beta\}_{\beta\in\mathcal{B}}$ a map $F : M \to N$ is* ***smooth*** *if the map*

$$\phi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap F^{-1}(U_\beta)) \to \mathbb{R}^n$$

is smooth for all $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.

Remark 18. *One thing from the two definitions above that is not immediately clear is whether smoothness is preserved on regions where coordinate patches overlap; as we shall now show, ensuring this condition can be thought of as the reason for asking transition maps to be smooth in the definition of a manifold in the first place. Consider $\alpha, \widetilde{\alpha} \in \mathcal{A}$ and $\beta, \widetilde{\beta} \in \mathcal{B}$ such that both $U_\alpha \cap U_{\widetilde{\alpha}} \neq \emptyset$ and $U_\beta \cap U_{\widetilde{\beta}} \neq \emptyset$. We then have that*

$$\phi_{\widetilde{\beta}} \circ F \circ \varphi_{\widetilde{\alpha}}^{-1} = (\phi_{\widetilde{\beta}} \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ F \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_{\widetilde{\alpha}}^{-1}),$$

from which we see that the left hand side of the above is smooth if and only if the second term on the right hand side is smooth (since the first and third terms on the right hand side are smooth transition maps). The moral here is that by requiring transition maps to be smooth, we ensure that smoothness is preserved when piecing together coordinate patches.

Both of the above definitions recover the notions for smooth maps in Euclidean space immediately since we can take an atlas consisting of the identity map. We will almost exclusively deal with smooth maps in this course, but let us check a few examples:

Example 16. *The identity map $I : M \to M$ on any n-manifold is smooth as, if $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ is an atlas on M, we have $\varphi_\alpha \circ I \circ \varphi_\alpha^{-1} = I_n$ the identity on $\mathbb{R}^n$ restricted to V_α, which is smooth.*

Example 17. *If $M \subset \mathbb{R}^n$ is a manifold (of any dimension $\leq n$) then the restriction to M of any smooth map on $\mathbb{R}^n$ is smooth. Similarly, if $M \subset \mathbb{R}^m$ and $\mathrm{N} \subset \mathbb{R}^n$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map with $f(M) \subset N$ then the restriction of f to M is smooth as a map between manifolds. For example, consider the* ***Hopf fibration*** *map $f : \mathbb{R}^4 \to \mathbb{R}^3$ defined for $x \in \mathbb{R}^4$ by setting*

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2x_1x_4 + 2x_2x_3, 2x_2x_4 - 2x_1x_3),$$

which is smooth. One can check that if $x \in S^3$ then $f(x) \in S^2$ and thus the restriction of the Hopf fibration to S^3 is a smooth map between S^3 and S^2.

Example 18. *Given a matrix group, G, which is a manifold, e.g. $GL(n,\mathbb{R})$ (we saw this was a manifold already), $SL(n,\mathbb{R}), O(n), SO(n)$ etc., one can consider the* ***multiplication map*** *$m : G \times G \to G$ taking a pair $(A, B) \in G \times G$ to $AB \in G$ and the* ***inversion map*** *$i : G \to G$ taking $A \in G$ to $A^{-1} \in G$; both of these maps are smooth maps (note that the first is a map from the product manifold $G \times G$ to G). The smoothness of the multiplication and inversion map on a group which is a manifold are what define them as* ***Lie groups****. Similarly, the left and right multiplication maps $L_A : G \to G$ with $L_A(B) = AB$ and $R_A : G \to G$ with $R_A(B) = BA$, determinant map $\det : G \to \mathbb{R}$, and trace map $\mathrm{tr} : G \to \mathbb{R}$ are also smooth.*

With the notion of smoothness established for maps between manifolds, we can now specify whether two manifolds are the same as we did for subsets of Euclidean space:

Definition 16. *Given manifolds M and N a map $F : M \to N$ is a* ***diffeomorphism*** *if it is smooth, bijective, and has smooth inverse (i.e. $F^{-1} : N \to M$ is well defined and smooth). If there is a diffeomorphism between M and N we say that the manifolds are* ***diffeomorphic****.*

Again this directly generalises the notion of a diffeomorphism between subsets of Euclidean space. We note however that the definition there required open sets in Euclidean spaces to be of the same dimension.

Remark 19. *Later on, we will in fact see that if two manifolds are diffeomorphic they necessarily have the same dimension; this can be checked for Euclidean spaces by using the chain rule and some linear algebra. The fact that open subsets of Euclidean space that were homeomorphic necessarily had to be of the same dimension implied that topological manifolds had uniquely defined dimension. One can replace homeomorphic by diffeomorphic in the previous sentence to show that a uniquely defined dimension holds for smooth manifolds in a simpler way.*

Let's look at some examples of diffeomorphic manifolds:

Example 19. *As we saw, charts for an manifold are smooth. Given an n-manifold, M, if $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ is an atlas on M then we have $\varphi_\alpha \circ \varphi_\alpha \circ I_n = I_n$ the identity on $\mathbb{R}^n$ restricted to V_α, which is smooth. We thus see that for each $\alpha \in \mathcal{A}$ that each chart, $\varphi_\alpha : U_\alpha \to V_\alpha$ is a diffeomorphism and so the manifolds U_α and V_α are diffeomorphic. This justifies the idea that manifolds locally 'look like' Euclidean space, in the sense that they are the 'same' as manifolds in each coordinate patch.*

Example 20. *The identity map $I_M : M \to M$ is a diffeomorphism as it is its own inverse.*

Example 21. *As a linear map on $\mathbb{R}^n$ corresponds to a smooth map in the usual sense for Euclidean space, this is a diffeomorphism if and only if it is invertible as a linear map. This corresponds to the matrix representation for the linear map being invertible, and thus the linear diffeomorphisms of $\mathbb{R}^n$ are given by the group $GL(n,\mathbb{R})$.*

Example 22. *The open interval $(0, 1)$ is diffeomorphic to $\mathbb{R}$ (as is any open interval).*

Example 23. *All of the tori we defined previously can be shown to be diffeomorphic.*

Example 24. *If F and G are diffeomorphisms from a manifold, M, to itself then so is $F \circ G$ and F^{-1}. The diffeomorphisms of a manifold thus form a group, which we will denote by $\mathrm{Diff}(M)$.*

2.3.3 Comparing smooth and topological manifolds

With the notion of diffeomorphism in hand, one can now ask how the theories of topological and smooth manifolds are related. We now discuss this briefly, but it can be safely ignored for all intents and purposes.

For dimensions $n \leq 3$ the theory is identical: every topological n-manifold admits an atlas with smooth transition maps and thus can be made into a smooth n-manifold. Additionally, if two topological n-manifolds are homeomorphic then they are necessarily diffeomorphic; thus any two atlases are equivalent and define the same smooth functions (see [Vir13] for $n = 1$, [Hat22] for $n = 2$, and [Mun60, Theorem 6.3] for $n = 3$).

In dimensions $n \geq 4$ however the situation is drastically different: there exist topological 4-manifolds that do not admit any atlas with smooth transition maps, and thus cannot be made into a smooth 4-manifold (see [Fre82]). There are also pairs of smooth n-manifolds which are homeomorphic but not diffeomorphic; showing that the atlases on each are not equivalent in the sense that they define different smooth structures. Historically speaking, it was first shown that there are 28 distinct smooth structures on the 7-sphere (see [Mil56, KM63]). It was also shown that there exists a smooth structure on $\mathbb{R}^4$ that is not diffeomorphic to the standard one (see [Don83]); this is not true in any other dimension, where the standard smooth structure on $\mathbb{R}^n$ for $n \neq 4$ is unique (see [Sta62]). Non-standard smooth structures are often referred to as **exotic structures**. This is still an active area of research; for instance, it seems to be an open problem to determine whether the 4-sphere admits an exotic structure (referred to as the smooth 4-dimensional Poincaré conjecture).

2.3.4 Manifolds as quotient spaces

As an application of the notion of diffeomorphisms we just introduced, we can now generate a wide range of examples of manifolds using quotient constructions.

Let us first make a couple of definitions:

Definition 17. *A group, G, **acts on a manifold** M **by diffeomorphisms** if for each $g \in G$ there is a diffeomorphism $F_g \in \mathrm{Diff}(M)$ such that:*

- *$F_e = I_M$ (i.e. the identity of G corresponds to the identity on M).*
- *$F_{g \circ h} = F_g \circ F_h$ for each $g, h \in G$.*

We will use group actions in order to construct manifolds, but need to impose two further restrictions on these actions to ensure the constructions work:

Definition 18. *A group is said to be **discrete** if it has at most countably many elements.*

Definition 19. *A discrete group, G, that acts on a manifold M by diffeomorphisms is said to **act freely and properly discontinuously** if both*

- *Each $x \in M$ belongs to an open set $U \subset M$ with $F_g(U) \cap U = \emptyset$ for each $g \in G \setminus \{e\}$.*
- *For any $x, y \in M$ where $x \neq F_g(y)$ for any $g \in G$ there are open sets $U, V \subset M$ with $x \in U$, $y \in V$, and such that $U \cap F_g(V) = \emptyset$ for all $g \in G$.*

The first property says that the action has no fixed points, and the second says that if no diffeomorphism takes y to x then the same is true for open sets containing y and x. With these definitions in hand we have the following effective way to construct manifolds as quotient spaces:

Theorem 2. *(Quotient manifold theorem) Let M be an n-manifold and G be a discrete group that acts on M by diffeomorphisms freely and properly discontinuously. Define an equivalence relation, $\sim$, on M by setting $x \sim y$ if and only if $x = F_g(y)$ for some $g \in G$. Then, the quotient space $M/\sim = M/G$ is an n-manifold.*

Proof. (*not covered in lectures*) We first check that the quotient topology induced on $M/\sim$ is both Hausdorff and second countable. To show this we will first establish that the projection $\pi : M \to M/\sim$ is an open map (i.e. takes open sets to open sets). Consider an open set $U \subset M$, we then observe that, by the definition of the equivalence relation, $\pi^{-1}(\pi(U)) = \cup_{g \in G} F_g(U)$. Since U is open and diffeomorphisms are homeomorphisms, each of the sets $F_g(U)$ are open and thus $\pi^{-1}(\pi(U))$ is open as the union of open sets. By the definition of the quotient topology this means that $\pi(U)$ is open in $M/\sim$ and thus π is an open map.

Consider two distinct point $x, y \in M/\sim$, by the definition of the equivalence relation this implies that $x \neq F_g(y)$ for all $g \in G$ (else we would have $x \sim y$). As G acts by diffeomorphisms freely and properly discontinuously (in particular by the second condition in the definition), there are thus open sets $U, V \subset M$ such that $U \cap F_g(V) = \emptyset$ for all $g \in G$. It thus follows that $\pi(U) \cap \pi(V) = \emptyset$ and as *pi* is an open map, both $\pi(U)$ and $\pi(V)$ are disjoint open sets containing x and y respectively; thus $M/\sim$ is Hausdorff.

As M is second countable, let $\{U_i\}_{i=1}^{\infty} \subset M$ be a countable base for the topology on M. As π is an open map, we can then define a countable base, $\{\pi(U_i)\}_{i=1}^{\infty}$, for $M/\sim$, showing that it is second countable. The fact that this collection is a countable base follows by taking an open $U \subset M/\sim$, giving an open set $\pi^{-1}(U) \subset M$ which we then write as a union of the original base, and then projecting this set to $M/\sim$.

Let $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$ denote the atlas for M. By the assumption that G acts freely and properly discontinuously, by potentially replacing the U_α, without loss of generality we may assume that each of the open sets $U_\alpha \subset M$ is such that $U_\alpha \cap F_g(U_\alpha) = \emptyset$ for all $g \in G \setminus \{e\}$. Since $M = \cup_{\alpha \in \mathcal{A}} U_\alpha$ we then have that $M/\sim = \cup_{\alpha \in \mathcal{A}} \pi(U_\alpha)$, where each of the $\pi(U_\alpha)$ are open as π is an open map. We then ensure that the restriction of π to each U_α, which we denote by $\pi_\alpha : U_\alpha \to \pi(U_\alpha)$, is a homeomorphism; the injectivity follows as $U_\alpha \cap F_g(U_\alpha) = \emptyset$ for all $g \in G \setminus \{e\}$ and its inverse is continuous as π is open. We can then define charts, $\phi_\alpha = \varphi_\alpha \circ \pi_\alpha^{-1} : \pi(U_\alpha) \to V_\alpha \subset \mathbb{R}^n$ (which are homeomorphisms as a composition of homeomorphisms), and thus an atlas for $M/\sim$ is given by $\{\phi_\alpha : \pi(U_\alpha) \to V_\alpha\}_{\alpha \in \mathcal{A}}$; this shows that $M/\sim$ is a topological n-manifold, it remains to prove that this atlas has smooth transition maps.

If $\pi(U_\alpha) \cap \pi(U_\beta) \neq \emptyset$ for some $\alpha, \beta \in \mathcal{A}$ then we need to check that the map $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth. As G acts by diffeomorphisms freely and properly discontinuously (in particular by the first condition in the definition), given a point $x \in \phi_\alpha(\pi(U_\alpha) \cap \pi(U_\beta))$ we see that $p \in \varphi_\alpha(U_\alpha \cap F_g(U_\beta))$ for a unique choice of $g \in G$. By definition of the charts we have that

$$\phi_\beta \circ \phi_\alpha^{-1} = \varphi_\beta \circ \pi_\beta^{-1} \circ \pi_\alpha \circ \varphi_\alpha^{-1}$$

and thus, by restricting to $\phi_\alpha(U_\alpha \cap F_g(U_\beta))$, it is enough to check that $\pi_\beta^{-1} \circ \pi_\alpha$ is smooth on the set $U_\alpha \cap F_g(U_\beta)$ (as the transition maps for M are smooth). Now if $y \in U_\alpha \cap F_g(U_\beta)$ then we have

$\widetilde{y} = \pi_\beta^{-1} \circ \pi_\alpha(y) \in U_\beta$ and thus $\pi_\beta(\widetilde{y}) = \pi_\alpha(y)$. By the definition of the equivalence relation this ensures that there is some $h \in G$ such that $F_h(\widetilde{y}) = y$. Since $y \in U_\alpha \cap F_g(U_\beta)$ and $\widetilde{y} \in U_\beta$ this implies that $y \in F_h(U_\beta) \cap F_g(U_\beta)$, but by the choice of atlas on M this ensures that $g = h$. We then note that as $y = F_g(\widetilde{y}) = F_g \circ (\pi_\beta^{-1} \circ \pi_\alpha)(y)$ we must have that $\pi_\beta^{-1} \circ \pi_\alpha = F_g^{-1}$ on the set $U_\alpha \cap F_g(U_\beta)$, which is smooth as it is a diffeomorphism. Thus we conclude that $\pi_\beta^{-1} \circ \pi_\alpha$ is smooth on $U_\alpha \cap F_g(U_\beta)$ and so the transition map $\phi_\beta^{-1} \circ \phi_\alpha$ is smooth as a composition of smooth maps; hence $M/\sim$ is an n-manifold. $\square$

Remark 20. *One can in fact define manifolds as quotient spaces more generally by considering Lie groups that act by diffeomorphisms freely and properly discontinuously (see [Lee12, Chapter 21]).*

We now put this theorem to use to produce a variety of examples of manifolds:

Example 25. *The group of two elements $\mathbb{Z}_2 = \{-1, 1\}$ acts on $\mathbb{R}^n$ by diffeomorphisms $F_{-1} = -I_n$ and $F_1 = I_n$ (the identity on $\mathbb{R}^n$); but does not act freely and properly discontinuously as $0 \in \mathbb{R}^n$ is fixed. One can check however that it does act freely and properly discontinuously on $\mathbb{R}^n \setminus \{0\}$, and indeed any manifold $M \subset \mathbb{R}^n \setminus \{0\}$ such that $-M = M$. This allows us to show that $\mathbb{R}P^n$, the Möbius band, and the Klein bottle are manifolds (applying the above theorem to appropriate manifolds in $\mathbb{R}^n \setminus \{0\}$).*

Example 26. *By considering the n-dimensional **hyperbolic space**, $\mathbb{H}^n = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$ (which is an n-manifold as it is an open subset of $\mathbb{R}^n$) and taking quotients by group actions one can define various **hyperbolic manifolds**.*

Example 27. *The group $\mathbb{Z}$ acts on $\mathbb{R}$ by diffeomorphisms freely and discontinuously for each $n \in \mathbb{Z}$ by setting $F_n(x) = x + n$ for $x \in \mathbb{R}$; we then have that $\mathbb{R}/\mathbb{Z}$ is diffeomorphic to the circle. Analogously, $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$ is diffeomorphic to the n-torus or a product of n circles.*

Example 28. *The group $\mathbb{Z}$ acting on $\mathbb{R}^2$ by diffeomorphisms freely and discontinuously for each $n \in \mathbb{Z}$ by setting $F_n(x, y) = (x + n, y)$ for $(x, y) \in \mathbb{R}^2$; we then have that $\mathbb{R}^2/\mathbb{Z}$ is diffeomorphic to a cylinder.*

Example 29. *The group $\mathbb{Z}$ acting on $\mathbb{R}^2$ by diffeomorphisms freely and discontinuously for each $n \in \mathbb{Z}$ by setting $F_n(x, y) = (x + n, (-1)^n y)$ for $(x, y) \in \mathbb{R}^2$; we then have that $\mathbb{R}^2/\mathbb{Z}$ is diffeomorphic to a Möbius band.*

Example 30. *The quotient constructions producing the Pac-Man torus and the Klein bottle discussed previously can also be realised in the same manner as quotients of $\mathbb{R}^2$.*

We have now shown that (aside from $SL(n, \mathbb{R})$ which we will address after building some more tools) every example discussed in the introduction is indeed a manifold.

2.4 Partitions of unity and the abundance of smooth functions

Before starting to make sense of what the derivative of a smooth function on a manifold really is, we first introduce some immensely useful analytic tools that we will exploit often throughout the course. For instance, as an immediate application of these tools we are able to easily construct a variety of smooth functions on any given manifold. For us, the most important class of smooth functions defined on manifolds will be those taking real values; we make some definitions:

Definition 20. *Given a manifold, M, we let $C^\infty(M) = \{f : M \to \mathbb{R} \,|\, f \text{ is smooth}\}$ denote the set of smooth real valued functions. We define the **support**, $\operatorname{supp}(f)$, of $f \in C^\infty(M)$ to be the set $\overline{\{x \in M \,|\, f(x) \neq 0\}}$ (i.e. the points where f does not vanish and the limits of such points).*

One immediate way to define a smooth function on a manifold is to use functions defined on Euclidean space and pull it back onto the manifold using charts. Precisely, if $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ is an atlas for a manifold M then given any smooth function $f : V_\alpha \to \mathbb{R}$ for some $\alpha \in \mathcal{A}$ we see that $f \circ \varphi_\alpha \in C^\infty(U_\alpha)$. One problem with this construction is that this function is only defined on a coordinate patch and it is unclear, unless $\mathrm{supp}(f \circ \varphi_\alpha) \subset U_\alpha$, how to extend this to one defined on all of the manifold. To do this we first define a helpful function on Euclidean space:

Lemma 1. *(Euclidean bump function) There is a smooth function $\eta \in C^\infty(\mathbb{R}^n)$ taking values in $[0,1]$ such that $\eta(x) = 1$ for $|x| \leq 1$ and $\mathrm{supp}(\eta(x)) \subset B_2(0)$.*

Proof. Let us define a function $f \in C^\infty(\mathbb{R})$ by setting

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

which one can directly verify is smooth, but not real analytic, at 0. We then use this to define a one-dimensional bump function $g \in C^\infty(\mathbb{R})$ taking values in $[0,1]$ by setting

$$g(t) = \frac{f(t)}{f(t) + f(1-t)},$$

which is such that $g(t) = 1$ if $t > 1$ and $g(t) = 0$ if $t < 0$. Finally, we define the Euclidean bump function $\eta \in C^\infty(\mathbb{R}^n)$ by setting $\eta(x) = g(2 - |x|)$; which has the desired properties. □

We can use this function to localise near a point on a manifold as follows:

Corollary 1. *(Bump function at a point) Let M be a manifold, $U \subset M$ an open set, and $x \in U$. There is a function $\eta_x \in C^\infty(M)$ taking values in $[0,1]$ such that $\eta_x(y) = 1$ for $y \in \widetilde{U}$ where $\widetilde{U} \subset U$ is an open set containing x and $\mathrm{supp}(\eta_x) \subset U$.*

Proof. Let $\varphi_\alpha : U_\alpha \to V_\alpha$ be a chart in the atlas for M with $x \in U_\alpha$. Set $y = \varphi_\alpha(x)$ and consider the open set $(\varphi_\alpha - y)(U \cap U_\alpha)$ which contains an open ball $B_\varepsilon(0)$ for some $\varepsilon > 0$. We can consider the open set $\frac{2}{\varepsilon}(\varphi_\alpha - y)(U \cap U_\alpha)$ which contains $B_2(0)$ and define the bump function at a point on $U \cap U_\alpha$ by setting

$$\eta_x = \eta \circ \left(\frac{2}{\varepsilon}(\varphi_\alpha - y)\right),$$

extended to be identically equal to zero on $M \setminus (U \cap U_\alpha)$; which then has the desired properties. □

We call the above the bump function at a point to emphasise that it depends on the point chosen in the manifold. Using this bump function at a point we are able to extend smooth functions defined on open sets, to ones that are defined on the whole manifold and agree with the original function at a point:

Corollary 2. *(Extending functions at a point) Let M be a manifold, $U \subset M$ an open set, $f \in C^\infty(U)$, and $x \in U$. There is an open set $\widetilde{U}_x \subset U$ containing x and a smooth function $\widetilde{f}_x \in C^\infty(M)$ that agrees with f on $\widetilde{U}_x$ (so in particular $\widetilde{f}_x(x) = f(x)$) and $\mathrm{supp}(\widetilde{f}_x) \subset U$.*

Proof. By considering η_x as in the previous corollary we can simply consider the function $\widetilde{f}_x = \eta_x \cdot f$ on U which agrees with f on the open set $\widetilde{U}_x$ and is identically zero on $M \setminus U$; which has the desired properties. □

Applying this latter corollary to the functions $f \circ \varphi_\alpha \in C^\infty(U_\alpha)$ discussed before, we produce a globally defined function on M; thus we see that there is an abundance of smooth functions on a given manifold (as we can pull back arbitrary smooth real value functions on Euclidean space by charts). The issue with the latter corollary however is that we cannot control the domain on which we extend the function to; in particular, we would like to extend smooth functions in such a way that they agree on prescribed subsets of the manifold. To do this we will need a way to piece together functions defined on each coordinate patch, for which we introduce the following indispensable tool:

Definition 21. *Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a cover of a manifold M (e.g. the domains of charts in the atlas). A **partition of unity subordinate to** $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of functions, $\{\rho_\alpha\} \subset C^\infty(M)$, taking values in $[0,1]$ such that:*

- $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ *for every $\alpha \in \mathcal{A}$.*
- $\{\mathrm{supp}(\rho_\alpha)\}_{\alpha \in \mathcal{A}}$ *is **locally finite** (i.e each point of M belongs to an open set intersecting only finitely many sets in the collection).*
- $\sum_{\alpha \in \mathcal{A}} \rho_\alpha(x) = 1$ *for every $x \in M$.*

Remark 21. *We observe that if $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{U_\beta\}_{\beta \in \mathcal{B}}$ are covers of a manifold with partitions of unity subordinate to the covers given by $\{\rho_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{\rho_\beta\}_{\beta \in \mathcal{B}}$ respectively, then $\{U_\alpha \cap U_\beta\}_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}}$ is a cover of M with a subordinate partition of unity given by $\{\rho_\alpha \rho_\beta\}_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}}$; this fact will be useful later when defining integration on manifolds.*

The requirement that the collection $\{\mathrm{supp}(\rho_\alpha)\}_{\alpha \in \mathcal{A}}$ is locally finite in the definition ensures that the sum $\sum_{\alpha \in \mathcal{A}} \rho_\alpha(x)$ is well defined at each point $x \in M$ (as only finitely many terms will be non-zero). We have the following general existence result:

Theorem 3. *(Existence of partitions of unity) There exists a partition of unity subordinate to any cover of a manifold.*

In order to show the existence of partitions of unity subordinate to a given cover we will need to introduce some other notions from topology; all of which will hold for manifolds. Let us postpone this for now and discuss an immediate application to strengthen our two previous corollaries for bump functions and extensions:

Corollary 3. *(Bump function on a set) Let M be a manifold, $A \subset M$ a closed set, and $U \subset M$ an open set containing A. There is a function, $\eta_A \in C^\infty(M)$ taking values in $[0,1]$ such that $\eta_A(x) = 1$ for $x \in A$ and $\mathrm{supp}(\eta_A) \subset U$.*

Proof. Consider the cover of M by the open sets $\{U, M \setminus A\}$ and let $\{\rho_U, \rho_{M \setminus A}\}$ be a partition of unity subordinate to this cover. As $\rho_{M \setminus A}$ is identically zero on A we must have that ρ_U identically equal to 1 on A; setting $\eta_A = \rho_U$ gives the desired function. $\square$

This now means that we can localise to any prescribed closed set on our manifold. To introduce general extensions we extend our definitions slightly and say that given an arbitrary set $A \subset M$ that a function $f : A \to \mathbb{R}$ is smooth if each point in $x \in A$ belongs to an open set, $\widetilde{U}_x \subset M$, and there is a smooth function $\widetilde{f}_x : \widetilde{U}_x \to \mathbb{R}$ that agrees with f on $\widetilde{U}_x \cap A$; we then say that $f \in C^\infty(A)$. The next result allows us to extend functions to the entire manifold so that they agree with the original function on any prescribed closed set:

Corollary 4. *(Extending functions on a set) Let M be a manifold, $A \subset M$ a closed set, and $f \in C^\infty(A)$. For any open set $U \subset M$ containing A there is a function $\widetilde{f}_A \in C^\infty(M)$ that agrees with f on A and such that* $\operatorname{supp}(\widetilde{f}_A) \subset U$.

Proof. As $f \in C^\infty(A)$, for each $x \in A$ we guarantee the existence of a function $\widetilde{f}_x \in C^\infty(M)$ that agrees with f on an open set $\widetilde{U}_x \subset M$ containing x. Potentially replacing $\widetilde{U}_x$ by $\widetilde{U}_x \cap U$ we may assume without loss of generality that $\widetilde{U}_x \subset U$. We then have a cover of M by the open sets $\{\widetilde{U}_x\}_{x\in A} \cup \{M \setminus A\}$, and let $\{\rho_x\}_{x\in A} \cup \{\rho_{M\setminus A}\}$ be a partition of unity subordinate to this cover. For each $x \in A$ we have functions $\rho_x \cdot \widetilde{f}_x \in C^\infty(M)$ by extending them to be identically zero outside of $\widetilde{U}_x$ (as $\operatorname{supp}(\rho_x) \subset \widetilde{U}_x$), and then define $\widetilde{f}_A = \sum_{x\in A} \rho_x \cdot \widetilde{f}_x$. As $\{\operatorname{supp}(\rho_x)\}_{x\in A}$ is locally finite, this sum is well defined with non-zero terms at each point $x \in M$. As $\rho_{M\setminus A}(x) = 0$ if $x \in A$ and $\widetilde{f}_x(y) = f(y)$ for each $y \in \operatorname{supp}(\rho_x) \subset \widetilde{U}_x$ we thus have that $\widetilde{f}_A$ agrees with f on A (as $\sum_{x\in A} \rho_x(y) = 1$ for $y \in A$). Finally, as

$$\operatorname{supp}(\widetilde{f}_A) = \overline{\cup_{x\in A} \operatorname{supp}(\rho_x)} = \cup_{x\in A} \operatorname{supp}(\rho_x) \subset \cup_{x\in A} \widetilde{U}_x \subset U,$$

where the second inequality follows from local finiteness, we have the extension as desired. □

These two consequences of the existence of partitions of unity will be incredibly useful later on. Some other consequences which we will see in this course include the Whitney embedding theorem, the existence of Riemannian metrics, and defining integration on manifolds.

(The rest of this section was not covered in lectures) We now introduce the necessary notions from topology that will allow us to prove the existence of partitions of unity subordinate to any cover of a manifold:

Definition 22. *We say that a cover $\{\widetilde{U}_j\}_{j\in\widetilde{J}}$ of a topological space is a **refinement** of another cover $\{U_i\}_{i\in I}$ of the topological space if for each $j \in J$ there is an $i \in I$ such that $\widetilde{U}_j \subset U_i$. We say that a topological space is **paracompact** if every cover has a locally finite refinement.*

As any subcover of a given cover provides a refinement, any compact topological space is thus paracompact (as every cover has a finite, hence locally finite, subcover). We have the following equivalent definitions of a topological manifold:

Theorem 4. *(Equivalence of second countable and paracompact assumption) In the definition of a topological manifold it is equivalent to replace the second countable assumption with the assumption that it is paracompact with countably many connected components.*

Remark 22. *The proof of the above theorem will in fact also show that it is equivalent to assume the existence of compact exhaustions or a countable cover by compact sets. We will also see that if we have a cover and a base (not necessarily countable) for a manifold, then there exists a countable, locally finite refinement of the cover consisting of elements of this base; this fact will be used in the proof of the existence of partitions of unity.*

Remark 23. *One can in fact also show that in any topological space in which single points are closed (sometimes called a T_1 space) being paracompact and Hausdorff is equivalent to the existence of partitions of unity subordinate to any cover.*

Proof. See [Tan14]. □

If you are not concerned with the topological assumptions needed to define a manifold you can just replace the second countable assumption with paracompact (or separable, or existence of compact exhaustions, or covered by compact sets, or T_1 and existence of partitions of unity subordinate to any cover, or ...) in their definition without further concern. The main point here is that knowing that manifolds are paracompact allows us to select covers that are favourable and the existence of partitions of unity is crucial for going from local constructions to global constructions. With the above established, we can now prove the general existence result for partitions of unity:

Proof of the existence of partitions of unity. We first observe that if we have a collection of non-negative functions $\{\rho_\alpha\}_{\alpha\in\mathcal{A}} \subset C^\infty(M)$ with $\sum_{\alpha\in\mathcal{A}} \rho_\alpha(x) > 0$ at every $x \in M$ we can then renormalise and consider instead the functions $\left\{\frac{\rho_\alpha}{\sum_{\alpha\in\mathcal{A}}\rho_\alpha(x)}\right\}_{\alpha\in\mathcal{A}} \subset C^\infty(M)$ so that the sum over $\alpha \in \mathcal{A}$ is equal to one at every point.

Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be a cover for a manifold M. As each $U_\alpha \subset M$ is an open set for each $\alpha \in \mathcal{A}$, U_α is a manifold and so (by the second countable assumption) we can choose a countable base, $\mathcal{B}_\alpha = \{B_i^\alpha\}_{i=1}^\infty$, for U_α with the property that each $B_i^\alpha \in \mathcal{B}_\alpha$ is the pre-image under a chart of a ball in Euclidean space with rational radius and centre. We then obtain a base (not necessarily countable) for M given by $\mathcal{B} = \cup_{\alpha\in\mathcal{A}}\mathcal{B}_\alpha$, which by the paracompactness of M has a countable, locally finite refinement given by some cover $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$ (see the remark following the equivalence theorem). Since the $\{B_i\}_{i=1}^\infty$ are locally finite we see that their closures $\{\overline{B_i}\}_{i=1}^\infty$ are also locally finite (by considering limit points). Note also that the cover $\{B_i\}_{i=1}^\infty$ is then a refinement of the original cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$.

For each integer $i \geq 1$, as B_i is the pre-image of the restriction some coordinate chart, which we will denote by φ_i, by stretching and translating $\varphi_i(B_i)$ (as in the construction of the bump function at a point) we may assume without loss of generality that $\varphi_i(B_i) = B_2(0)$. We then define non-negative $\rho_i \in C^\infty(M)$ for each integer $i \geq 1$ by setting $\rho_i = \eta \circ \varphi_i$ on every open set containing $\overline{B_i}$ (one can for instance choose the pre-image under a chart of a slightly bigger ball in Euclidean space) and ρ_i identically zero on $M \setminus \overline{B_i}$; where here $\eta \in C^\infty(\mathbb{R})$ is the Euclidean bump function. Note that in particular $\operatorname{supp}(\rho_i) = \overline{B_i}$, and hence $\{\operatorname{supp}(\rho_i)\}_{i=1}^\infty = \{\overline{B_i}\}_{i=2}^\infty$ is locally finite as above. This means that the sum $\sum_{i=1}^\infty \rho_i(x)$ is well defined at each point $x \in M$ (as only finitely many terms are non-zero), and strictly positive at each point as the $\{B_i\}_{i=1}^\infty$ are a cover of M and $\rho_i(x) > 0$ if $x \in B_i$. By renormalising as described above, so that now $\sum_{i=1}^\infty \rho_i(x) = 1$, all that remains is to re-index the functions $\{\rho_i\}_{i=1}^\infty$ in terms of our cover $\mathcal{A}$ in order to complete the construction.

As the cover $\{B_i\}_{i=1}^\infty$ is a refinement of the original cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, for each integer $i \geq 1$ we can choose an $\alpha(i) \in \mathcal{A}$ such that $B_i \subset U_{\alpha(i)}$. We then define $\rho_\alpha \in C^\infty(M)$ by setting $\rho_\alpha = \sum_{\{i\,|\,\alpha(i)=\alpha\}} \rho_i$ (where we choose the zero function for ρ_α if no there are no i with $\alpha(i) = \alpha$). We then have that $\sum_{\alpha\in\mathcal{A}} \rho_\alpha = \sum_{i=1}^\infty \rho_i = 1$ (after renormalising above), $\{\operatorname{supp}(\rho_\alpha)\}_{\alpha\in\mathcal{A}}$ is locally finite, and moreover

$$\operatorname{supp}(\rho_\alpha) = \overline{\cup_{\{i\,|\,\alpha(i)=\alpha\}} B_i} = \cup_{\{i\,|\,\alpha(i)=\alpha\}} \overline{B_i} \subset U_\alpha;$$

where the second equality above follows since $\cup_{\{i\,|\,\alpha(i)=\alpha\}}\overline{B_i}$ is closed by the local finiteness of the $\{\overline{B_i}\}_{i=1}^\infty$. This completes the construction with $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ as the desired functions. □

3 Tangents to manifolds

We have defined smooth functions on and between manifolds, but it is still not clear what the derivative of such a map should be. In order to make sense of the derivative we will first study tangent vectors to a manifold, a generalisation of the notion of a tangent line or plane to a curve or surface in Euclidean space which does not rely on the manifold sitting inside of an ambient space.

3.1 Tangent vectors and the differential of a map

In a first calculus course the derivative of a function of one variable has the geometric interpretation of being the slope of a tangent line to a curve (i.e. a line that touches but not cross the curve nearby the point of interest), with this curve acting as a linear approximation of the object. Similarly, for multivariable functions the derivative is a linear map whose image provides a linear approximation of the graph of the function at each point; which we call the tangent plane. Geometrically, this idea still makes sense if even if we do not require a surface to be the graph of some function; for instance the tangent plane to a point $x \in S^2 \subset \mathbb{R}^3$ on the sphere consists of all of the vectors perpendicular to x. In general, the tangent plane represents a vector space of the same dimension as the surface, which we often think of as being 'attached' to the surface at the point where we took the derivative.

In order to generalise this to manifolds, which are not required to sit in some ambient space, we want to associate to each point in the manifold a vector space (which we will call the tangent space) which should act as a linear approximation of the manifold at this point. The derivative at a point of a map between manifolds should then be a linear map between these tangent spaces; i.e. the derivative at a point should be a linear map between vector spaces which approximate the manifolds. In order to do this we will first need to shift our perspective.

Let us hereafter consider denote by $I = (-\varepsilon, \varepsilon) \subset \mathbb{R}$ (for some small $\varepsilon > 0$) any interval containing the origin. Consider again a point $x \in S^2 \subset \mathbb{R}^3$ and a smooth curve $\gamma : I \to S^2$ (i.e. a smooth curve whose image lies in S^2) with $\gamma(0) = x$. The derivative of the curve at 0, $\gamma'(0)$, will then be a vector perpendicular to x; in other words $\gamma'(0)$ lies in the tangent plane to S^2 at x; varying over all possible choices of γ, we recover the entire tangent plane. We can thus think of the tangent plane to a surface at a point as being defined by the collection of all smooth curves in the surface that pass through that point.

For a general manifold, M, and a smooth curve $\gamma : I \to M$ it is unclear how to take its derivative, but by composing this with a function $f \in C^\infty(M)$ we get a smooth function of one variable, $f \circ \gamma \in C^\infty(I)$, which we can differentiate; i.e. we can consider $(f \circ \gamma)'(0)$! This gives us a new perspective on the tangent space, namely instead of thinking of curves as defining tangent vectors we think of curves as giving us a way to differentiate functions.

Remark 24. *In Euclidean space this idea recovers the directional derivative: given $x, v \in \mathbb{R}^n$ we can consider the smooth curve $\gamma(t) = x + tv$ which is such that $\gamma(0) = x$ and $\gamma'(0) = v$, so that for each $f \in C^\infty(\mathbb{R}^n)$ we have*

$$(f \circ \gamma)'(0) = \frac{d}{dt}\bigg|_{t=0} f(x + tv) = v \cdot J_f(x) = \sum_{i=1}^{n} v_i \cdot \frac{\partial f}{\partial x_i}(x).$$

Given a point $x \in M$ in a manifold and a smooth curve $\gamma : I \to M$ with $\gamma(0) = x \in M$, we define the linear functional $D_\gamma : C^\infty(M) \to \mathbb{R}$ by setting $D_\gamma(f) = (f \circ \gamma)'(0)$ for each $f \in C^\infty(M)$; the fact that this map is linear follows by linearity of the derivative for one variable functions. We observe that this linear map also inherits the product rule in the sense that given $f, g \in C^\infty(M)$ we have

$$D_\gamma(fg) = (fg \circ \gamma)'(0) = ((f \circ \gamma)(g \circ \gamma))'(0) = g(x)D_\gamma(f) + f(x)D_\gamma(g).$$

For $x \in S^2 \subset \mathbb{R}^3$ and a smooth curve $\gamma : I \to S^2$ with $\gamma(0) = x$ we have $D_\gamma(I_3) = (I_3 \circ \gamma)'(0) = \gamma'(0)$. Let us turn this change in perspective into a formal definition:

Definition 23. *Let $x \in M$ be a point in a manifold. The* ***tangent space*** *to M at x, denoted T_xM, is the collection of all* ***tangent vectors****, which we define as linear functionals $D : C^\infty(M) \to \mathbb{R}$ with the property that*

$$D(fg) = g(x)D(f) + f(x)D(g)$$

for each $f, g \in C^\infty(M)$.

One can check directly that the tangent space is indeed a vector space; though it is not yet clear whether this is finite dimensional or whether it shares the same dimension as the manifold itself! Sometimes the space of tangent vectors are referred to as derivations, and the product rule as the Liebniz rule.

Tangent vectors are now thought of as ways to differentiate functions, which the following lemma helps solidify:

Lemma 2. *(Properties of tangent vectors) Let $x \in M$ be a point in a manifold, $f, g \in C^\infty(M)$, and $D \in T_xM$. Then we have the following:*

1. *If f is constant then $D(f) = 0$.*
2. *If $f(x) = g(x) = 0$ then $D(fg) = 0$.*
3. *If $f = g$ in an open set containing x then $D(f) = D(g)$.*

Proof. For the first property we observe that

$$D(1) = D(1 \cdot 1) = 2 \cdot D(1)$$

and so $D(1) = 0$; the result then follows by linearity of D.

For the second property we compute directly that

$$D(fg) = g(x)D(f) + f(x)D(g) = 0.$$

For the third property we consider $h = f - g \in C^\infty(M)$ which is identically zero on an open set containing x, which we call U. Let η be a bump function identically equal to 1 on $\operatorname{supp}(h)$ with $\operatorname{supp}(\eta) \subset M \setminus \{x\}$ so that $\eta \cdot h = h$ on M and $\eta(x) = 0$. By the second propertyof this lemma just proved we have

$$D(f - g) = D(h) = D(\eta \cdot h) = 0,$$

and thus by linearity of D we have $D(f) = D(g)$. $\square$

This lemma is telling us that tangent vectors do indeed behave like derivatives of functions, and importantly that they only are determined locally in the sense that they depend on an arbitrarily small neighbourhood of any point. We now would like to check that the tangent space we have, somewhat abstractly, defined indeed makes sense for Euclidean space which is linear, so its tangent space should be itself!

Proposition 1. *(Euclidean tangent space) Given $x \in \mathbb{R}^n$ the tangent space $T_x\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ via the map taking $v \in \mathbb{R}^n$ to the derivation $D_\gamma \in T_x\mathbb{R}^n$ associated to the curve $\gamma(t) = x+tv$. Moreover, a basis for $T_x\mathbb{R}^n$ is $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n \subset T_x\mathbb{R}^n$ which are defined for each $i = 1, \ldots, n$ and $f \in C^\infty(M)$ by $\frac{\partial}{\partial x_i}\big|_x(f) = \frac{\partial f}{\partial x_i}(x)$.*

Proof. Given $v \in \mathbb{R}^n$ we saw that the associated derivation, D_γ, acts on $f \in C^\infty(\mathbb{R}^n)$ by

$$D_\gamma(f) = \sum_{i=1}^n v_i \cdot \frac{\partial f}{\partial x_i}(x).$$

The linearity of the map taking $v \in \mathbb{R}^n$ to D_γ is then immediate.

To prove injectivity we suppose that $v \in \mathbb{R}^n$ is such that $D_\gamma(f) = 0$ for all $f \in C^\infty(\mathbb{R}^n)$. Consider the smooth coordinate functions $x^j : \mathbb{R}^n \to \mathbb{R}$ for each $j = 1, \ldots, n$ defined by $x^j(w) = w_j$ for each $w \in \mathbb{R}^n$. We then have for each $j = 1, \ldots, n$ that

$$0 = D_\gamma(x^j) = \sum_{i=1}^n v_i \cdot \frac{\partial x^j}{\partial x_i} = v_j;$$

as $\frac{\partial x^j}{\partial x_i} = \delta_{ij}$ (equal to one if $i = j$ and zero otherwise). We thus conclude that $v = 0$ and so the map is injective.

To prove surjectivity, for each $D \in T_x\mathbb{R}^n$ and $j = 1, \ldots, n$ we consider $D(x_j) = v_j$ for the smooth coordinate functions as above and set $v = \sum_{i=1}^n v_i e_i$; where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$. We now show that $D = D_\gamma$ for the curve $\gamma(t) = x + tv$. For each $f \in C^\infty(\mathbb{R}^n$ we have by Taylor's theorem that

$$f(y) = f(x) + \sum_{i=1}^n (y_i - x_i) \cdot \frac{\partial f}{\partial x_i}(x) + \sum_{i,j=1}^n (y_i - x_i) \cdot (y_j - x_j) \cdot \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x + t(y-x))\, dt.$$

We now note that since $f(x)$ is a constant and both $(y_i - x_i)$ and $(y_j - x_j)$ are smooth functions (viewing $y_i = x^i(y)$ and $y_j = x^j(y)$) that vanish at x, which by the lemma on the properties of tangent vectors we have that the first and third terms above are zero under the action of D and thus

$$D(f) = \sum_{i=1}^n D(x^j - x_i) \cdot \frac{\partial f}{\partial x_i}(x) = \sum_{i=1}^n v_i \cdot \frac{\partial f}{\partial x_i} = D_\gamma(f),$$

by construction of v; thus the map is surjective and we are done. □

Having defined tangent spaces for manifolds, our definition of the derivative of a map between manifolds at a point is essentially determined by asking it to be linear between the tangent spaces:

Definition 24. *Given a smooth map $F : M \to N$ between manifolds we define for $x \in M$ the* ***differential*** *of F at the point x, denoted by*

$$d_x F : T_x M \to T_{F(x)} N,$$

to be the linear map given by setting

$$(d_x F(D))(g) = D(g \circ F)$$

for each $D \in T_x M$ and $g \in C^\infty(N)$.

One can check that the differential is indeed well defined and linear by its definition. Sometimes the differential is referred to as a pushforward map as it 'pushes' tangent vectors from one manifold onto another. Let's establish some properties:

Lemma 3. *(Properties of the differential) Let $F : M \to N$ and $G : N \to P$ be smooth maps between manifolds and $x \in M$. Then we have the following:*

1. *The chain rule holds, i.e. $d_x(G \circ F) = d_{F(x)}G \circ d_xF : T_xM \to T_{G(F(x))}P$.*

2. *$d_xI_M = I_{T_xM} : T_xM \to T_xM$.*

3. *If F is a diffeomorphism $d_xF : T_xM \to T_{F(x)}N$ is an isomorphism and $(d_xF)^{-1} = d_{F(x)}(F^{-1})$.*

4. *If $U \subset M$ is an open set containing x and $\iota : U \to M$ is the smooth inclusion map then $d_x\iota : T_xU \to T_xM$ is an isomorphism.*

Proof. For the first property we note that for each $D \in T_xM$ and $h \in C^\infty(P)$ we have

$$d_x(G \circ F)(D)(h) = D(h \circ G \circ F) = d_xF(D)(h \circ G) = d_{F(x)}G(d_xF(D))(h),$$

and so $d_x(G \circ F) = d_{F(x)}G \circ d_xF$ as desired.

For the second property we note that for each $D \in T_xM$ and $f \in C^\infty(M)$ we have

$$d_xI_M(D)(f) = D(f \circ I_M) = D(f) = I_{T_xM}(D)(f),$$

and so $d_xI_M = I_{T_xM}$ as desired.

For the third property we note that both $F^{-1} \circ F = I_M$ and $F \circ F^{-1} = I_N$ and so by combining the first and second property proved above we have that both

$$I_{T_xM} = d_x(F^{-1} \circ F) = d_{F(x)}(F^{-1}) \circ d_xF$$

and

$$I_{T_{F(x)}N} = d_{F(x)}(F \circ F^{-1}) = d_xF \circ d_{F(x)}(F^{-1}).$$

We thus see that d_xF is linear and invertible, with inverse $(d_xF)^{-1} = d_{F(x)}(F^{-1})$ as desired.

For the fourth property we first suppose that $D \in T_xU$ is such that $d_x\iota(D) = 0 \in T_xM$ to show injectivity. Let $f \in C^\infty(U)$ and consider any smooth extension, $\widetilde{f} \in C^\infty(M)$, of f that agrees with f on an open set in U containing x. As f and $\widetilde{f}$ agree at x, by the lemma on the properties of tangent vectors we have that

$$D(f) = D(\widetilde{f}\big|_U) = D(\widetilde{f} \circ \iota) = d_x\iota(D)(\widetilde{f}) = 0,$$

by assumption. As f was arbitrary we thus have that $D = 0 \in T_xU$, thus $d_x\iota$ is injective. For surjectivity we let $\widetilde{D} \in T_xM$ and define $D \in T_xU$ on $f \in C^\infty(U)$ by setting $D(f) = \widetilde{D}(\widetilde{f})$ where $\widetilde{f}$ is any extension, $\widetilde{f} \in C^\infty(M)$, that agrees with f on an open set in U containing x. This is a derivation as D is and is well defined independently of the extension chosen by virtue of the lemma on the properties of tangent vectors. Now if $g \in C^\infty(M)$ then we have by the lemma on tangent vectors, denoting an arbitrary extension by a tilde, that

$$\widetilde{D}(g) = \widetilde{D}(\widetilde{g \circ \iota}) = D(g \circ \iota) = d_x\iota(D)(g),$$

so $\widetilde{D} = d_x\iota(D)$ and thus $d_x\iota$ is surjective; with the injectivity above we have the desired conclusion. □

Remark 25. *Given the fourth property of the above lemma we can safely identify T_xU with T_xM without any further confusion.*

Example 31. *The final property in the lemma above tells us in particular that if $M \subset \mathbb{R}^n$ is an open set which is a manifold then T_xM is isomorphic to $\mathbb{R}^n$ for each $x \in M$; in particular for each $A \in GL(n, \mathbb{R})$ we have $T_A GL(n, \mathbb{R})$ is isomorphic to $\mathbb{R}^{n^2}$, which we identify with the space of $n \times n$ matrices.*

Let us use these properties of the differential to finish our desired generalisation:

Proposition 2. *If M is an n-manifold and $x \in M$ then T_xM is n-dimensional.*

Proof. Let $x \in M$ and $\varphi : U \to V$ be a chart on M with $x \in U$. As φ is a diffeomorphism, by the third property in the lemma on the properties of the differential we have that

$$d_x\varphi : T_xU \to T_{\varphi(x)}V$$

is an isomorphism. By the fourth property in the lemma on the properties of the differential we have that T_xU is isomorphic to T_xM and $T_{\varphi(x)}V$ is isomorphic to $T_{\varphi(x)}\mathbb{R}^n$. Combining the above we conclude that T_xM is isomorphic to $T_{\varphi(x)}\mathbb{R}^n$, which is n-dimensional by the proposition on the Euclidean tangent space, hence T_xM is n-dimensional (and hence also isomorphic to $\mathbb{R}^n$). $\square$

We have thus generalised all of the properties of the usual derivative we desired our derivative of a smooth map between manifolds to have: each point in the manifold is identified with a vector space of the same dimension as the manifold, and the derivative of a smooth map between manifolds gives us a linear map between these spaces.

3.2 Coordinate representations

We did not write down a specific basis of the tangent space when proving it shared the same dimension as the manifold, but we now do so by looking at coordinate patches. Consider a chart, $\varphi : U \to V$, in the atlas of a manifold, M, and $x \in U$. As φ is a diffeomorphism by the properties of the differential established above, we have that $d_x\varphi : T_xM \to T_{\varphi(x)}\mathbb{R}^n$ is a linear isomorphism, with $\{\frac{\partial}{\partial x_i}\big|_{\varphi(x)}\}_{i=1}^n$ a basis of $T_{\varphi(x)}\mathbb{R}^n$ (we are implicitly using the identification of $T_{\varphi(x)}V$ with $T_{\varphi(x)}\mathbb{R}^n$ here). For each $i = 1, \dots, n$ we can define a basis of T_xM by

$$\frac{\partial}{\partial x_i}\bigg|_x = (d_x\varphi)^{-1}\left(\frac{\partial}{\partial x_i}\bigg|_{\varphi(x)}\right) = d_{\varphi(x)}(\varphi^{-1})\left(\frac{\partial}{\partial x_i}\bigg|_{\varphi(x)}\right),$$

which act on $f \in C^\infty(M)$ by

$$\frac{\partial}{\partial x_i}\bigg|_x (f) = \frac{\partial}{\partial x_i}\bigg|_{\varphi(x)} (f \circ \varphi^{-1}) = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(x)) = \frac{\partial f}{\partial x_i}(x).$$

In other words, the basis of T_xM takes the partial derivative in the coordinate patch around x.

Definition 25. *We call the basis $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n$ **coordinate vectors** of T_xM, and for $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\big|_x$ we call v_i for $i = 1, \dots, n$ the **components** of $v \in T_xM$.*

While the definition of the coordinate vectors depends on the choice of chart containing $x \in M$, the way in which we defined T_xM is independent of the choice of chart. One can see how the coordinate vectors and components change between overlapping coordinate patches by looking at the transition maps between the charts, this calculation is similar to the ones that follow so we omit it here and leave it to the homework.

We next see what the differential looks like in coordinates, first examining it in Euclidean space. If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $F : U \to V$ is smooth and $x \in U$, we compute the matrix of the differential $d_xF : T_x\mathbb{R}^n \to T_{F(x)}\mathbb{R}^m$ in the standard basis: writing $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with basis $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n$ for $T_x\mathbb{R}^n$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ with basis $\{\frac{\partial}{\partial y_j}\big|_y\}_{j=1}^m$ for $T_x\mathbb{R}^m$ we have for each $i = 1, \ldots, n$ and $g \in C^\infty(\mathbb{R}^m)$ that

$$d_xF\left(\frac{\partial}{\partial x_i}\bigg|_x\right)(g) = \frac{\partial}{\partial x_i}\bigg|_x (g \circ F) = \sum_{j=1}^m \frac{\partial g}{\partial y_j}\frac{\partial F^j}{\partial x_i}(x) = \left(\sum_{j=1}^m \frac{\partial F^j}{\partial x_i}(x)\frac{\partial}{\partial y_j}\bigg|_{F(x)}\right)(g),$$

and so

$$d_xF\left(\frac{\partial}{\partial x_i}\bigg|_x\right) = \sum_{j=1}^m \frac{\partial F^j}{\partial x_i}(x)\frac{\partial}{\partial y_j}\bigg|_{F(x)}.$$

Thus, the matrix representation of d_xF is thus the Jacobian matrix of F; i.e. d_xF is precisely D_xF!

Suppose now that $F : M \to N$ is a smooth map between manifolds, and $x \in M$, $\varphi : U \to V$ is a chart with $x \in U$, and $\phi : \widetilde{U} \to \widetilde{V}$ is a chart with $F(x) \in \widetilde{U}$. We then have the **coordinate representation of** F denoted

$$\widetilde{F} = \phi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(\widetilde{U})) \to \phi(\widetilde{U})$$

has differential, $d_{\varphi(x)}F$, represented as a matrix by the Jacobian of $\widetilde{F}$ at $\varphi(x)$. We have by the chain rule that

$$d_xF\left(\frac{\partial}{\partial x_i}\bigg|_x\right) = d_xF\left(d_{\varphi(x)}(\varphi^{-1})\left(\frac{\partial}{\partial x_i}\bigg|_{\varphi(x)}\right)\right) = d_{\varphi(x)}(F \circ \varphi^{-1})\left(\frac{\partial}{\partial x_i}\bigg|_{\varphi(x)}\right),$$

and as $F \circ \varphi^{-1} = \phi^{-1} \circ \widetilde{F}$ by definition we have

$$d_xF\left(\frac{\partial}{\partial x_i}\bigg|_x\right) = d_{\varphi(x)}(\phi^{-1} \circ \widetilde{F})\left(\frac{\partial}{\partial x_i}\bigg|_{\varphi(x)}\right) = d_{\widetilde{F}(\varphi(x))}(\phi^{-1}))\left(d_{\varphi(x)}\widetilde{F}\left(\frac{\partial}{\partial x_i}\bigg|_{\varphi(x)}\right)\right).$$

Now by our calculation in Euclidean space above we see that

$$d_xF\left(\frac{\partial}{\partial x_i}\bigg|_x\right) = d_{\widetilde{F}(\varphi(x))}(\phi^{-1})\left(\sum_{j=1}^m \frac{\partial \widetilde{F}^j}{\partial x_i}(\varphi(x))\frac{\partial}{\partial y_j}\bigg|_{\widetilde{F}(\varphi(x))}\right) = \sum_{j=1}^m \frac{\partial \widetilde{F}^j}{\partial x_i}(\varphi(x))\frac{\partial}{\partial y_j}\bigg|_{F(x)}.$$

We therefore conclude that the differential d_xF is represented in coordinates by the Jacobian of its coordinate representative. By using the identity, $I_M : M \to M$, in the above and following the calculation, one sees how coordinate vectors and their components change between charts; precisely we get that $\widetilde{F}$ in the above is a transition map!

3.3 Immersions, submersions, and embeddings

We now want to see what the differential of a smooth map between manifolds tells us about the map itself. We first have the following generalisation of the Euclidean inverse function theorem:

Theorem 5. *(Manifold inverse function theorem) If $F : M \to N$ is a smooth map between manifolds with d_xF invertible at some point $x \in M$, then there exist open sets $U \subset M$ and $V \subset N$ such that $F : U \to V$ is a diffeomorphism.*

Proof. As d_xF is invertible, we must have that the dimensions of M and N agree since the dimensions of T_xM and $T_{F(x)}N$ must also. Let $\varphi : U \to V$ and $\phi : \widetilde{U} \to \widetilde{V}$ be charts around x on M and $F(x)$ on N respectively. We then have that the smooth coordinate representation $\widetilde{F} = \phi \circ F \circ \varphi$ is such that $d_{\varphi(x)}\widetilde{F}$ is invertible by the chain rule since the charts are diffeomorphisms and d_xF is invertible by assumption. Applying the Euclidean inverse function theorem to $\widetilde{F}$ we ensure the existence of open sets $V_{\varphi(x)} \subset V$ and $\widetilde{V}_{\phi(F(x))} \subset \widetilde{V}$ containing $\varphi(x)$ and $\phi(F(x))$ respectively such that $\widetilde{F} : V_{\varphi(x)} \to V_{\phi(F(x))}$ is a diffeomorphism. Defining open sets $U_x = \varphi^{-1}(V_{\varphi(x)}) \subset M$ and $\widetilde{U}_{F(x)} = \phi^{-1}(\widetilde{V}_{\phi(F(x))}) \subset N$ we conclude that $F : U_x \to \widetilde{U}_{F(x)}$ is a diffeomorphism as desired. ∎

Note that as d_xF is invertible, M and N have the same dimension. This result says that if the differential is invertible at a point then the smooth map is a local diffeomorphism. As we saw in the proof, the coordinate representation, $\widetilde{F} = \phi \circ F \circ \varphi^{-1}$, of F was a diffeomorphism and thus $\widetilde{F} \circ \varphi = \phi \circ F$ is also a chart on M; i.e. with respect to the charts $\phi \circ F$ on M and ϕ on N, we have that the coordinate representation of F is the identity!

We try to generalise this idea of understanding how the coordinate representation of a smooth maps behaves when the dimensions of the manifolds differ. We will do this by analysing the rank (the dimension of the image) of the differential, a notion which is independent of the basis/coordinates chosen:

Definition 26. *If $L : V \to W$ is a linear map between finite dimensional vector spaces, we say that L has **maximal rank** if the rank of L is equal to $\min\{\dim(V), \dim(W)\}$.*

Observe that if $\dim(V) \leq \dim(W)$ then L is injective, if $\dim(V) \geq \dim(W)$ then L is surjective, and if $\dim(V) = \dim(W)$ then L is invertible. Up to a change of basis for the vector spaces there are only two examples:

Example 32. *For $n \leq m$ we have the **standard immersion**, $\iota : \mathbb{R}^n \to \mathbb{R}^m$, defined by*

$$\iota(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^m.$$

Example 33. *For $n \geq m$ we have the **standard submersion**, $\sigma : \mathbb{R}^n \to \mathbb{R}^m$, defined by*

$$\iota(x_1, \ldots, x_n) = (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

With this notion of maximal rank, one can thus reinterpret the manifold inverse function theorem as saying that if the differential of a smooth map between manifolds of the same dimension is of full rank, then the map is a local diffeomorphism. Let's generalise this:

Theorem 6. *(Rank theorem) If $F : M \to N$ is a smooth map between manifolds with d_xF of maximal rank at some point $x \in M$, then there exist charts on M and N around x and $F(x)$ respectively such that the coordinate representation of F is the restriction of the standard immersion or submersion.*

Proof. We first establish the theorem for maps between Euclidean spaces which will imply the result in general by pre and post composing with charts, i.e. we establish it directly for the coordinate representation. To this end let $F : \mathbb{R}^n \to \mathbb{R}^m$ be smooth with $d_x F$ of maximal rank for some $x \in \mathbb{R}^n$. We now address each case depending on whether $n \leq m$ or $n \geq m$. Without of loss of generality, indeed up to translation and a change of basis, we may assume that $x = 0 \in \mathbb{R}^n$, $F(x) = 0 \in \mathbb{R}^m$, and $d_0 F$ is the standard immersion or submersion.

If $n \leq m$, we write $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$ and define a smooth map $\widetilde{F} : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ by setting

$$\widetilde{F}(x_1, \ldots, x_m) = F(x_1, \ldots, x_n) + (0^n, x_{n+1}, \ldots, x_m),$$

where 0^n denotes n zero entries. We then have by assumption that $d_0 \widetilde{F}$ is the identity on $\mathbb{R}^m$ as $d_0 F$ is the standard immersion. By the Euclidean inverse function theorem we ensure the existence of open sets $\widetilde{U}, \widetilde{V} \subset \mathbb{R}^m$ containing the origin such that $\widetilde{F} : \widetilde{U} \to \widetilde{V}$ is a diffeomorphism. By setting $U = \widetilde{U} \cap \mathbb{R}^n$ which is open in $\mathbb{R}^n$ we ensure that if $y \in U$ then

$$\widetilde{F}^{-1}(F(y)) = \widetilde{F}^{-1}(F(y) + 0^m) = \widetilde{F}^{-1}(\widetilde{F}(y, 0^{m-n})) = (y, 0^{m-n});$$

thus in some coordinates we have that F is the restriction of the standard immersion.

If $n \geq m$, we write $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and define a smooth map $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ by setting

$$\widetilde{F}(x) = (F(x), \pi(x)),$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is the smooth projection to the final $n - m$ coordinates. We then have by assumption that $d_0 \widetilde{F}$ is the identity on $\mathbb{R}^n$ as $d_0 F$ is the standard submersion. By the Euclidean inverse function theorem we ensure the existence of open sets $\widetilde{U}, \widetilde{V} \subset \mathbb{R}^n$ containing the origin such that $\widetilde{F} : \widetilde{U} \to \widetilde{V}$ is a diffeomorphism. We then note that if $y \in \widetilde{V}$ then $y = (F(x), \pi(x))$ for some $x \in \widetilde{U}$ and thus in particular, $F(x) = (y_1, \ldots, y_m)$. Observe now that

$$F(\widetilde{F}^{-1}(y)) = F(\widetilde{F}^{-1}(F(x), \pi(x))) = F(x) = (y_1, \ldots, y_m);$$

thus in some coordinates we have that F is the restriction of the standard submersion.

As mentioned at the beginning of the proof, if we had a smooth map $F : M \to N$ between manifolds with $d_x F$ of maximal rank for some $x \in M$, then by considering charts around x and $F(x)$ we can apply the above reasoning to the coordinate representation of F to yield the result. $\square$

We can use the rank theorem to establish a method for producing a large number of examples of manifolds and compute their tangent spaces:

Theorem 7. *(Regular value theorem) Let $F : M \to N$ be a smooth map between manifolds of dimension m and n respectively. If $y \in N$ is such that $F^{-1}(y) \neq \emptyset$ and $d_x F : T_x M \to T_y N$ is surjective for all $x \in F^{-1}(y)$ then $F^{-1}(y)$ is an $(m-n)$-manifold with $T_x(F^{-1}(y)) = \mathrm{Ker}(d_x F)$ for each $x \in F^{-1}(y)$.*

Proof. First note that $m \geq n$ or the statement is vacuous. We will equip $F^{-1}(y)$ with the subspace topology so that it is Hausdorff and second countable since M is. We now define an atlas of charts on $F^{-1}(y)$ with smooth transition maps. Given $x \in F^{-1}(y)$, since $d_x F$ is surjective by the rank theorem we may choose some chart, $\varphi_x : U_x \to V_x$, on M around x so that some coordinate representation of F is given by the restriction of the standard submersion. We then note that $F^{-1}(y) \cap U_x$ corresponds to the points in U_x with first n coordinates in $V_x \subset \mathbb{R}^m$ equal to zero; this is since $F(x) = y$ maps to the origin in $\mathbb{R}^n$ in this coordinate representation. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-n}$ be the smooth projection to the final $n - m$ coordinates and define a chart around x by

$$\pi \circ \varphi_x : F^{-1}(y) \cap U_x \to \mathbb{R}^{m-n},$$

which is a homeomorphism as π restricts to a homeomorphism on $V_x \cap \{0^m\} \times \mathbb{R}^{m-n} \subset \mathbb{R}^m$. Varying over all $x \in F^{-1}(y)$ gives an atlas for $F^{-1}(y)$, and we note that the transition maps are smooth since the the transition maps for M are.

As each tangent vector arises as the velocity of a curve (homework 2), given $v \in T_xF^{-1}(y)$ for some $x \in F^{-1}(y)$ as above we thus have some $\gamma : I \to F^{-1}(y)$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Note that as $\gamma(I) \subset F^{-1}(y)$ we have $(F \circ \gamma)(t) = y$ for each $t \in I$, and so $d_xF(v)(f) = d_xF(\gamma'(0))(f) = (f \circ F \circ \gamma)'(0) = 0$; thus $v \in \mathrm{Ker}(d_xF)$ and so $T_xF^{-1}(y) \subset \mathrm{Ker}(d_xF)$. On the other hand, since $\dim(\mathrm{Ker}(d_xF)) = m - n = \dim(T_xF^{-1}(y))$ we must have $T_x(F^{-1}(y)) = \mathrm{Ker}(d_xF)$ as desired. □

Remark 26. *We call points, $y \in N$, in the hypothesis of the regular value theorem* ***regular values****. Sard's theorem (not proved in this course) for manifolds implies that almost every (in the measure theoretic sense) point of $F(M) \subset N$ is a regular value; in other words, if we take a point in the image of the map at random then with probability one its preimage will be a $(m - n)$-manifold.*

Let's apply the regular value theorem to a number of examples, allowing us a slicker method to verify familiar objects are indeed manifolds and find their tangent spaces:

Example 34. *Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be the smooth map defined by setting $F(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$; we then have that, for each $r > 0$, $F^{-1}(r)$ is the n-sphere of radius $\sqrt{r}$ in $\mathbb{R}^{n+1}$. We see that for each $x \in \mathbb{R}^{n+1}$ we have*

$$d_xF = (2x_1, \ldots, 2x_{n+1})$$

which is surjective whenever $x \neq 0$, thus by the regular value theorem the sphere of radius r in $\mathbb{R}^{n+1}$ is an n-manifold (note we use $n + 1 - 1 = n$ here to find the dimension); in particular S^n is an n-manifold. We also see from the above that

$$T_xS^n = \mathrm{Ker}(d_xF) = \left\{ v \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} v_i \cdot x_i = 0 \right\};$$

and so in particular the tangent space to a point $x \in S^n$ is precisely the set of vectors perpendicular to x.

Example 35. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and consider the smooth map $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by $F(x, t) = f(x) - t$; we then have that $F^{-1}(0) = \mathrm{graph}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \,|\, y = f(x)\} \subset \mathbb{R}^{n+1}$. We see that $d_{(x,t)}F = (d_xf, -1)$ which is of rank one and thus surjective, thus by the regular value theorem we conclude that $F^{-1}(0) = \mathrm{graph}(f)$ is an n-manifold. We also see from the above that*

$$T_{(x,f(x))}\mathrm{graph}(f) = \mathrm{Ker}(d_{(x,f(x))}F) = \{(u, v) \in \mathbb{R}^n \times \mathbb{R} \,|\, v = d_xf(u)\} = \mathrm{graph}(d_xf).$$

Example 36. *Let $\Delta : \mathbb{R}^{n^2} \to \mathbb{R}$ denote the smooth determinant map, $\det : GL(n, \mathbb{R}) \to \mathbb{R}$, acting on Euclidean space. For each $A \in GL(n, \mathbb{R})$ let $\gamma : I \to GL(n, \mathbb{R})$ be defined by $\gamma(t) = (1 + t)A$, so that $\Delta(\gamma(t)) = (1 + t)^n\Delta(A)$. We then compute that (taking $f(x) = x$ identity on $\mathbb{R}$) we have*

$$(d_A\Delta)(\gamma'(0))(x) = (\Delta \circ \gamma)'(0) = n\Delta(A);$$

we thus have that $d_A\Delta$ is surjective for each A with non-zero determinant, thus by the regular value theorem $\Delta^{-1}(1) = SL(n, \mathbb{R})$ is an $(n^2 - 1)$-manifold. One can in fact also show that the tangent space to the identity is the set of trace free matrices!

The regular value theorem produces manifolds (level sets of smooth maps) that are subsets of some other manifold (the domain of the smooth map). We would like to understand how these manifolds, or indeed manifolds in general, 'sit' inside one another. Motivated by the rank theorem we introduce the following notions:

Definition 27. *We say that a smooth map $F : M \to N$ between manifolds is an:*

- ***immersion*** *if d_xF is injective for each $x \in M$. We call the image of an immersion an **immersed submanifold**.*

- ***submersion*** *if d_xF is surjective for each $x \in M$. We call the image of a submersion a **submersed submanifold***

- ***embedding*** *if F is a diffeomorphism onto its range (or equivalently an immersion which is a homeomorphism onto its range). We call the image of an embedding, $F(M)$, simply a **submanifold** of N and often denote this simply by $M \subset N$.*

Remark 27. *The equivalence of the definitions of embeddings follows since if F is homeomorphic onto its range then $F(M)$ is also a manifold of the same dimension; this also implies that F^{-1} is well defined on $F(M)$. The injectivity of the differential implies surjectivity by the rank-nullity theorem, since the dimensions of the tangent spaces must then agree, and thus the inverse function theorem guarantees that F^{-1} is also smooth on $F(M)$; hence F is a diffeomorphism onto its image. The other direction of the equivalence is immediate since a diffeomorphism is by definition a homeomorphism that must have injective differential.*

Remark 28. *By the rank theorem, any immersion is locally injective and any submersion is locally surjective (since its coordinate representations will be). Any embedding is necessarily an injective immersion, but the converse is not true (unless it is a homeomorphism onto its range) by considering the image of $[0, 1)$ in $\mathbb{R}^n$ as the curve that looks like a 6.*

Let's look at some examples:

Example 37. *(immersions) The donut torus as the image of $\mathbb{R}^2$ is an immersed submanifold (but the immersion is not injective so it is not an embedding, this also is clear since the domain is not compact but the image is). The figure of eight or lemniscate curve that looks like an 8 is an immersed submanifold in $\mathbb{R}^2$ as an immersion from S^1. Similarly the Klein bottle and $\mathbb{R}P^2$ can be realised as immersed submanifolds of $\mathbb{R}^3$ (the latter is called the Boy surface). Note however that these latter two examples are not able to be realised as submanifolds of $\mathbb{R}^3$ (one can show this by tropological methods).*

Example 38. *(submersions) Prototypical examples of submersions are provided by projection maps, e.g. $\pi_M : M \times N \to M$ or the Hopf fibration map. Note that the differential of these maps cannot be injective if the dimension of the domain is larger than that of the codomain. We will see more examples of submersions when discussing vector bundles later in the course.*

Example 39. *(embeddings) The prototypical embedding is provided by the smooth inclusion map $\iota : M \to N$ which is a diffeomorphism onto its image (this in particular shows that the map from the standard torus to the donut torus from homework 1 is an embedding); a concrete example of which is the embedding $\iota_M : M \to M \times N$ by inclusion. Also, given a smooth map $F : M \to N$ and a regular value $y \in F(M)$, the proof of the regular value theorem shows that $F^{-1}(y)$ is a submanifold of M (by inclusion).*

Given a submanifold $S \subset M$ and a smooth map $F : M \to N$ between manifolds the restriction map $F|_S = F \circ \iota$ is smooth by composition. Similarly, given $f \in C^\infty(S)$ one can use our results on extending smooth functions to show that there exists some open set $U \subset M$ containing M and an extension $\widetilde{f} \in C^\infty(U)$ that agrees with f on S. Using this, and the fact that every tangent vector arises from a curve, one can show that for each $x \in M$ we have $T_xS \subset T_xM$ (or one can simply consider the differential of the inclusion map $\iota : S \to M$).

Since we understand calculus in Euclidean space, if we can embed any manifold into some Euclidean space then we can perform calculus on the manifold simply by viewing it as Euclidean calculus restricted to the manifold. We establish a very weak formulation of this embedding for compact manifolds:

Theorem 8. *Every compact manifold can be embedded into Euclidean space.*

Proof. Let M be a compact manifold of dimension n. Since M is covered by the domains of the charts of any atlas, since M is compact we may extract a finite collection of charts, $\{\varphi_i : U_i \to V_i\}_{i=1}^k$, which is still an atlas for M with smooth transition maps. We will show that M embeds in $\mathbb{R}^{k(n+1)}$.

We extend each chart to a smooth functions on all of M in such a way that they agree inside of the domain of each chart. We further define bump functions, $\{\rho_i\}_{i=1}^k \subset C^\infty(M)$, such that ρ_i is identically equal to one on U_i for each $i = 1, \dots, k$. Consider the map $F : M \to \mathbb{R}^{k(n+1)}$ defined by setting

$$F(x) = (\rho_1(x)\varphi_1(x), \dots, \rho_k(x)\varphi_k(x), \rho_1(x), \dots, \rho_k(x)),$$

which is smooth since the charts and bump functions defined above are; we will show that F is an embedding.

We have that F is an immersion since, by the Liebniz/product rule for differentials we have that if $x \in U_i$ for some $i = 1, \dots, k$ then $d_x(\rho_i\varphi_i) = d_x\varphi_i$, which is injective as charts are diffeomorphisms; hence d_xF is injective and thus F is an immersion.

We also see that F is injective since if $F(x) = F(y)$ then we must have $x \in U_i$ for some $i = 1, \dots, k$ and thus

$$\varphi_i(x) = \rho_i(x)\varphi_i(x) = \rho_i(y)\varphi_i(y) = \varphi_i(y),$$

from which we see that $x = y$ as charts are diffeomorphisms; thus F^{-1} is well defined on $F(M)$.

We conclude by showing that F is a homeomorphism onto its image; we know that F^{-1} is defined on the $F(M)$ since F is injective. As any closed set $A \subset M$ is compact, since M is compact, we ensure that, as F is smooth and hence continuous, $F(A)$ is compact and hence closed in $\mathbb{R}^{k(n+1)}$ by the Heine–Borel theorem. Thus, since $(F^{-1})^{-1}(A) = F(A)$, we see that F^{-1} is continuous since F is a closed map and hence F is a homeomorphism. We thus have that F is an embedding as it is an immersion which is homeomorphic onto its image. $\square$

Remark 29. *The proof actually shows that any n-manifold covered by finitely many, say k, charts can be embedded into Euclidean space, namely $\mathbb{R}^{k(n+1)}$. This embedding is 'wasteful' in some sense, it embeds S^n into $\mathbb{R}^{2n+2}$ but we know it embeds into $\mathbb{R}^{n+1}$! Whitney used the second countability assumption (compact exhaustions specifically) along with a projection technique based on Sard's theorem (mentioned above) to show that every n-manifold (not necessarily compact) in fact embeds into $\mathbb{R}^{2n+1}$; referred to as the weak Whitney embedding theorem. Using topological methods this was improved to* *embeddings in $\mathbb{R}^{2n}$; referred to as the strong Whitney embedding theorem. If one allows for immersions then the weak and strong Whitney immersion theorems allow for immersions into $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n-1}$ respectively. See [Lee12, Chapter 6] for proofs of the proceeding facts. Moreover, if one has a notion of distance on a manifold, namely a Riemannian metric which we will introduce later in the course, then the Nash embedding theorem guarantees that one can embed any such manifold into some high dimensional Euclidean space isometrically (i.e. in a way that preserves distances).*

3.4 The tangent bundle, global differential, and vector fields

We now want to study vector fields on manifolds just as we did for Euclidean space; namely assigning a tangent vector to each point of the manifold in a smoothly varying way. To make sense of such a notion we will need the following:

Definition 28. *Given a manifold, M, the **tangent bundle** of M, denoted TM, is the disjoint union of the tangent spaces of M; i.e. we have*

$$TM = \bigcup_{x \in M} T_x M.$$

We write elements of TM either as pairs $(x, v) \in M \times T_x M$ or more commonly just as $v \in T_x M$. The tangent bundle comes with a natural projection map $\pi : TM \to M$ defined for $(x, v) \in TM$ by $\pi(x, v) = x$ (projection to the first factor).

Let us look at a couple of examples:

Example 40. *Since we have that $T_x \mathbb{R}^n$ is canonically identified with $\mathbb{R}^n$ we have*

$$T\mathbb{R}^n = \bigcup_{x \in \mathbb{R}^n} T_x \mathbb{R}^n \simeq \bigcup_{x \in \mathbb{R}^n} \mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n},$$

where we used the symbol $\simeq$ to denote isomorphisms. We thus see that the tangent bundle to n-dimensional Euclidean space is a $2n$-manifold.

Example 41. *We will show that TS^1 is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$: Consider the map $f : S^1 \times \mathbb{R} \to TS^1$ defined by $f(\cos(\theta), \sin(\theta), \lambda) = \lambda(-\sin(\theta), \cos(\theta)) \in T_{(\cos(\theta), \sin(\theta))} S^1$ which is smooth with smooth inverse, and hence a diffeomorphism. It is not true that TS^2, or the tangent bundle of any even dimensional sphere, is diffeomorphic to $S^2 \times \mathbb{R}^2$ however, by virtue of the "hairy ball" theorem (more on this later).*

Remark 30. *Since manifolds may lack any linear structure, we cannot canonically identify each tangent space with a copy of Euclidean space and thus in general we do not expect that TM is diffeomorphic to $M \times \mathbb{R}^n$; for example, this is true for TS^1 as seen above but not for TS^2. We will discuss this notion further when looking at vector bundles later on in the course.*

While the definition of the tangent bundle is just as a disjoint union of vector spaces, it actually possesses a manifold structure determined by the underlying manifold itself:

Proposition 3. *Given an n-manifold, M, its tangent bundle, TM, is a $2n$-manifold and the natural projection map is a smooth submersion.*

Proof. We define a topology on TM by declaring subsets $V \subset TM$ open if $V = \pi^{-1}(U)$ for some open $U \subset M$. One then has that the Hausdorff and second countability of TM follows from that of M.

Given an atlas of charts, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$, for M we define charts, $\widetilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2n}$, on TM for each $\alpha \in \mathcal{A}$ by setting $\widetilde{\varphi}_\alpha(x, v) = (\varphi_\alpha(x), d_x\varphi_\alpha(v))$ for each pair $(x, v) \in TM$; these are homeomorphisms onto their image since charts are diffeomorphisms. We then have for $\alpha, \beta \in \mathcal{A}$ and $(y, \eta) \in T\mathbb{R}^n$ that

$$\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}(y, \eta) = \widetilde{\varphi}_\beta(\varphi_\alpha^{-1}(y), d_y(\varphi_\alpha^{-1})(\eta)) = (\varphi_\beta \circ \varphi_\alpha^{-1}(y), d_y(\varphi_\beta \circ \varphi_\alpha^{-1})(\eta))$$

where in the second equality we used the fact that $(d_{\varphi_\alpha^{-1}(y)}\varphi_\beta) \circ d_y(\varphi_\alpha^{-1})(\eta) = d_y(\varphi_\beta \circ \varphi_\alpha^{-1})(\eta)$ by the chain rule for the differential. Since the transition maps, $\varphi_\beta \circ \varphi_\alpha^{-1}$, are smooth for M the maps, $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$, are thus smooth and hence $\{\widetilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2n}\}_{\alpha \in \mathcal{A}}$ is an atlas of charts for TM with smooth transition maps; so TM is a $2n$-manifold. To see that $\pi : TM \to M$ is a smooth submersion it suffices to note that its coordinate representation is the smooth submersion. $\square$

Immediately from the proof we then have that:

Corollary 5. *If an n-manifold, M, has a one chart atlas then TM is diffeomorphic to $M \times \mathbb{R}^n$.* $\square$

With the definition of the tangent bundle we can now generalise the notion of vector fields:

Definition 29. *Given a manifold, M, a* ***vector field*** *on M is a smooth map $X : M \to TM$ such that $X_x = X(x) \in T_xM$ for each $x \in M$ (or equivalently $\pi_M \circ X = I_M$). We denote the collection of vector fields on M by $\Gamma(TM)$ or $\mathfrak{X}(M)$ which are often referred to as* ***sections*** *of TM.*

We now look at some examples:

Example 42. *In $\mathbb{R}^n$ we have vector fields $\{\frac{\partial}{\partial x_i}\}_{i=1}^n \subset \Gamma(T\mathbb{R}^n)$ defined such that $\frac{\partial}{\partial x_i}(x) = \frac{\partial}{\partial x_i}\big|_x$ for each $x \in \mathbb{R}^n$ (i.e. these vector fields act on smooth functions by differentiation in the ith coordinate direction at a point $x \in \mathbb{R}^n$). Since the $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n$ form a basis of the tangent space, which is just $\mathbb{R}^n$, at each point, $x \in \mathbb{R}^n$, we then have that any vector field, $X \in \Gamma(T\mathbb{R}^n)$, is such that $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ where $X_i \in C^\infty(\mathbb{R}^n)$ for each $i = 1, \ldots, n$.*

Example 43. *Similarly to the above example, given a point in a manifold, $x \in M$, and a chart, $\varphi : U \to V$, around x we have the* ***coordinate vector fields****, $\{\frac{\partial}{\partial x_i}\}_{i=1}^n \subset \Gamma(TU)$ (which we can extend to $\Gamma(TM)$), defined such that $\frac{\partial}{\partial x_i}(x) = \frac{\partial}{\partial x_i}\big|_x$ for each $x \in U$. Since the $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n$ form a basis of the tangent space, T_xM, at each point, $x \in U$, have that each $X \in \Gamma(TM)$ locally takes the form $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ where $X_i \in C^\infty(U)$ are the* ***component functions*** *for each $i = 1, \ldots, n$.*

Example 44. *If $M \subset \mathbb{R}^n$ is a submanifold, then any vector field on M is the restriction of some vector field, $X \in \Gamma(T\mathbb{R}^n)$, such that $X_x \in T_xM$ for each $x \in M$. E.g. define $X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$ which restricts to a vector field on $S^1 \subset \mathbb{R}^2$ acting on $f \in C^\infty(S^1)$ by $X_{(x_1,x_2)}(f) = -x_2\frac{\partial f}{\partial x_1} + x_1\frac{\partial f}{\partial x_2}$.*

Remark 31. *We have that $\Gamma(TM)$ is a vector space (in fact since we can multiply a vector field by a smooth function it is also a module over $C^\infty(M)$).*

Remark 32. *Using our tools on extending smooth functions we can extend vector fields defined on subsets of a manifold to the entire manifold.*

Given a manifold, M, and both a vector field, $X \in \Gamma(TM)$, and a smooth function, $f \in C^\infty(M)$, we can define another smooth function, $X(f) \in C^\infty(M)$, by setting $X(f)(x) = X_x(f)$ for each $x \in M$. This is well defined since $X_x = X(x) \in T_xM$ for each $x \in M$ and smooth since both X and f are. Moreover, we note that given $f, g \in C^\infty(M)$ the product rule for tangent vectors applied to the above construction shows that $X(fg) = fX(g) + gX(f)$; i.e. vector fields also satisfy the product rule (note here that we are viewing the vector field as a map $X : C^\infty(M) \to C^\infty(M)$). This perspective yields the following characterisation of vector fields:

Proposition 4. *Given a manifold, M, any linear map $D : C^\infty(M) \to C^\infty(M)$ such that $D(fg) = fD(g) + gD(f)$ for each $f, g \in C^\infty(M)$ (i.e. a derivation between the set of smooth functions on a manifold) corresponds to a unique vector field, $X \in \Gamma(TM)$. In other words $D(f) = X(f)$ for each $f \in C^\infty(M)$.*

Proof. Given D as in the statement we can define tangent vectors, $X_x \in T_xM$, for each $x \in M$ by considering $X_x(f) = D(f)(x)$ for each $f \in C^\infty(M)$. We then have that the map $X : M \to TM$ is such that $D(f) = X(f)$, and hence X is smooth since D is. The uniqueness follows since X is entirely determined by the action of D. $\square$

With the above in mind, any derivation between smooth functions on a manifold gives rise to a vector field. Using this idea we can construct new vector fields:

Example 45. *(The Lie bracket) Given a manifold, M, and two vector fields $X, Y \in \Gamma(TM)$ then it is not necessarily true that $XY \in \Gamma(TM)$, where for $f \in C^\infty(M)$ we are defining $XY(f) = X(Y(f)) \in C^\infty(M)$ as in the above discussion. Explicitly we can consider, on $\mathbb{R}^2$, the example $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial y}$ and the smooth functions $f(x, y) = x$, $g(x, y) = y$; then we have that $XY(fg) = 2x$ but $fXY(g) + gXY(f) = x$ and hence XY is not a derivation between the set of smooth functions!*

The precise reason for the failure of the composition of vector fields to be a vector field is due to the appearance of a second derivative term (which one can verify in coordinates) in XY. However, since second derivatives commute we can consider the difference $XY - YX$ which will be a vector field! To verify this we can compute directly that given $f, g \in C^\infty(M)$ we have

$$XY(fg) - YX(fg) = f(XY - YX)(g) + g(XY - YX)(f),$$

*thus $XY - YX$ defined a derivation between smooth functions and hence by the characterisation above is a vector field. We denote this vector field by $[X, Y] = XY - YX \in \Gamma(TM)$ and call it the **Lie bracket** of X and Y. This vector field is very useful in geometry.*

Using the tangent bundle we can also extend the notion of the differential of a smooth map, which is defined at a given point, to one defined globally:

Definition 30. *If $F : M \to N$ is a smooth map between manifolds, then the **global differential** of F, denoted $dF : TM \to TN$, is defined for $(x, v) \in TM$ by*

$$dF(x, v) = (F(x), d_xF(x));$$

in other words dF is the unique map whose restriction to each tangent space, T_xM, is the differential at that point, $d_xF : T_xM \to T_{F(x)}N$.

Note that for maps between Euclidean spaces this is the usual notion of the global derivative. Since they agree at points of the manifold, the global differential inherits properties from the usual differential:

Proposition 5. *(Properties of the global differential) Let $F : M \to N$ and $G : N \to P$ be smooth maps between manifolds. Then we have the following:*

1. *$dF : TM \to TN$ is smooth as a map between manifolds.*

2. *$d(G \circ F) = dG \circ dF$.*

3. *$d(I_M) = I_{TM}$.*

4. *If F is a diffeomorphism then dF is a diffeomorphism with $(dF)^{-1} = d(F^{-1})$.*

Proof. The first property follows by virtue of the fact that the coordinate representation of d_xF depends smoothly on the point $x \in M$. The second, third, and fourth property follow directly from the properties of the differential previously established. □

Just as we alternatively called the differential of a smooth map at a point a pushforward, we have the following notion:

Definition 31. *If $F : M \to N$ is a diffeomorphism between manifolds, then the* ***pushforward*** *of F, denoted $F_* : \Gamma(TM) \to \Gamma(TN)$, is defined for $X \in \Gamma(TM)$ by $F_*(X) = dF(X)$; i.e. $F_*(X)(F(x)) = d_xF(X_x)$ for each $x \in M$.*

Remark 33. *We require F to be a diffeomorphism in the definition above since if F were not injective then it would not be well defined, and if F was not surjective then it would not be defined on all of N.*

3.5 Cotangent bundle, 1-forms, and the line integral

A physical interpretation of the integral along a curve in Euclidean space is that of the 'work' done moving along the curve. Precisely, given a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ with $\gamma(a) = x$ and $\gamma(b) = y$, the integral along this curve should provide us with a number quantifying how much 'work' (or energy etc.) was done moving from x to y along γ. We can discretise this curve by considering vectors, v_i, joining points, x_{i-1} to x_i, which lie on γ so that $x_0 = a$ and $x_n = b$, and then assign an amount of 'work', $\omega_i \in \mathbb{R}$, taken to move along the vector v_i from x_{i-1} to x_i. The 'work' done along this discrete path of vectors from a to b is then equal to $\sum_{i=1}^n \omega_i$. If we imagine taking a limit of this process we would need an object, ω, which inputs vectors tangent to the curve and outputs a number at each point of γ. We should then have by a limiting process that the 'work' done to move from a to b along γ, or the integral of ω along γ, is given by

$$\int_\gamma \omega = \lim_{n\to\infty} \sum_{i=1}^n \omega_i.$$

See [Tao20] for more motivation in this vein (which is where I took the idea for this brief motivation from). In this section we will make this above idea precise.

In order to make sense of the above discussion we will introduce objects that are "dual" to tangent vectors and fields, for which we need to recall some linear algebra:

Definition 32. *Given a finite dimensional vector space, V, over $\mathbb{R}$, a* ***covector*** *is a linear functional $\omega : V \to \mathbb{R}$; i.e. a linear map from V to $\mathbb{R}$. The space of all covectors is the* ***dual space*** *of V, which we denote by V^*. Given a basis, $\{e_i\}_{i=1}^n \subset V$, of V we define the* ***dual basis****, $\{\mathcal{E}^i\}_{i=1}^n \subset V^*$, for V^**

*to be the covectors such that $\mathcal{E}^i(e_j) = \delta_{ij}$ (equal to 1 if $i = j$ and zero otherwise); thus the dimension of V and V^* are the same. Given a linear map $L : V \to W$ between finite dimensional vector spaces over $\mathbb{R}$ we define the **dual map**, $L^* : W^* \to V^*$, by setting $L^*\omega(v) = \omega(L(v))$ for each $\omega \in W^*$ and $v \in V$.*

Example 46. *Given the standard basis, $\{e_i\}_{i=1}^n \subset \mathbb{R}^n$ (i.e. equal to one in the ith position and zero elsewhere), of $\mathbb{R}^n$ then the dual basis, $\{\mathcal{E}^i\}_{i=1}^n \subset \mathbb{R}^n$, is such that $\mathcal{E}^i(v) = v_i$ for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.*

We are now ready to define objects dual to tangent vectors on manifolds:

Definition 33. *Let $x \in M$ be a point in a manifold. The **cotangent space** to M at x, denoted T_x^*M, is the dual space to T_xM. Elements of T_x^*M are called **cotangent vectors**.*

In coordinates we have:

Definition 34. *Given local coordinates around a point in a manifold, $x \in M$, we had a basis of coordinate vectors, $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n \subset T_xM$, for T_xM. We then denote its dual basis of **coordinate 1-forms** by $\{dx^i|_x\}_{i=1}^n \subset T_x^*M$, and for $\omega = \sum_{i=1}^n \omega_i dx^i$ we call ω_i for $i = 1, \ldots, n$ the **components** of $\omega \in T_x^*M$.*

Remark 34. *We note that the components of covectors change in the opposite way to vectors. This motivates why we call tangent vectors **covariant**, since their components transform in the same way as the coordinate partial derivatives, while we call cotangent vectors **contravariant**, since their components transform in the oppose way to the coordinate partial derivatives (see homework 3 question 2).*

Example 47. *One important example of a covector is determined by the differential of a smooth function: i.e. given $f \in C^\infty(M)$ then the differential $d_xf : T_xM \to \mathbb{R}$, which is a linear map, can be viewed as an element of T_x^*M; in coordinates we then have that $d_xf = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \cdot dx^i$.*

Just as we did for tangent vectors, we can consider the collection of all cotangent spaces:

Definition 35. *Given a manifold, M, the **cotangent bundle** of M, denoted T^*M, is the disjoint union of the cotangent sapces of M; i.e. we have*

$$T^*M = \bigcup_{x \in M} T_x^*M.$$

We write elements of T^*M either as pairs $(x, \omega) \in M \times T_x^*M$ or more commonly just as $\omega \in T_x^*M$. The cotangent bundle comes with a natural projection map $\pi : T^*M \to M$ defined for $(x, \omega) \in T^*M$ by $\pi(x, \omega) = x$ (projection to the first factor).

Just as for the tangent bundle, the cotangent bundle has a manifold structure determined by the underlying manifold itself:

Proposition 6. *Given an n-manifold, M, its cotangent bundle, T^*M, is a $2n$-manifold and the natural projection map is a smooth submersion.*

Proof. The proof is essentially the same idea we used to show that TM was a manifold. We define a topology on T^*M by declaring subsets $V \subset TM$ open if $V = \pi^{-1}(U)$ for some open $U \subset M$. One then has that the Hausdorff and second countability of T^*M follows from that of M.

Given an atlas of charts, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$, for M we define charts, $\widetilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2n}$, on T^*M for each $\alpha \in \mathcal{A}$ by setting $\widetilde{\varphi}_\alpha(x,\omega) = (\varphi_\alpha(x), (d_{\varphi_\alpha(x)}(\varphi_\alpha^{-1}))^*(\omega))$ for each pair $(x,\omega) \in TM$; these are homeomorphisms onto their image since charts are diffeomorphisms. Here we are using the fact that φ_α is a diffeomorphism to ensure that $d_{\varphi(x)}(\varphi_\alpha^{-1}) : \mathbb{R}^n \to T_xM$ is an isomorphism, and hence its dual map is an isomorphism $(d_{\varphi(x)}(\varphi_\alpha^{-1}))^* : T_x^*M \to \mathbb{R}^n$.

We then have for $\alpha, \beta \in \mathcal{A}$ and $(y,\eta) \in T^*\mathbb{R}^n$ that

$$\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}(y,\eta) = \widetilde{\varphi}_\beta(\varphi_\alpha^{-1}(y), (d_{\varphi_\alpha^{-1}(y)}\varphi_\alpha)^*(\eta)) = (\varphi_\beta \circ \varphi_\alpha^{-1}(y), (d_y(\varphi_\beta \circ \varphi_\alpha^{-1}))^*(\eta)).$$

In the above we computed, noting $(A \circ B)^* = B^* \circ A^*$ for the dual map, that

$$(d_{\varphi_\beta\circ\varphi_\alpha^{-1}(y)}(\varphi_\beta^{-1}))^* \circ (d_{\varphi_\alpha^{-1}(y)}\varphi_\alpha)^*(\eta) = (d_{\varphi_\alpha^{-1}(y)}\varphi_\alpha \circ d_{\varphi_\beta\circ\varphi_\alpha^{-1}(y)}(\varphi_\beta^{-1}))^*(\eta) = (d_{\varphi_\beta\circ\varphi_\alpha^{-1}(y)}(\varphi_\alpha \circ \varphi_\beta^{-1}))^*(\eta),$$

using the chain rule for the second equality, and concluded by noting that

$$d_{\varphi_\beta\circ\varphi_\alpha^{-1}(y)}(\varphi_\alpha \circ \varphi_\beta^{-1}) = d_y(\varphi_\beta \circ \varphi_\alpha^{-1})$$

since $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a diffeomorphism.

Since the transition maps, $\varphi_\beta \circ \varphi_\alpha^{-1}$, are smooth for M the maps, $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$, are thus smooth and hence $\{\widetilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^{2n}\}_{\alpha\in\mathcal{A}}$ is an atlas of charts for T^*M with smooth transition maps; so T^*M is a $2n$-manifold. To see that $\pi : T^*M \to M$ is a smooth submersion it suffices to note that its coordinate representation is the smooth submersion. □

Immediately from the proof we then have that:

Corollary 6. *If an n-manifold, M, has a one chart atlas then T^*M is diffeomorphic to $M \times \mathbb{R}^n$.* □

Analogously to the notion of a vector field we have the following:

Definition 36. *Given a manifold, M, a* ***covector field*** *or more commonly* ***1-form*** *on M is a smooth map, $\omega : M \to T^*M$, such that $\omega_x = \omega(x) \in T_x^*M$ for each $x \in M$ (or equivalently $\pi \circ \omega = I_M$). We denote the collection of 1-forms on M by $\Gamma(T^*M)$ or $\mathfrak{X}^*(M)$ which are often referred to as* ***sections*** *of T^*M.*

Remark 35. *We have that $\Gamma(T^*M)$ is a vector space (in fact since we can multiply a vector field by a smooth function it is also a module over $C^\infty(M)$).*

Remark 36. *Using our tools on extending smooth functions we can extend 1-forms defined on subsets of a manifold to the entire manifold.*

Remark 37. *Note that for each $\omega \in \Gamma(T^*M)$ and $X \in \Gamma(TM)$ we have that $\omega(X) \in C^\infty(M)$, where $\omega(X)(x) = \omega_x(X_x)$ for each $x \in M$.*

We now look at some examples:

Example 48. *In $\mathbb{R}^n$ we have 1-forms $\{dx^i\}_{i=1}^n \subset \Gamma(T^*\mathbb{R}^n)$ defined such that $dx^i(x) = dx^i|_x$ for each $x \in \mathbb{R}^n$. Since the $\{dx^i|_x\}_{i=1}^n$ form a basis of the cotangent space, which is just $\mathbb{R}^n$, at each point, $x \in \mathbb{R}^n$, we then have that any 1-form, $\omega \in \Gamma(T^*\mathbb{R}^n)$, is such that $\omega = \sum_{i=1}^n \omega_i \frac{\partial}{\partial x_i}$ where $\omega_i \in C^\infty(\mathbb{R}^n)$ for each $i = 1, \ldots, n$.*

Example 49. *Similarly to the above example, given a point in a manifold, $x \in M$, and a chart, $\varphi : U \to V$, around x we have the* ***coordinate*** **1*-forms***, $\{dx^i\}_{i=1}^n \subset \Gamma(T^*U)$ *(which we can extend to $\Gamma(TM)$), defined such that $dx^i(x) = dx^i|_x$ for each $x \in U$. Since the $\{dx^i|_x\}_{i=1}^n$ form a basis of the cotangent space, T_x^*M, at each point, $x \in U$, have that each $\omega \in \Gamma(T^*M)$ locally takes the form $\omega = \sum_{i=1}^n \omega_i \frac{\partial}{\partial x_i}$ where $\omega_i \in C^\infty(U)$ are the* ***component functions*** *for each $i = 1, \ldots, n$.*

Example 50. *If $M \subset \mathbb{R}^n$ is a submanifold, then any 1-form on M is the restriction of some 1-form, $\omega \in \Gamma(T^*\mathbb{R}^n)$, such that $\omega_x \in T_x^*M$ for each $x \in M$. E.g. recall that we defined $X_{(x_1,x_2)} = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$, which restricted to a vector field on $S^1 \subset \mathbb{R}^2$ acting on $f \in C^\infty(S^1)$ by $X_{(x_1,x_2)}(f) = -x_2\frac{\partial f}{\partial x_1} + x_1\frac{\partial f}{\partial x_2}$. If we consider the 1-form $\omega_{(x_1,x_2)} = \frac{x_1 dx^2 - x_2 dx^1}{x_1^2 + x_2^2}$ defined on $\mathbb{R}^n \setminus \{0\}$ restricted to $S^1 \subset \mathbb{R}^n \setminus \{0\}$, then we have $\omega(X) = 1$!*

By using the dual map, one can pullback covectors on the target to the domain, in the opposite manner to which vectors were pushed forwards by the differential. We then have the following notion:

Definition 37. *If $F : M \to N$ is a smooth map between manifolds, then the* ***pullback*** *of F, denoted $F^* : \Gamma(T^*N) \to \Gamma(T^*M)$, is defined for $\omega \in \Gamma(T^*N)$ by $F^*\omega = (d_xF)^*(\omega)$; i.e. $F^*\omega(x) = (d_xF)^*(\omega_x)$ for each $x \in M$.*

As mentioned above, a smooth function, $f \in C^\infty(M)$, determines a covector at every point, $x \in M$ given by its differential at that point, $d_xf \in T_x^*M$. By combining each of these differentials we get a smooth 1-form, $df \in \Gamma(T^*M)$, which we also refer call the **differential** of f, defined at each point by $df(x) = d_xf$ (it is worth noticing that this coincides with the definition of the global differential since we can canonically identify the tangent spaces of $\mathbb{R}$ with each other). In coordinates we then have the expression

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i,$$

which in one dimension yields the familiar expression $df = \frac{df}{dx}dx$ from elementary calculus. We can use this coordinate expression to explicitly compute some examples:

Example 51. *If $x^j : U \to \mathbb{R}$ is the smooth jth coordinate function then we have $d(x^j) = dx^j$.*

Example 52. *If $f(x, y) = x^2\cos(x)y$ on $\mathbb{R}^2$ then*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (2x\cos(x)y - x^2\sin(x)y)dx + (x^2\cos(x))dy.$$

Example 53. *We can also use the coordinate expression to relate coordinate 1-forms with respect to different coordinates. E.g. on $\mathbb{R}^2$ we have Cartesian coordinates, (x, y), and polar coordinates, (r, θ), which are related by the formulae $x = r\cos(\theta)$, $y = r\sin(\theta)$. We can then compute that their coordinate 1-forms are related by the usual formulae*

$$dx = \cos(\theta)dr - r\sin(\theta)d\theta,\ dy = \sin(\theta)dr + r\cos(\theta)d\theta.$$

Since the coordinate components for the differential of a smooth function are simply partial derivatives of the function (on the manifold) it inherits properties from the usual derivative in Euclidean space, the proofs of which are then immediate since they hold in Euclidean space:

Proposition 7. *(Properties of the differential 1-form) Given a manifold, M, $f, g \in C^\infty(M)$, and $\lambda \in \mathbb{R}$. Then we have the following:*

1. $d(f + \lambda g) = df + \lambda dg$.

2. $d(fg) = gdf + fdg$.

3. $df = 0$ *if and only if* f *is constant on each connected component of* M. □

We can now define the line integral of a 1-form, beginning in one dimension. Consider a 1-manifold, M, a smooth curve $\gamma : [a, b] \to M$ (here we mean the restriction of some smooth curve defined on an open set containing this closed interval), and a 1-form $\omega \in \Gamma(T^*M)$. In coordinates we have that $\omega = fdx$ for some smooth function, f, of one variable; we then simply define the line integral of ω along γ to be

$$\int_\gamma \omega = \int_a^b f(t)\, dt$$

in the usual sense from one variable calculus; notice however that dt is not a 1-form here, so the notation is not too precise! More generally, if M is a manifold of any dimension, $\gamma : [a, b] \to M$ is a smooth curve, and $\omega \in \Gamma(T^*M)$, we can define a smooth one variable function $\omega_{\gamma(t)}(\gamma'(t))$ for each $t \in [a, b]$ (this is well defined since $\omega_{\gamma(t)} \in T^*_{\gamma(t)}M$ and $\gamma'(t) \in T_{\gamma(t)}M$ for each $t \in [a, b]$).

Definition 38. *Given a manifold,* M*, the* ***line integral*** *of* $\omega \in \Gamma(T^*M)$ *along a smooth curve* $\gamma : [a, b] \to M$*, denoted* $\int_\gamma \omega$*, is defined to be*

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt.$$

Remark 38. *Notice that if we consider* $-\gamma : [0, 1] \to M$ *(setting* $a = 0$ *and* $b = 1$ *for brevity and without loss of generality) defined by* $-\gamma(t) = \gamma(1 - t)$ *i.e. by reversing the direction of* γ*, then we have*

$$\int_{-\gamma} \omega = -\int_\gamma \omega.$$

Remark 39. *One can relate this definition of the line integral for* 1*-forms to the usual one for vector fields in Euclidean space by means of the dot product (see homework 3).*

Remark 40. *Some authors, e.g. in [Lee12, Chapter 11], use the pullback of* 1*-forms and define* $\int_\gamma \omega = \int_a^b \gamma^*\omega$*, noting then that* $\gamma^*\omega$ *is then a* 1*-form on* $\mathbb{R}$ *and hence is of the form* fdx *for some smooth one variable function,* f*; this agrees with our definition since*

$$(\gamma^*\omega)(t) = (d_t\gamma)^*(\omega_{\gamma(t)}) = \omega_{\gamma(t)}(\gamma'(t)).$$

Remark 41. *If instead one relaxes the smoothness condition on the curve to piecewise smooth then one can define* $\int_\gamma \omega = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \omega_{\gamma(t)}(\gamma'(t))dt$ *if* $\gamma : [a, b] \to M$ *is smooth on each interval* $[x_i, x_{i+1}]$ *for each* $i = 0, \dots, k - 1$*, where* $x_0 = a$ *and* $x_{n-1} = b$*. See [Lee12, Chapter 11] for more on this.*

Let us now compute an example:

Example 54. *Consider the* 2*-manifold given by the punctured plane,* $M = \mathbb{R}^2 \setminus \{0\}$*, the smooth closed curve* $\gamma : [0, 2\pi] \to M$ *defined by* $\gamma(t) = (\cos(t), \sin(t))$*, and* 1*-form,* $\omega \in \Gamma(T^*M)$*, defined by* $\omega_{(x,y)} = \frac{xdy - ydx}{x^2 + y^2}$*. We have that* $\gamma'(t) = (-\sin(t), \cos(t))$ *and thus along* γ *we have* $dx = -\sin(t)dt$*,* $dy = \cos(t)dt$*. By the definition above, we compute that*

$$\int_\gamma \omega = \int_0^{2\pi} \omega_{\gamma'(t)}(\gamma'(t))\, dt = \int_0^{2\pi} \frac{\cos(t)(\cos(t)dt) - \sin(t)(-\sin(t)dt)}{\cos^2(t) + \sin^2(t)} = \int_0^{2\pi} 1\, dt = 2\pi;$$

we will revisit this example later on in the course.

When we consider the 1-form given by the differential of a smooth function, we have the following generalisation of the fundamental theorem of calculus for line integrals:

Theorem 9. *(FTC for line integrals) Given a manifold, M, a smooth curve $\gamma : [a, b] \to M$, and a smooth function, $f \in C^\infty(M)$ we have*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

In particular if γ is closed then $\int_\gamma df = 0$.

Proof. We compute directly from the definitions that

$$\int_\gamma df = \int_a^b d_{\gamma(t)} f(\gamma'(t))\, dt = \int_a^b (f \circ \gamma)'(t)\, dt = f(\gamma(b)) - f(\gamma(a)),$$

where we have used the one variable FTC in the final equality. If $\gamma(a) = \gamma(b)$ then $\int_\gamma df = 0$. □

The FTC for line integrals established above allows for integrals of 1-forms along curves to be computed easily, provided we know that they arise as the differential of a function. In what will be relevant terminology later on in the course, we say that a 1-form, $\omega \in \Gamma(T^*M)$, is **exact** if it arises as the differential of a smooth function $f \in C^\infty(M)$; i.e. if $\omega = df$. If $\omega = df$ then we call f the **potential** for ω (notice then that potentials are unique up to the addition of a constant). As a further consequence we also know then that the line integral of any exact 1-form over a smooth closed curve is zero, we refer to such forms as **conservative** (one can in fact show that 1-forms are exact if and only if they are conservative, e.g. see [Lee12, Chapter 11]). Observe that the example computed above on the punctured plane is not conservative, as its integral around a closed loop was non-zero, and thus it cannot be exact; we will revisit these ideas when discussing de Rham cohomology later on in the course.

4 Constructions on manifolds

Using all of the theory introduced so far we can now start to build up constructions on manifolds that will ultimately aid us in defining orientations.

4.1 Vector bundles

Both the tangent and cotangent bundles were examples of collections of vector spaces that formed a manifold with a natural projection map to some underlying manifold. Here we generalise this construction and introduce the following notion:

Definition 39. *A **vector bundle** of rank k over an n-manifold, M, called the **base** is an $(n+k)$-manifold, E, called the **total space** along with a smooth submersion, $\pi : E \to M$, such that:*

- *For each $x \in M$ we have that $E_x = \pi^{-1}(x)$, the **fibre** over x, is a k-dimensional vector space.*
- *For each $x \in M$ there is an open set $U \subset M$ containing x and a diffeomorphism, $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, the **local trivialisation** near x, with $\pi_U \circ \psi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \to U$ is smooth projection to the first factor) and $\psi : E_y \to y \times \mathbb{R}^k$ a linear isomorphism for each $y \in U$.*

Remark 42. *If $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ is an atlas with smooth transition maps for M then, potentially shrinking the domains of the charts, that there is a* ***locally trivialising*** *atlas with smooth transition maps for E given by $\{\psi_\alpha : \pi^{-1}(U_\alpha) \to V_\alpha \times \mathbb{R}^k\}_{\alpha\in\mathcal{A}}$.*

Let us look at some examples:

Example 55. *We have seen that the tangent bundle, TM, and the cotangent bundle, T^*M, are both vector bundles of rank n.*

Example 56. *The* ***trivial*** *bundle of rank k for M is given by the product manifold $M \times \mathbb{R}^k$. In particular we saw that the tangent bundle, TS^1, of the circle, S^1, is diffeomorphic to the trivial bundle $S^1 \times \mathbb{R}$.*

Example 57. *We can consider the Möbius band, B, as introduced in the introductory examples of manifolds as a vector bundle of rank 1 over the circle, S^1. We note that the Möbius band is not diffeomorphic to $S^1 \times \mathbb{R}$ however!*

As direct generalisations of the notions of vector fields and 1-forms we introduce the following notion:

Definition 40. *Given a vector bundle, $\pi : E \to M$, a* ***section*** *of E is a smooth map, $s : M \to E$, such that $s(x) \in E_x$ for each $x \in M$ (or equivalently $\pi \circ s = I_M$). We denote the collection of sections of E by $\Gamma(E)$.*

Let us look at some examples:

Example 58. *We have seen that vector fields are sections of the tangent bundle and 1-forms are sections of the cotangent bundle.*

Example 59. *Given a smooth function $f : M \to \mathbb{R}^k$ we can view this as a section of the trivial bundle $M \times \mathbb{R}^k$ by considering $s(x) = (x, f(x))$ for each $x \in M$; we then have that $\mathrm{graph}(f) = s(M)$ is embedded in $M \times \mathbb{R}^k$.*

We now introduce a notion allowing us to determine whether two vector bundles are the same, namely a diffeomorphism which respects the vector bundle structure:

Definition 41. *We say that two vector bundles $\pi : E \to M$ and $\widetilde{\pi} : \widetilde{E} \to M$ are* ***isomorphic*** *if there exists a diffeomorphism $\psi : E \to \widetilde{E}$ with $\widetilde{\pi} \circ \psi = \pi$ and such that $\psi : E_x \to \widetilde{E}_x$ is a linear isomorphism for each $x \in M$; ψ is called a* ***bundle isomorphism****. Moreover, we say that a vector bundle of rank k, $\pi : E \to M$, is* ***trivial*** *if it is isomorphic to the trivial bundle $M \times \mathbb{R}^k$.*

We saw that the tangent bundle for the circle, TS^1, is trivial but that the Möbius bundle is not. We would like to understand a way to determine whether a vector bundle is trivial or not, for which we relate the triviality to the existence of nowhere vanishing sections that form a basis:

Lemma 4. *A vector bundle of rank k, $\pi : E \to M$, is trivial if and only if there exist sections $s_1, \ldots, s_k \in \Gamma(E)$ such that $\{s_i(x)\}_{i=1}^k$ is a basis of E_x for each $x \in M$.*

Proof. If we have an isomorphism $\psi : E \to M \times \mathbb{R}^k$ then we can define sections $s_i(x) = \psi(x, e_i)$ for each $x \in M$ and $i = 1, \ldots, k$ where $\{e_i\}_{i=1}^k$ is a basis of $\mathbb{R}^k$. On the other hand if we have sections $s_1, \ldots, s_k \in \Gamma(E)$ such that $\{s_i(x)\}_{i=1}^k$ is a basis of E_x for each $x \in M$ then we can define a bundle isomorphism $\Psi : M \times \mathbb{R}^k \to E$ by setting $\Psi(x, (\lambda_1, \ldots, \lambda_k)) = \sum_{i=1}^k \lambda_i s_i(x)$. □

The 'hairy ball theorem' for the even-dimensional spheres shows that there exist no continuous non-zero maps into their tangent bundle; see [Mil78] for a concise proof. The lemma above thus tells us that the tangent bundles of even dimensional spheres cannot be trivial! We make a specific definition in this case, capturing the idea that we can compare whether tangent vectors at all points of the manifold are parallel:

Definition 42. *We say that a manifold, M, is* ***parallelisable*** *if its tangent bundle, TM, is trivial; i.e. if TM is isomorphic to $M \times \mathbb{R}^n$.*

The lemma allows us to check this for several examples:

Example 60. *As noted above, by application of the lemma we have that S^{2n} is not parallelisable for each $n \geq 1$. We saw that S^1 is parallelisable, and in fact one can show that the only parallelisable spheres are S^0, S^1, S^3, and S^7; this is related to the existence of the complex numbers, quaternions, and octonians.*

Example 61. *We will show that any lie group (recall this is a manifold with diffeomorphisms for group actions), G, is parallelisable; in particular no even dimensional sphere could be a Lie group (on homework 3 we will see that S^3 is). Given each $g \in G$ we have that the left multiplication map, $L_g : G \to G$, is a smooth diffeomorphism and thus $d_e L_g : T_e G \to T_g G$ is a linear isomorphism (where $e \in G$ is the identity). We can thus take some basis, $\{v_i\}_{i=1}^n$, of $T_e G$ and form a basis, $\{d_e L_g(v_i)\}_{i=1}^n$, for $T_g G$; this provides smooth sections satisfying the conditions of the above lemma, hence G is parallelisable.*

Shortly, we will require a way to construct vector bundles given some collection of vector spaces associated to points of a manifold, for which the following result will be indispensable:

Lemma 5. *(Vector bundle chart lemma) Let M be a smooth manifold and for each $x \in M$ a vector space E_x of dimension k. Let $E = \bigsqcup_{x \in M} E_x$ and define $\pi : E \to M$ be the projection (i.e. $\pi(E_x) = x$ for each $x \in M$). If the following hold:*

- *There is an open cover, $\{U_\alpha\}_{\alpha \in \mathcal{A}}$, of M such that for each $\alpha \in \mathcal{A}$ there is a bijection $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, with $\psi_\alpha : E_x \to x \times \mathbb{R}^k$ a linear isomorphism for each $x \in U_\alpha$.*

- *For each $\alpha, \beta \in \mathcal{A}$ with $U_\alpha \cap U_\beta \neq \emptyset$ there is a smooth map $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$, such that for each $(x, v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ we have*

  $$\psi_\beta \circ \psi_\alpha^{-1}(x, v) = (x, \tau_{\beta\alpha}(x)v);$$

  notice that $\tau_{\beta\alpha}(x)$ exists for each $x \in M$ by the first bullet point, this condition says that the assignment is smooth as we vary x.

Then E has a unique topology and atlas with smooth transition maps making it a rank k vector bundle over M with $\pi : E \to M$ a smooth submersion and ψ_α the local trivialisations.

Proof. The proof can be found in [Lee12, Chapter 10], it is very similar to the proofs that the tangent and cotangent bundles were manifolds. □

4.2 Some multilinear algebra

In order to proceed further with developing constructions on manifolds we will need to introduce some notions from (multi)linear algebra. We restrict to the cases of interest for us, though much more general constructions are possible (e.g. see [Lee12, Chapter 12]):

Definition 43. *Given a finite dimensional real vector space, V, a map, $F : V \times \cdots \times V \to \mathbb{R}$ is said to be k-**multilinear** (sometimes called a **covariant** k-**tensor**) on V if is linear in each input; i.e. for $v_1, \ldots, v_k, \widetilde{v} \in V$ and $\lambda \in \mathbb{R}$ we have for each $i = 1, \ldots, k$ that*

$$F(v_1, \ldots, v_{i-1}, v_i + \lambda\widetilde{v}, v_{i+1}, \ldots, v_k) = F(v_1, \ldots, v_k) + \lambda F(v_1, \ldots, v_{i-1}, \widetilde{v}, v_{i+1}, \ldots, v_k).$$

*If $k = 1$ this gives a **linear** map, if $k = 2$ this gives a **bilinear** map. We denote the collection of all k-multilinear maps as $\otimes^k V^*$ (sometimes this is written as $T^k(V^*)$).*

Remark 43. *We have that $\otimes^k V^*$ is a vector space under addition and scalar multiplication. Note that $\otimes^1 V^* = V^*$ and by convention we set $\otimes^0 V^* = \mathbb{R}$.*

We already have two familiar examples and introduce another:

Example 62. *The dot product on $\mathbb{R}^n$, $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $\cdot(x, y) = x \cdot y$ is a bilinear map on $\mathbb{R}^n$; used to define angles and lengths of vectors.*

Example 63. *The determinant, as a function on $\mathbb{R}^{n^2}$, is a multilinear map, $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ by defining $\det(v_1, \ldots, v_n)$ to be the determinant of the matrix with rows given by the $v_1, \ldots, v_n$; used to detect linear independence and compute volumes of parallelepipeds spanned by the vectors.*

Example 64. *Given two covectors, $\omega, \eta \in V^*$, for a finite dimensional real vector space, V, we can define the **tensor product**, $\omega \otimes \eta : V \times V \to \mathbb{R}$, to be the bilinear map defined on $(v, w) \in V \times V$ by setting $\omega \otimes \eta(v, w) = \omega(v) \cdot \eta(w)$. For example, given the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$ for $\mathbb{R}^2$ we have that if $v, w \in \mathbb{R}^2$ then $e_1 \otimes e_2(v, w) = v_1 \cdot w_2$.*

Since the multilinear maps form a vector space we would like to know what a basis for it is. For this we will first generalise the third example above to higher dimensions:

Definition 44. *Given a finite dimensional real vector space, V, with $F \in \otimes^k V^*$ and $G \in \otimes^l V^*$ we define the **tensor product** of F and G, denoted $F \otimes G \in \otimes^{k+l} V^*$, for $v_1, \ldots, v_k, w_1, \ldots, w_l \in V$ by setting*

$$F \otimes G(v_1, \ldots, v_k, w_1, \ldots, w_l) = F(v_1, \ldots, v_k) \cdot G(w_1, \ldots, w_l).$$

Remark 44. *One can check that the tensor product operation is bilinear, so $(F + G) \otimes H = F \otimes H + G \otimes H$, and associative, so $(F \otimes G) \otimes H = F \otimes (G \otimes H)$, for arbitrary multilinear maps, F, G, and H.*

In particular, if $\omega_1, \ldots, \omega_k \in V^*$ then we have that $\omega_1 \otimes \cdots \otimes \omega_k \in \otimes^k V^*$ defined for $v_1, \ldots, v_k \in V$ by setting

$$\omega_1 \otimes \cdots \otimes \omega_k(v_1, \ldots, v_k) = \omega_1(v_1) \cdot \cdots \cdot \omega_k(v_k).$$

With this definition we can now establish a basis:

Proposition 8. *Given a finite dimensional real vector space, V, and a basis, $\{\mathcal{E}^i\}_{i=1}^n$, for V^* a basis for $\otimes^k V^*$ is given by*

$$\{\mathcal{E}^{i_1} \otimes \cdots \otimes \mathcal{E}^{i_k} \mid i_j = 1, \ldots, n \text{ and } j = 1, \ldots, k\},$$

and thus the dimension of $\otimes^k V^*$ is n^k.

Proof. See [Lee12, Proposition 12.4]; this is mainly an exercise in keeping track of notation. □

Remark 45. *It is important to note that not element of $\otimes^k V^*$ can be written as a **simple** tensor, i.e. as $\omega_1 \otimes \cdots \otimes \omega_k$ for some $\omega_1, \ldots, \omega_k \in V^*$; for instance if V is of dimension 2 with a basis $\{\mathcal{E}^1, \mathcal{E}^\in\}$ for V^* then $\otimes^2 V^* = \mathrm{Span}\{\mathcal{E}^1, \mathcal{E}^2\}$ and hence $\mathcal{E}^1 \otimes \mathcal{E}^1 + \mathcal{E}^2 \otimes \mathcal{E}^2$ is not simple.*

We now seek to generalise the properties of the dot product and determinant, for which we introduce the following notions:

Definition 45. *Given a finite dimensional vector space, V, a tensor $\alpha \in \otimes^k V^*$ is said to be:*

- ***symmetric*** *if whenever $v_1, \ldots, v_k \in V$ we have*

$$\alpha(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{j-1}, v_j, v_{j+1}, v_k) = \alpha(v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, v_k)$$

  for each $i, j = 1, \ldots, k$; i.e. swapping entries doesn't change the output. We denote the collection of all symmetric k-tensors by $S^k V^*$.

- ***alternating*** *if whenever $v_1, \ldots, v_k \in V$ we have*

$$\alpha(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{j-1}, v_j, v_{j+1}, v_k) = -\alpha(v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, v_k)$$

  for each $i, j = 1, \ldots, k$; i.e. swapping entries gives negative the output. We denote the collection of all alternating k-tensors by $\Lambda^k V^*$.

Remark 46. *We have that both $S^k V^*$ and $\Lambda^k V^*$ are vector subspaces of $\otimes^k V^*$. Note that $S^1 V^*$ is just the zero map, $\Lambda^1 V^* = V^*$, and by convention we set $\Lambda^0 V^* = \mathbb{R}$. We also observe that if k is larger than the dimension of V, then $\Lambda^k V^*$ is also just the zero map.*

We have that the dot product is a symmetric 2-tensor on $\mathbb{R}^n$ and the determinant is an alternating n-tensor on $\mathbb{R}^n$. We now introduce maps that allow us to turn arbitrary tensors into symmetric and alternating tensors:

Definition 46. *For $\sigma \in \mathrm{Sym}(k)$ (bijections from $\{1, \ldots, k\}$ to itself) and $\alpha \in \otimes^k V^*$ we define ${}^\sigma\alpha \in \otimes^k V^*$ on $v_1, \ldots, v_k \in V$ by setting ${}^\sigma\alpha(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$. We then have the **symmetrising** map*

$$\mathrm{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in \mathrm{Sym}(k)} {}^\sigma\alpha \in S^k V^*,$$

*and **alternating** map*

$$\mathrm{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in \mathrm{Sym}(k)} \mathrm{sign}(\sigma) \cdot {}^\sigma\alpha \in \Lambda^k V^*,$$

where $\mathrm{sign}(\sigma)$ is ± 1 if σ is composed of an even or odd number of transpositions (swaps) respectively.

Remark 47. *If $\alpha \in S^k V^*$ then $\mathrm{Sym}(\alpha) = \alpha$ and if $\beta \in \Lambda^k V^*$ then $\mathrm{Alt}(\beta) = \beta$.*

Example 65. *If $\alpha \in \otimes^2 V^*$ then for $v, w \in V$ we have that*

$$\mathrm{Sym}(\alpha)(v, w) = \frac{1}{2}(\alpha(v, w) + \alpha(w, v)),\ \mathrm{Alt}(\alpha)(v, w) = \frac{1}{2}(\alpha(v, w) - \alpha(w, v)).$$

To conclude this segue into multilinear algebra we introduce a way to combine alternating tensors of different rank in a manner which generalises the determinant; ultimately allowing us to compute areas of higher dimensional parallelepipeds:

Definition 47. *Given a finite dimensional real vector space, V, $\omega \in \Lambda^k V^*$, and $\eta \in \Lambda^l V^*$, we define the **wedge product** of ω and η, denoted by $\omega \wedge \eta \in \Lambda^{k+l} V^*$, by setting*

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\omega \otimes \eta).$$

Let us briefly try to justify the mysterious factor in front of the alternating map in the definition of the wedge product. If $\alpha, \beta \in \Lambda^1 V^* = V^*$ then $\alpha \wedge \beta \in \Lambda^2 V^*$ is given by

$$\alpha \wedge \beta = 2\mathrm{Alt}(\alpha \otimes \beta) = \alpha \otimes \beta - \beta \otimes \alpha.$$

In the special case that $V = \mathbb{R}^2$ with $\alpha = \mathcal{E}^1$ and $\beta = \mathcal{E}^2$ the dual basis for the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$ of $\mathbb{R}^2$ we thus have for each $v = (a, b) \in \mathbb{R}^2$ and $w = (c, d) \in \mathbb{R}^2$ that

$$\mathcal{E}^1 \wedge \mathcal{E}^2(v, w) = ad - bc = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix};$$

and thus the wedge product of the standard dual basis of $\mathbb{R}^2$ acts like the determinant! This is more general and is recorded as one of the following general properties of the wedge product:

Proposition 9. *(Properties of the wedge product) Given a finite dimensional real vector space, V, tensors, $\omega, \widetilde{\omega}, \eta, \widetilde{\eta}, \zeta$, and $\lambda \in \mathbb{R}$. Then we have the following:*

1. $(\omega + \lambda\widetilde{\omega}) \wedge \eta = \omega \wedge \eta + \lambda(\widetilde{\omega} \wedge \eta)$ *and* $\omega \wedge (\eta + \lambda\widetilde{\eta}) = \omega \wedge \eta + \lambda(\omega \wedge \widetilde{\eta})$*; i.e. the wedge product is bilinear.*

2. $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$*; i.e. the wedge product is associative.*

3. *If $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$ then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$; i.e. the wedge product is anticommutative.*

4. *If $\omega_1, \ldots, \omega_k \in V^*$ and $v_1, \ldots, v_k \in V$ then*

$$\omega_1 \wedge \cdots \wedge \omega_k(v_1, \ldots, v_k) = \det((\omega^j(v_i))_{ij});$$

   where $(\omega^j(v_i))_{ij}$ denotes the matrix with ijth entry $\omega^j(v_i)$.

5. *If $\{\mathcal{E}^i\}_{i=1}^n$ is a basis for V^* then a basis for $\Lambda^k V^*$ is given by*

$$\{\mathcal{E}^{i_1} \wedge \cdots \wedge \mathcal{E}^{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\},$$

   and thus the dimension of $\Lambda^k V^*$ is $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$.

Proof. See [Lee12, Proposition 14.8 and 14.11]; requires a careful examination of the definitions. ☐

Remark 48. *In particular if k is the dimension of V, say n as in the proposition, then $\Lambda^n V^*$ is one-dimensional; e.g. if $n = 2$ and $\{\mathcal{E}^1, \mathcal{E}^2\}$ is a basis for V^*, then $\Lambda^2 V^*$ is spanned by $\mathcal{E}^1 \wedge \mathcal{E}^2$! We will utilise this fact several times later on in the course.*

4.3 Tensor bundles

We now combine the results of the previous two subsections, specifically to the vector spaces given by the tangent spaces to a manifold, to form the following vector bundles via the vector bundle chart lemma:

Definition 48. *Given a manifold, M, we define the:*

- ***covariant** k**-tensor bundle** $\otimes^k T^*M = \bigsqcup_{x\in M} \otimes^k T_x^*M$.*
- ***symmetric** k**-tensor bundle** $S^k T^*M = \bigsqcup_{x\in M} S^k T_x^*M$.*
- ***alternating** k**-tensor bundle** $\Lambda^k T^*M = \bigsqcup_{x\in M} \Lambda^k T_x^*M$.*

Each comes equipped with a natural projection map to M.

Remark 49. *We have $\Lambda^1 T^*M = T^*M$, $\Lambda^0 T^*M = M \times \mathbb{R}$ (so sections are just smooth functions on M), and if M is of dimension n then $\Lambda^k T^*M$ is a rank $\begin{pmatrix} n \\ k \end{pmatrix}$ bundle. In particular $\Lambda^n T^*M$ is rank 1.*

We will study sections of these bundles, first to introduce geometry to manifolds (namely notions of length and angles), second to introduce appropriate higher dimensional objects for integration, and finally to introduce the notion of orientations.

4.4 Riemannian metrics

Here we briefly introduce a notion of geometry on our manifolds, further study of which could consist of a first course in Riemannian geometry. In order to define lengths and angles in Euclidean space one relies on the dot product, which we saw was a symmetric 2-tensor on $\mathbb{R}^n$. The dot product also has the feature that it assigns a vector to have length zero if and only if it is the zero vector, i.e. it is positive definite. We now introduce a notion of smoothly varying dot or inner product on a manifold in the same way:

Definition 49. *A **Riemannian metric** on a manifold, M, is a section, $g \in \Gamma(S^2 T^*M)$, that is positive definite on each fibre; i.e. for each $x \in M$ we have that $g_x(v, v) = 0$ if and only if $v = 0 \in T_xM$. A **Riemannian manifold** is a pair (M, g) where M is a manifold and g is a Riemannian metric on M.*

Remark 50. *The terminology 'metric' is no in the sense of metric spaces, though it is related as we will see shortly.*

Remark 51. *By relaxing the positive definite assumption to simply non-degenerate, one can define the notion of a **pseudo-Riemannian metric**. A special case of such metrics on 4-manifolds is given by the **Lorentzian metrics**, which are central objects of study in general relativity.*

Locally in coordinates we can express a Riemannian metric, g, in the form

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j$$

where $(g_{ij})_{ij}$ is a symmetric positive definite matrix. We now look at some examples:

Example 66. *On $\mathbb{R}^n$ the **standard metric** is given by*

$$\overline{g} = \sum_{i,j=1}^{n} \delta_{ij}\, dx^i \otimes dx^j = \sum_{i=1}^{n} dx^i \otimes dx^i = (dx^1)^2 + \cdots + (dx^n)^2.$$

Thus given $x \in \mathbb{R}^n$ and $v, w \in T_x\mathbb{R}^n = \mathbb{R}^n$ we have

$$\overline{g}_x(v, w) = \sum_{i=1}^{n} dx^i \otimes dx^i(v, w) = \sum_{i=1}^{n} v_i \cdot w_i = v \cdot w,$$

recovering the dot product!

Example 67. *If $M \subset N$ is a submanifold, then we can restrict a Riemannian metric on N to M, which we call the **induced metric**. For example, restricting the standard metric to the sphere, $S^n \subset \mathbb{R}^{n+1}$, gives the so called **round metric**.*

Before seeing how a Riemannian metric allows us to define notions of lengths and angles between tangent vectors, we would first like to know whether they always exist:

Theorem 10. *Every manifold admits a Riemannian metric.*

Proof. The idea here is to pull back the dot product from Euclidean space in each chart, then patch this up using a partition of unity. Precisely, given an atlas, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$, for M with smooth transition maps, in each U_α we can define a Riemannian metric, g^α, by setting

$$g^\alpha_x(v, w) = d_x\varphi_\alpha(v) \cdot d_x\varphi_\alpha(w),$$

for each $x \in U_\alpha$ and $v, w \in T_xM$; which is symmetric and positive definite since the dot product is. We now let $\{\rho_\alpha\}_{\alpha \in \mathcal{A}}$ be a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and define a Riemannian metric, g, on M by setting

$$g_x(v, w) = \sum_{\alpha \in \mathcal{A}} \rho_\alpha(x) g^\alpha_x(v, w)$$

for each $x \in M$ and $v, w \in T_xM$. By the construction of the partition of unity this is well defined, with only finitely many non-zero terms in the sum at each point, and is both symmetric and positive definite since the g^α are. $\square$

While the round metric on the sphere recovers the usual symmetric sphere, we can choose other Riemannian metrics on the sphere that give it weirder shapes. Morally speaking, the choice of Riemannian metric determines the 'shape' of the manifold; thus diffeomorphic manifolds can be given wildly different shapes. Understanding which Riemannian metrics can be put on a given manifold is a central focus of the subject of Riemannian geometry. We conclude this subsection by introducing some geometric notions that can be defined from a given Riemannian metric:

Definition 50. *Given a Riemannian manifold, (M, g), we define the:*

- ***norm** of $v \in T_xM$ by $|v|_g = g_x(v, v)$.*
- ***angle**, θ, between $v, w \in T_xM$ by $\cos(\theta) = \frac{g_x(v,w)}{|v|_g |w|_g}$. In particular, if $\theta = \frac{\pi}{2}$ or equivalently that $g_x(v, w) = 0$, we say that v and w are **orthogonal**.*
- ***length**, $L_g(\gamma)$, of a smooth curve $\gamma[a, b] \to M$ by $L_g(\gamma) = \int_a^b |\gamma'(t)|_g \, dt$.*

- ***distance***, $d_g(x, y)$, *between* $x, y \in M$ *by*

$$d_g(x, y) = \inf\{L(\gamma) \mid \gamma : [a, b] \to M \textit{ is a smooth curve with } \gamma(a) = x, \gamma(b) = y\},$$

where we make the convention that $d_g(x, y) = \infty$ *if there are no such curves.*

The definition of orthogonal vectors allows for the notion of a **normal bundle** to be defined for submanifolds of Riemannian manifolds (c.f. homework 3). Also, the notion of the length of a smooth curve agrees with the familiar notion of length in Euclidean space for by considering the induced metric on submanifolds of Euclidean space. Finally, the notion of distance turns M into a metric space; with curves realising the infimum in the definition being an instance of a **geodesic** or locally length minimising path between points. One can also ask whether a diffeomorphism between manifolds preserves the angles/distances on a Riemannian manifold; this leads to the notion of **isometry**. It is known, see [Nas56], that every Riemannian manifold can be isometrically embedded into a high dimensional Euclidean space; namely, there exists an embedding of any Riemannian manifold into some Euclidean space that preserves lengths of curves (and hence angles between tangent vectors). Further study of Riemannian manifolds could occupy the entirety of one or more courses of study, and would rely heavily on the various manifold notions introduced in this course.

4.5 Differential forms

We saw that sections of the cotangent bundle, which we called 1-forms, provided a means to define the line integral over a smooth curve or equivalently a 1-manifold. By studying sections of the alternating tensor bundles we now define objects, k-forms, that we will be able to integrate over k-manifolds:

Definition 51. *A **k-form** on a manifold,* M*, is a section,* $\omega \in \Gamma(\Lambda^k T^*M)$*. We denote the collection of* k*-forms on* M *by* $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$*. Given* $\omega \in \Omega^k(M)$ *and* $\eta \in \Omega^l(M)$ *we define their **wedge product**,* $\omega \wedge \eta \in \Omega^{k+l}(M)$*, by setting* $(\omega \wedge \eta)_x = \omega_x \wedge \eta_x$ *for each* $x \in M$*.*

Remark 52. *We have that* $\Omega^k(M)$ *is a real vector space under addition and scalar multiplication. Notice that* $\Omega^k(M)$ *is just the zero map for* k *bigger than the dimension of* M*. Since* $\Lambda^0 T^*M = M \times \mathbb{R}$ *we have that* $\Omega^0(M) = C^\infty(M)$*, i.e. that* 0*-forms are smooth functions with* $f \wedge \omega = f\omega \in \Omega^k(M)$ *for* $f \in C^\infty(M)$ *and* $\omega \in \Omega^k(M)$*. Since* $\Lambda^1 T^*M = T^*M$ *we also have that* $\Omega^1(M)$ *is the set of* 1*-forms as defined previously!*

In order to work with k-forms locally we first introduce some convenient notation:

Definition 52. *Let* $\mathcal{I}(n, k)$ *be the set of* k*-tuples,* $I = (i_1, \ldots, i_k)$*, such that* $1 \le i_1 < \cdots < i_k \le n$*. For* $I \in \mathcal{I}(n, k)$ *we will then denote* $dx^I = dx^{i_1} \wedge \ldots dx^{i_k}$ *where* $\{dx^i\}_{i=1}^n$ *are coordinate* 1*-forms.*

Remark 53. *In [Lee12, Chapter 14] different notation is used where they use an apostrophe to denote sums over increasing* k*-tuples.*

Using this notation, locally we can write each k-form, $\omega \in \Omega^k(M)$, on a n-manifold, M, as:

$$\omega = \sum_{I \in \mathcal{I}(n,k)} \omega_I dx^I = \sum_{I \in \mathcal{I}(n,k)} \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $\omega_I = \omega\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}}\right)$ if $I = (i_1, \ldots, i_k)$ is a smooth function. Note that this expression is guaranteed since the basis of $\Lambda^k T^*_x M$ is provided by increasing wedge products of the coordinate 1-forms at each $x \in M$. Let's see some examples:

Example 68. *On $\mathbb{R}^3$ we have that:*

- *0-forms are $f \in C^\infty(\mathbb{R}^3)$.*
- *1-forms are of the form $fdx + gdy + hdz$ for $f, g, h \in C^\infty(\mathbb{R}^3)$.*
- *2-forms are of the form $fdx \wedge dy + gdx \wedge dz + hdy \wedge dz$ for $f, g, h \in C^\infty(\mathbb{R}^3)$.*
- *3-forms are of the form $fdx \wedge dy \wedge dz$ for $f \in C^\infty(\mathbb{R}^3)$*
- *k-forms for $k \geq 4$ are all zero!*

Just as we defined pullbacks for 1-forms we can extend this to k-forms:

Definition 53. *If $F : M \to N$ is a smooth map between manifolds, then the* ***pullback*** *of F, denoted $F^* : \Omega^k(N) \to \Omega^k(M)$, acts on $\omega \in \Omega^k(N)$ by letting $F^*\omega \in \Omega^k(M)$ be defined by*

$$(F^*w)_x(v_1, \ldots, v_k) = \omega_{F(x)}(d_xF(v_1), \ldots, d_xF(v_k))$$

for each $x \in M$ and $v_1, \ldots, v_k \in T_xM$. We then have that for $\omega \in \Omega^k(N)$ and $\eta \in \Omega^l(N)$ that $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$. Moreover, if $F : M \to N$ and $G : N \to P$ then $(G \circ F)^* = F^* \circ G^*$.

Remark 54. *Notice that if $f \in \Omega^0(N) = C^\infty(N)$ is a 0-form then $F^*f = f \circ F \in C^\infty(M)$*

Remark 55. *In the special case that $M = N = \mathbb{R}^n$ as $\Lambda^n T^*M$ is 1-dimensional, every n-form or* ***top dimensional form****, ω, is of the form $\omega = dx^1 \wedge \cdots \wedge dx^n$ and thus for any smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ we have by the definition above and the properties of the wedge product that $F^*\omega = \det(d_xF)\omega$.*

We now compute an example:

Example 69. *Consider the 2-form $\omega = dx \wedge dy$ on $\mathbb{R}^2$ and the change to polar coordinates given by $x = r\cos(\theta)$, $y = r\sin(\theta)$. Using the above remark for top dimensional forms we compute that*

$$\omega = dx \wedge dy = d(r\cos(\theta)) \wedge d(r\sin(\theta)) = rdr \wedge d\theta,$$

showing how ω looks with respect to another choice of coordinates.

4.6 Orientations

The notion of an orientation or direction is familiar in Euclidean space, where we use the standard basis to impose a global set of coordinate axes with a direction (from negative to positive infinity). We want to generalise this idea to manifolds, where for example we want the sphere to be oriented (as it has a consistent inside/outside and or choice of continuous normal) but manifolds like the Möbius band or Klein bottle to not be oriented (as they do not have a consistent inside/outside and or choice of continuous normal). To capture this notion we will need some more linear algebra in order to compare bases of a given vector space:

Definition 54. *Given a real vector space, V, of dimension n we say that two bases $E = \{E_1, \ldots, E_n\}$ and $F = \{F_1, \ldots, F_n\}$ for V are* ***consistent*** *if the linear map $L : V \to V$ taking the basis E to the basis F has positive determinant. This forms an equivalence relation on the set of bases of V and we say that a choice of an equivalence class of bases is an* ***orientation*** *for V.*

Remark 56. *Precisely, the matrix representation, $L = (L_{ij})_{ij}$, of $L : V \to V$ is such that $F_i = \sum_{j=1}^n L_{ij} E_i$ and $\det(L) > 0$. In the case that $n = 0$ an orientation for V is simply a choice of ± 1. Since every vector space of dimension n is isomorphic to $\mathbb{R}^n$, an orientation always exists by the following example.*

We have familiar examples in Euclidean space:

Example 70. *On $\mathbb{R}^n$ we have the* ***standard orientation*** *given by the equivalence class of bases consistent with the standard basis $\{e_1, \dots, e_n\}$. For $n = 0$ this is a choice of ± 1, for $n = 1$ this simply gives the standard positive direction for the x-axis, for $n = 2$ this gives the anti-clockwise Cartesian axes, and for $n = 3$ this gives the axes determined by the 'right-hand rule'.*

We now provide a connection between orientations of vector spaces and top dimensional alternating tensors:

Proposition 10. *Given a real vector space, V, of dimension n, every non-zero alternating n-tensor, $\omega \in \Lambda^n V^* \setminus \{0\}$, determines an orientation, $\mathcal{O}_\omega$, of V as follows:*

- *if $n = 0$ then $\mathcal{O}_\omega = +1$ if $\omega > 0$ and $\mathcal{O}_\omega = -1$ if $\omega < 0$.*
- *if $n \geq 1$ then $\mathcal{O}_\omega$ is the set of bases $\{E_1, \dots, E_n\}$ for V such that $\omega(E_1, \dots, E_n) > 0$.*

We call such an ω an ***oriented n-covector****. Any two oriented n-covectors determine the same orientation if and only if they are positive multiples of eachother.*

Proof. The case $n = 0$ follows by noting that then $\Lambda^n V^* = \mathbb{R}$ and thus $\omega \in \mathbb{R} \setminus \{0\}$. For the case $n \geq 1$ we simply need to show that the definition of $\mathcal{O}_\omega$ provides an equivalence class. We note that if $L : V \to V$ is linear and $v_1, \dots, v_n \in V$ then

$$\omega(Lv_1, \dots, Lv_n) = \det(L)\omega(v_1, \dots, v_n);$$

this follows by the properties of the wedge product and multilinearity of ω (which one can check on basis elements). Thus if two bases $E = \{E_1, \dots, E_n\}$ and $F = \{F_1, \dots, F_n\}$ for V are related by a linear map $L : V \to V$ then

$$\omega(F_1, \dots, F_n) = \omega(LE_1, \dots, LE_n) = \det(L)\omega(E_1, \dots, E_n);$$

and thus E and F are consistent if and only if ω gives them the same sign; i.e. if and only if they lie in the same equivalence class. The final statement thus follows as positive multiples do not affect the sign. $\square$

Thus we see that choosing an orientation for a vector space is equivalent to choosing a non-zero top dimensional alternating tensor. We now apply this to tangent spaces in order to define orientations on manifolds:

Definition 55. *Given an n-manifold, M, a* ***pointwise orientation*** *is a choice of orientation for each tangent space. For a given pointwise orientation, we say that local vector fields, $\{E_i\}_{i=1}^n$, forming a basis of the tangent spaces at each point in their domain are an* ***oriented local frame*** *if the basis, $\{E_i|_x\}_{i=1}^n$, lies in the orientation of $T_x M$ for each x in their domain (i.e. if they lie in the equivalence class given by the pointwise orientation). A pointwise orientation is thus said to be* ***continuous*** *if each point of M lies in the domain of an oriented local frame. We say that M is* ***orientable*** *if there exists a continuous pointwise orientation, simply* ***orientation*** *for M, and that M is* ***non-orientable*** *if such an orientation does not exist.*

Remark 57. *While local vector fields that form a basis at each point always exist, since the tangent bundle is locally trivial (as it is a vector bundle), the choice of pointwise orientation could vary wildly from point to point. The notion of an oriented local frame ensures that the assignment of pointwise orientations are locally consistent with one another. Note that we can thus always find an open set in which is orientable, but being orientable ensures that these oriented local frames patch up continuously on the entire manifold, so that the consistency of the pointwise orientation is global.*

Let's check some easy examples:

Example 71. *Every* 0*-manifold is orientable, we just assign a choice of* ± 1 *to each point which is vacuously continuous.*

Example 72. *Euclidean space,* $\mathbb{R}^n$*, is orientable since we can take the standard orientation (equivalence class of bases consistent with the standard basis) which is consistent with the basis given by the coordinate vector fields* $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$*.*

Example 73. *With the above example in mind, any manifold with an atlas with one chart is orientable.*

In principle, whether a given manifold is oriented or not is hard to check. We will now discuss two more equivalent definitions or orientability which are in practice often easier to verify:

Proposition 11. *Given an* n*-manifold,* M*, a nowhere vanishing* n*-form,* $\omega \in \Omega^n(M)$*, determines an orientation for* M*. Conversely, an orientation for* M *determines a nowhere vanishing* n*-form. We call such a nowhere vanishing* ω *an* ***orientation form****.*

Proof. By the above proposition, ω defines a pointwise orientation for M (since ω_x is an orientation n-covector on T_xM for each $x \in M$). This is continuous since locally ω takes the form $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some $f \in C^\infty(U) \setminus \{0\}$. As each point of M lies in the domain of a chart, ω thus determines an orientation for M (precisely here we use the fact that ω takes either a positive or negative sign, but not both, on each connected component of M here).

For the converse we refer to [Lee12, Proposition 15.5]; the core idea is to take a positive orientation n-covector on each tangent space and patch this up to give an orientation form on the manifold using partitions of unity. ☐

We thus see that being oriented is equivalent to the existence of an orientation form. This allows us to check several examples:

Example 74. *We can use the proposition to show that the* n*-spheres are orientable by defining an* n*-form on* $\mathbb{R}^{n+1}$ *that restricts to* S^n*. For* $n = 1$ *we can let* $\omega = x_1 dx^2 - x_2 dx^1$ *and for* $n = 2$ *we can define* $\omega = x_1 dx^2 \wedge dx^3 - x_2 dx^1 \wedge dx^3 + x_3 dx^2 \wedge dx^3$*. A similar construction works for general* $n \geq 3$*.*

Example 75. *As another way to see that Euclidean space,* $\mathbb{R}^n$*, is orientable we can simply consider* $\omega = dx^1 \wedge \cdots \wedge dx^n$*.*

Example 76. *Any parallelisable manifold, and hence any Lie group, is orientable since if* TM *is trivial, then* T^*M *is also. We can thus consider linearly independent* 1*-forms,* $\omega^1, \ldots, \omega^n$ *and form an orientation form by considering their wedge product* $\omega^1 \wedge \cdots \wedge \omega^n$ *(notice that this is non-zero since the* 1*-forms are linearly independent).*

We can also characterise orientability in terms of an atlas:

Proposition 12. *Given a smooth manifold, M, if there is an atlas whose transition map's Jacobians have positive determinant at each point, then M is orientable. Conversely, an orientation for M ensures the existence of such an atlas. We call such an atlas an* ***orientation atlas****.*

Proof. The coordinate vector fields provide a local oriented frame and hence determine a pointwise orientation for M at each point of the domain of a chart. As seen in homework 2, whenever the domains of two charts overlap, the linear map relating coordinate vector fields in each set of coordinates is provided by the Jacobian of the transition map for the charts. If this is assumed to have positive determinant, the pointwise orientation provided by each chart agrees. This then determines a continuous pointwise orientation for M as they agree on the overlaps, hence M is continuous.

Conversely, if M is oriented then we can consider local orientated frames provided by the coordinate vector fields. By potentially composing the chart with a reflection in a coordinate axes we can always ensure that this g frame is oriented, this then ensures that on the overlaps between the charts their transition maps have positive Jacobian determinant, thus providing an orientation atlas. □

We can use this characterisation to show the following:

Lemma 6. *Any manifold covered by an atlas with smooth transition maps consisting of two charts is orientable.*

Proof. The determinant of the transition map's Jacobian must have a fixed sign on the overlap of the domains of the charts (since the transition map is a diffeomorphism). By potentially composing one of the charts with a reflection in a coordinate axes we can always ensure that this determinant is positive, and hence determine an orientation atlas for the manifold. □

This immediately gives some examples of orientable manifolds and some ideas about non-orientable manifolds:

Example 77. *The n-spheres, S^n, have atlases with smooth transition maps provided by stereographic projection from the north and south pole, and hence are orientable.*

Example 78. *One can find an atlas with smooth transition maps with only two charts for the n-torus, T^n, (I encourage you to think about what this would look like) and hence is orientable.*

Remark 58. *By the above discussions, any non-orientable manifold must have an atlas with smooth transition maps consisting of at least three charts!*

We now conclude this section with some further remarks:

Remark 59. *Any open subset of an oriented manifold is oriented; one can see this by any of the three characterisations provided above.*

Example 79. *On homework 4 you will show that the product of manifolds is orientable if and only if each of the factors is orientable, as well as show explicitly that certain real projective spaces are and are not orientable (which also shows that the Möbius band and Klein bottle are non-orientable by the same reasoning).*

5 Integration on manifolds

By combining the various constructions on manifolds in the previous section, we are able to now properly define integration. Just as we integrated 1-forms along curves (which are 1-dimensional), we will integrate n-forms on n-dimensional manifolds.

5.1 Integration of forms

We first observe that there is no coordinate independent way in which one can define the integral, even in Euclidean space. To see this one can consider integrating the function identically equal to one on any closed set. Integrating this function should return the volume of this set, but by rescaling (i.e. choosing different coordinates on Euclidean space) this volume value changes. On the other hand, we saw that 1-forms could be integrated in a coordinate independent way. To generalise this to higher dimensional spaces we make use of n-forms on n-dimensional manifolds; we observe that heuristically these should be good candidates for objects to integrate since, by their multilinearity, they respect the scaling and orientation of the parallelepiped spanned by n-vectors (which the n-form assigns a value which one can think of as the volume).

We will first make the definition and then check why we require its hypotheses and make sure that it is well defined:

Definition 56. *Given an orientable n-manifold, M, and $\omega \in \Omega^n(M)$ with compact support, an orientation atlas, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$, for M with smooth transition maps, and a partition of unity, $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$, subordinate to the domains of the charts. If* $\mathrm{supp}(\omega) \subset U_\alpha$ *for some $\alpha \in \mathcal{A}$ then we define*

$$I_\alpha(\omega) = \int_M \omega = \int_{V_\alpha} f_\alpha \circ \varphi_\alpha^{-1}(x)\, dx,$$

as an integral in $\mathbb{R}^n$, where $\omega = f_\alpha dx^1 \wedge \cdots \wedge dx^n$ for $f \in C^\infty(M)$ with $\mathrm{supp}(f_\alpha) \subset U_\alpha$. *In general, we define the **integral of** ω **on** M by setting*

$$\int_M \omega = \sum_{\alpha\in\mathcal{A}} I_\alpha(\rho_\alpha\omega) = \sum_{\alpha\in\mathcal{A}} \int_M \rho_\alpha\omega.$$

Remark 60. *The integral is well defined since the sum is locally finite and only finitely many of the domains of the charts meet the support of ω since it is assumed to be compact. One can extend the definition of integral to include forms that are not compactly supported under appropriate hypotheses, but we will not do this here.*

Remark 61. *The necessity of the requirement that the manifold be oriented will be made clear shortly. We note that the value of the integral explicitly depends on the orientation that is chosen for the manifold. However, one can define the integral in various ways for manifolds that are not oriented, but we will not address them here. For instance one could consider integrating on the 'oriented double cover' (see [Lee12, Chapter 15] for a definition) of the manifold and halving the value. Alternatively, one can use 'densities' as shown in [Lee12, Chapter 16].*

While the definition of the integral is well defined for a given atlas and partition of unity, it is not clear that it is independent of these choices; we will now show however that this is in fact the case.

Firstly, if we have another orientation atlas, $\{\phi_\beta : U_\beta \to V_\beta\}_{\beta\in\mathcal{B}}$, for M (giving the same orientation) with smooth transition maps then we want to check that if $\mathrm{supp}(\omega) \subset U_\alpha \cap U_\beta$ then $I_\alpha(\omega) = I_\beta(\omega)$ (i.e. that the definition is independent of the choice of chart!). As usual we let $\{dx^i\}_{i=1}^n$ and $\{dy^i\}_{i=1}^n$ be coordinate 1-forms for U_α and U_β respectively, which are related by the Jacobian of the transition map $\varphi_\alpha \circ \phi_\beta^{-1}$. We then have that

$$\omega = f_\alpha dx^1 \wedge \cdots \wedge dx^n = f_\beta dy^1 \wedge \cdots \wedge dy^n,$$

and so we see that $f_\beta = \Delta_{\alpha\beta} f_\alpha$ if we write $\Delta_{\alpha\beta}$ for the determinant of the Jacobian of the transition map (notice that this function never vanishes and is positive since the atlases determine the same orientation). We then compute that

$$I_\beta(\omega) = \int_{V_\beta} f_\beta \circ \phi_\beta^{-1}(y)\, dy = \int_{V_\beta} \Delta_{\alpha\beta}(y)(f_\alpha \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \phi_\beta^{-1})(y)\, dy,$$

which by the standard change of variables formula yields,

$$I_\beta(\omega) = \int_{V_\alpha} \frac{\Delta_{\alpha\beta}(y)}{|\Delta_{\alpha\beta}(y)|} f_\alpha \circ \varphi_\alpha^{-1}(x)\, dx = I_\alpha(\omega),$$

as desired. Notice that in the final line we used that fact that $\Delta_{\alpha\beta} > 0$ since the atlases yield the same orientation for M; this shows the necessity of the orientability of M in the definition of integration! There is thus no ambiguity about integrating forms that are supported in the overlaps of charts, we will now use this to show that the definition of the integral is globally independent of the choice of atlas and partition of unity.

As we remarked when defining partitions of unity, if $\{\rho_\beta\}_{\beta\in\mathcal{B}}$ is a partition of unity subordinate to the cover $\{U_\beta\}_{\beta\in\mathcal{B}}$ then $\{\rho_\alpha\rho_\beta\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$ is a partition of unity subordinate to the cover $\{U_\alpha \cap U_\beta\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$. Noting that if $\operatorname{supp}(\omega) \subset U_\alpha$ then we can write

$$I_\alpha(\omega) = \sum_{\beta\in\mathcal{B}} I_\alpha(\rho_\beta\omega),$$

we thus have for a general compactly supported $\omega \in \Omega^n(M)$ that

$$\sum_{\alpha\in\mathcal{A}} I_\alpha(\rho_\alpha\omega) = \sum_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} I_\alpha(\rho_\alpha\rho_\beta\omega) = \sum_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} I_\beta(\rho_\alpha\rho_\beta\omega) = \sum_{\beta\in\mathcal{B}} I_\beta(\rho_\beta\omega);$$

where in the second inequality we used the fact that $\operatorname{supp}(\rho_\alpha\rho_\beta\omega) \subset U_\alpha \cap U_\beta$ which ensures that $I_\alpha(\rho_\alpha\rho_\beta\omega) = I_\beta(\rho_\alpha\rho_\beta\omega)$. The above shows that the integral is thus well defined independently of the choice of atlas or the choice of partition of unity (one can simply take the second atlas the same as the first and choose a different partition of unity for the latter case).

With the definition established, let us now look at some examples:

Example 80. *For a 0-form or smooth function, f, on a 0-manifold, M, we simply have that*

$$\int_M f = \sum_{x\in M} \pm f(x),$$

where only finitely many terms are non-zero and the choice of the ± 1 factor is precisely the orientation assigned to each point of M.

Example 81. *If $\gamma : [a,b] \to M$ is a smooth curve such that $\gamma([a,b])$ is an embedded 1-manifold, then this definition agrees with our previous definition of the line integral of a 1-form (see [Lee12, Proposition 16.8] for details).*

Example 82. *If $M = (a,b)$ and $\omega = dx$ then $\int_M \omega = \int_a^b dx = b-a$, the length of (a,b) as we expect! Similarly, integrating $dx \wedge dy$ on a bounded open subset of $\mathbb{R}^2$ gives the area and integrating*

*$dx \wedge dy \wedge dz$ on a bounded open subset of $\mathbb{R}^3$ gives the volume. More generally, if ω is an orientation form for a compact manifold M then we define the **volume** of M to be $\mathrm{Vol}(M) = \int_M \omega$. We can then integrate $f \in C^\infty(M)$ against this form and write $\int_M f\,dV = \int_M f\omega$; this is at odds with the notation we will use for the exterior derivative of forms later, so we will avoid this here and only mention that this is common notation in Riemannian geometry using a **Riemannian volume form** (see [Lee12, Chapter 15]).*

We also record some properties of the integral, the first two follow from the corresponding result for integration in Euclidean space and the final two follow by examination of the definitions:

Proposition 13. *(Properties of the integral) Given oriented n-manifolds M, N, compactly supported n-forms $\omega, \eta \in \Omega^n(M)$, and $\lambda \in \mathbb{R}$. Then we have the following:*

1. $\int_M (\omega + \lambda\eta) = \int_M \omega + \lambda \int_M \eta$.
2. *If $-M$ denotes M equipped with its opposite orientation then* $\int_{-M} \omega = -\int_M \omega$.
3. *If ω is an orientation form for M, then* $\int_M \omega > 0$.
4. *If $F : N \to M$ is an orientation preserving/reversing diffeomorphism (see homework 4) then* $\int_N F^*\omega = \pm \int_M \omega$.

As an example of the above we can compute the following example:

Example 83. *If $f : (0, 2\pi) \to S^1$ is defined by $f(\theta) = (\cos(\theta), \sin(\theta))$ then this is an orientation preserving diffeomorphism onto its image (which misses only one point of S^1). If $\omega = x_1 dx^2 - x_2 dx^1$ on S^1, so that $f^*\omega = d\theta$, then we see that*

$$\int_{S^1 \setminus \{pt\}} \omega = \int_0^{2\pi} f^*\omega = \int_0^{2\pi} d\theta = 2\pi.$$

This says that the length of the circle minus a point is 2π with respect to the usual volume form. Of course adding a point will not affect the value of the integral and so the length of the circle is 2π with this volume form, as expected! One can perform similar calculations to find the surface area of the sphere (see homework 4).

5.2 Manifolds with boundary

One thing that the fundamental theorem of calculus as well as the Green, Gauss, and Kelvin–Stokes theorems all have in common is that, via integration, they relate the rate of change, i.e. derivative, of some quantity on the ‘interior’ of some space, to the quantity itself on the ‘boundary’ of this space. We want to generalise this idea to manifolds, and in doing so derive a result that subsumes all of those previously mentioned. To do this however, we will first need to make sense of what the ‘boundary’ of a manifold should be.

In the case of a closed interval, $[a, b]$, (to which we may apply the fundamental theorem of calculus) we have a good notion that the ‘interior’ region is the open interval, (a, b), and the ‘boundary’ is the endpoints $\{a\} \cup \{b\}$; notice in particular that the boundary is a manifold of one dimension less than the interior here. Another example is provided by the closure of the n-dimensional hyperbolic space, $\overline{\mathbb{H}}^n = \{x \in \mathbb{R}^n \,|\, x_n \geq 0$, which has an ‘interior’ formed by $\mathbb{H}^n$ and a ‘boundary’ given by

$\mathbb{R}^{n-1} \times \{0\} = \{x \in \mathbb{R}^n \mid x_n = 0\}$ which we will denote by $\partial\overline{\mathbb{H}}^n$. Finally, the closed n-dimensional unit ball, $\overline{B}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, has an 'interior' formed by B^n itself and a 'boundary' formed by S^{n-1}. Notice that the first and last of these examples each is comprised of regions that can locally be smoothly deformed to look like regions in $\overline{\mathbb{H}}^n$.

The problem with trying to apply the theory we have developed so far however lies in the fact that closed intervals are not manifolds; in particular half open intervals are not homeomorphic to open intervals, but are homeomorphic to open subsets of $\overline{\mathbb{H}}^1$! With this in mind we extend our definition of manifold to include a notion of boundary by changing the 'model' space, namely Euclidean space, to the closure of hyperbolic space as follows:

Definition 57. *An* n***-dimensional topological manifold with boundary*** *or* ***topological*** n***-manifold with boundary*** *is a Hausdorff, second countable topological space for which every point belongs to an open set homeomorphic to an open subset of* $\overline{\mathbb{H}}^n$*. A homeomorphism* $\varphi : U \to V$ *between open subsets* $U \subset M$ *and* $V \subset \overline{\mathbb{H}}^n$ *is a* ***chart*** *on* M*, and an* ***atlas*** *is a collection of charts whose domains cover* M*. An* n***-dimensional smooth manifold with boundary****, hereafter an* n***-manifold with boundary****, is a topological* n*-manifold with boundary admitting an atlas, with smooth transition maps. We then distinguish the following sets:*

- *The* ***interior*** *of* M *is* $M^o = \{x \in M \mid \varphi(x) \in \mathbb{H}^n$ *for every chart in the atlas of* $M\}$*.*
- *The* ***boundary*** *of* M *is* $\partial M = \{x \in M \mid \varphi(x) \in \partial\overline{\mathbb{H}}^n$ *for every chart in the atlas of* $M\}$*.*

Remark 62. *The notion of interior and boundary for a manifold with boundary are well defined as one can show that if the image of some point lies in* $\mathbb{H}^n$ *or* $\partial\overline{\mathbb{H}}^n$ *for some chart, it lies in it for every chart (see [Lee12, Theorem 1.37]). We then have that both* $M = M^o \cup \partial M$ *and* $M^o \cap \partial M = \emptyset$*. Moreover,* M^o *is an* n*-manifold, and* ∂M*, whenever it is not empty, is an* $(n-1)$*-manifold in our original sense (from which it follows that a manifold with boundary is a manifold if and only if it has empty boundary and moreover that* $\partial(\partial M) = \emptyset$*)!*

We can now verify our motivating examples discussed above are indeed manifolds with boundary:

Example 84. *As a sanity check, we indeed see that* n*-dimensional hyperbolic space,* $\overline{\mathbb{H}}^n$*, is an* n*-manifold with boundary by taking the one chart atlas of the identity map. As expected we have* $(\overline{\mathbb{H}}^n)^o = \mathbb{H}^n$ *and* $\partial\overline{\mathbb{H}}^n = \mathbb{R}^{n-1} \times \{0\} = \mathbb{R}^{n-1}$*.*

Example 85. *Any closed interval,* $[a, b]$*, is then a* 1*-manifold with boundary by taking charts mapping that half open intervals* $[a, b)$ *and* $(a, b]$ *to* $\overline{\mathbb{H}}^1$*. We then see that* $[a, b]^o = (a, b)$ *and* $\partial[a, b] = \{a\} \cup \{b\}$*. With the above, one can actually now (e.g. see [Vir13] or [Lee12, Problem 15-13]) classify all* 1*-manifolds with boundary up to diffeomorphism as one of* $\mathbb{R}, S^1, [a, b], (a, b), [a, b), (a, b]$ *(notice that* a *or* b *can be infinity here).*

Example 86. *The closed* n*-dimensional unit ball,* $\overline{B}^n$*, is an* n*-manifold with boundary by, for example, taking charts that 'straighten' out the boundary (we will not do this too precisely here but it is worth thinking about what these maps look like). We then see that* $(\overline{B}^n)^o = B^n$ *and* $\partial\overline{B}^n = S^{n-1}$*.*

All of the notions that we have developed for manifolds so far carry over, with appropriate modifications, to manifolds with boundary; for example, this can be seen explicitly in each instance in [Lee12] where manifolds with boundary are discussed at the same time throughout. One thing that we want to examine precisely is the way in which orientations are induced on the boundary:

Proposition 14. *An orientation for a manifold with boundary determines an orientation, called the* ***Stokes*** *or* ***induced*** *orientation, for its boundary.*

Proof. Consider an n-manifold with boundary, M, which we can assume is such that $\partial M \neq \emptyset$; else there is nothing to prove. Given an orientation atlas, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$, for M we produce an atlas for ∂M by restricting the atlas for M to the boundary, let us denote this atlas by $\{\widetilde{\varphi}_\alpha : \widetilde{U}_\alpha \to \widetilde{V}_\alpha\}_{\alpha \in \mathcal{A}}$ (so that $\widetilde{U}_\alpha = U_\alpha \cap \partial M$ for each $\alpha \in \mathcal{A}$). We will now show that this is an orientation atlas for ∂M, and call the orientation it determines the Stokes or induced orientation.

For fixed $\alpha, \beta \in \mathcal{A}$ we denote $f = \varphi_\alpha \circ \varphi_\beta^{-1}$ and $\widetilde{f} = \widetilde{\varphi}_\alpha \circ \widetilde{\varphi}_\beta^{-1}$ (i.e. the restriction of f to $\partial \overline{\mathbb{H}}^n$). Since f is a diffeomorphism, it must map $\partial \overline{\mathbb{H}}^n$ into itself and thus the inward pointing normal, $\frac{\partial}{\partial x_n}$, must remain inward pointing when mapped by the differential of f; i.e. we must have both that $\frac{\partial f_n}{\partial x_n} > 0$ and $f_n(x_1, \dots, x_{n-1}, 0) = 0$ if $f = (f_1, \dots, f_n)$. We then compute that the Jacobian, J_f, of f along $\partial \overline{\mathbb{H}}^n$ is of the form

$$J_f = \begin{pmatrix} J_{\widetilde{f}} & ? \\ 0 & \frac{\partial f_n}{\partial x_n} \end{pmatrix},$$

which is block upper triangular and so $\det(J_f) = \det(J_{\widetilde{f}}) \frac{\partial f_n}{\partial x_n}$. Since both $\det(J_f) > 0$ (since we have an orientation atlas for M) and $\frac{\partial f_n}{\partial x_n} > 0$ as argued above, we must have that $\det(J_{\widetilde{f}}) > 0$ also. Since the choice of $\alpha, \beta \in \mathcal{A}$ was arbitrary, this shows that the restriction of an orientation atlas for M to ∂M provides an orientation atlas, and hence an orientation, for ∂M as desired. ☐

Another way to find the induced orientation for the boundary is by the use of the following:

Definition 58. *Given a manifold with boundary, M, $X \in \Gamma(TM)$, and $\omega \in \Omega^n(M)$ we define the* ***interior product*** *of ω with X to be $\iota_X : \Omega^n(M) \to \Omega^{n-1}(M)$ defined for $Y_1, \dots, Y_{n-1} \in \Gamma(TM)$ by setting $\iota_X \omega(Y_1, \dots, Y_{n-1}) = \omega(X, Y_1, \dots, Y_{n-1})$ (i.e. place X in the first entry). We then obtain the* ***Stokes*** *or* ***induced*** *orientation form for ∂M by considering $\iota_\nu \omega$ were ω is the orientation form for M and ν is an outward pointing normal vector field.*

Remark 63. *Notice that this gives the expected orientations for the boundary of a closed interval $[a, b]$, assigning $+1$ to $\{b\}$ and -1 to $\{a\}$ in agreement with the fundamental theorem of calculus; i.e. we have that $\int_{[a,b]} df = f(b) - f(a) = \int_{\{a\} \cup \{b\}} f$ for each $f \in C^\infty(\mathbb{R})$ which can be seen as the simplest case of the generalised Stokes theorem we will prove soon!*

We can use this to insert the outward pointing normal to a manifold with boundary as another way to recover the induced orientation, we show this on some examples but one can consult [Lee12, Proposition 15.24] for a proof:

Example 87. *If we consider the standard orientation form $dx^1 \wedge \cdots \wedge dx^n$ for $\overline{\mathbb{H}}^n$ and the outward pointing normal given by $-\frac{\partial}{\partial x_n}$ we get the induced orientation on $\partial \overline{\mathbb{H}}^n$ as $\iota_{-\frac{\partial}{\partial x_n}}(dx^1 \wedge \cdots \wedge dx^n) = (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}$ (where we get $n-1$ multiples of -1 from swapping indices and another from the negative sign on the normal). Notice that if n is even then the orientation form is the same (omitting dx^n), while if n is odd then the orientation 'flips'.*

Example 88. *The standard orientation form $dx^1 \wedge dx^2$ for $\mathbb{R}^2$ restricts to an orientation form on $\overline{B}^2$ which has outward pointing normal given by $\nu = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$. We then see that by interior multiplication the induced orientation form for the boundary, S^1, is $\iota_\nu(dx^1 \wedge dx^2) = x_1 dx^2 - x_2 dx^1$ as expected. The same idea goes through for general n also to deduce an orientation form on S^{n-1}, providing yet another proof that the spheres are oriented!*

5.3 The exterior derivative on forms

In order to continue developing the machinery to develop a general result encompassing our classical theorems from calculus, we need to develop a notion of rate of change, i.e. a 'derivative', for differential forms. For smooth functions, namely 0-forms, we have a natural candidate for the derivative, namely the differential; as we have already noted the fundamental theorem of calculus follows in this case by integrating over a closed interval. We will state the definition as a theorem and discuss the notion as we establish the result:

Theorem 11. *(The exterior derivative on forms) Given an n-manifold with boundary, M, there is a unique family of linear maps, denoted $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for each k, called the **exterior derivative** satisfying the following properties:*

1. *If $f \in \Omega^0(M)$ then $df \in \Omega^1(M)$ is the differential of f.*
2. *If $f \in \Omega^0(M)$ and $\omega \in \Omega^k(M)$ then $d(f\omega) = df \wedge \omega + f d\omega$.*
3. *$d^2 = d \circ d$ is the zero map, i.e. $d^2\omega = 0$ for each $\omega \in \Omega^k(M)$.*
4. *If $U \subset M$ is open then $(d\omega)|_U = d(\omega|_U)$.*

We first make some remarks on the definition before proving the result, from which the precise definition of the exterior derivative will become clear:

Remark 64. *The first property just states that the exterior derivative of a smooth function is the differential of this function. The second property can been seen as a sort of product rule for the exterior derivative; indeed it does imply one for general forms as we will state after the proof. The third property, as we will see in the proof, is related to the fact that partial derivatives commute and, as we will later see from Stokes' theorem, that the boundary of the boundary of a manifold is empty (since the boundary of a manifold with boundary is a manifold). Finally, the fourth property ensures that the exterior derivative is a local definition, as we expect for the derivative of a function depending only on an arbitrarily small neighbourhood of a point.*

Proof. We first show that, if the exterior derivative exists then it is uniquely determined. By the fourth property it suffices to check the uniqueness of the exterior derivative in the domains of charts.

We will use the claim that if $k \geq 1$ and $f_1, \dots, f_k \in \Omega^0(M)$ then it holds that

$$d(df_1 \wedge \cdots \wedge df_k);$$

which follows by induction on k as we now show. The base case $k = 1$ of the claim follows from property two of the exterior derivative. Assuming the claim holds up to $k - 1$ we see by property three that

$$d(f_1 df_2 \wedge \cdots \wedge df_k) = df_1 \wedge \cdots \wedge df_k + f_1 d(df_2 \wedge \cdots \wedge df_k) = df_1 \wedge \cdots \wedge df_k,$$

and thus, by property two, the claim holds by taking the exterior derivative of both sides of the above relation.

Locally we express each $\omega \in \Omega^k(M)$ in the form $\omega = \sum_{I \in \mathcal{I}(n,k)} \omega_I dx^I$ and thus, by the linearity of the exterior derivative, the above claim, and property three we see that

$$d\omega = \sum_{I \in \mathcal{I}(n,k)} d\omega_I \wedge dx^I;$$

thus the exterior derivative is unique provided it exists! We now show the existence of the exterior derivative by using the above formula as a local definition on each differential form (we of course now need to check that this is well defined and satisfies the desired properties).

Let us first show that the above local definition satisfies the first three properties; we note that the linearity of the definition is clear since the differential of smooth functions is linear. For the first property, if $f \in \Omega^0(M)$ then the above definition clearly implies that df is the differential of f as the local representation of f it simply itself!

For the second property, by the product rule for the differential of smooth functions if we have $f \in \Omega^0(M)$ and $\omega \in \Omega^k(M)$ then locally for each $I \in \mathcal{I}(n,k)$ we have

$$d(f\omega_I dx^I) = d(f\omega_I) \wedge dx^I = df \wedge (\omega_I dx^I) + f d\omega_I \wedge dx^I;$$

the second property then follows by summing over all $I \in \mathcal{I}(n,k)$ by the linearity of the local definition.

For the third property, for each $I \in \mathcal{I}(n,k)$ we may compute the differential of the component functions as $d\omega_I = \sum_{j=1}^n \frac{\partial \omega_I}{\partial x_j} dx^j$ and thus

$$d^2(\omega_I dx^I) = d\left(\sum_{j=1}^n \frac{\partial \omega_I}{\partial x_j} dx^j \wedge dx^I\right) = \sum_{i,j=1}^n \frac{\partial \omega_I}{\partial x_i \partial x_j} dx^i \wedge dx^j \wedge dx^I.$$

By summing only over terms $i < j$ and using the fact that partial derivatives commute we may write the above as

$$d^2(\omega_I dx^I) = \sum_{i<j,i,j=1}^n \left(\frac{\partial \omega_I}{\partial x_i \partial x_j} - \frac{\partial \omega_I}{\partial x_j \partial x_i}\right) dx^i \wedge dx^j \wedge dx^I = 0;$$

the third property then follows by summing over all $I \in \mathcal{I}(n,k)$ by the linearity of the local definition.

Having shown that the first three properties hold using the local definition, we now extend this definition to all of M and check it is well defined. Given $\omega \in \Omega^k(M)$ and $x \in M$ we define $d\omega(x) = d(\omega|_U)(x)$ where U is the domain of any chart containing x; namely, we use the above local definition. If $x \in U \cap \widetilde{U}$ for the domains of two charts, then by the uniqueness of the definition on charts shown at the beginning of the proof we conclude that

$$d(\omega|_U)(x) = d(\omega|_{U\cap\widetilde{U}})(x) = d(\omega|_{\widetilde{U}})(x);$$

thus the above definition is well defined on all of M and satisfies the fourth property by the way in which we have defined it. Since the first three properties hold for the local definition, they now hold with our global definition; this concludes the proof. $\square$

Remark 65. *An alternate manner in which one can check that the local definition of the exterior derivative extends to a well defined global definition is by showing that the formula is invariant under changes of coordinates.*

We now state some properties of the exterior derivative which one can check directly from the definition:

Proposition 15. *(Properties of the exterior derivative) Given a manifold with boundary, M, then we have the following:*

1. *If $\omega = df$ for some $f \in \Omega^0(M)$ then $d\omega = 0$ (i.e. every exact form is closed).*

2. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ (i.e. the product rule holds).*

3. *If $F : N \to M$ is a smooth map between manifolds and $\omega \in \Omega^k(M)$ then $F^*(d\omega) = d(F^*\omega)$ (i.e. the exterior derivative commutes with the pullback).*

We conclude this subsection by analysing the exterior derivative on $\mathbb{R}^3$, making clear the connections to previous notions from calculus:

Example 89. *On $\mathbb{R}^3$ we have that:*

- *If $f \in \Omega^0(M)$ then $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \in \Omega^1(M)$ is the usual differential of f; notice that the coefficient functions are the entries of the gradient of the function f!*

- *If $\omega = f dx + g dy + h dz \in \Omega^1(M)$ then one can compute that*

  $$d\omega = df \wedge dx + dg \wedge dy + dh \wedge dz = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy + \left(\frac{\partial f}{\partial x} - \frac{\partial h}{\partial z}\right) dz \wedge dx + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz;$$

  notice that the coefficient functions are (up to permutation) the entries of the curl of the vector field (f, g, h)!

- *If $\omega = f dx \wedge dy + g dz \wedge dx + h dy \wedge dz \in \Omega^2(M)$ then one can compute that*

  $$d\omega = \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right) dx \wedge dy \wedge dz;$$

  notice that the coefficient is (up to a permutation) the divergence of the vector field (f, g, h)!

The above example shows that in $\mathbb{R}^3$ the exterior derivative encodes the notions gradient, curl, and divergence; this was one of the original motivations to introduce differential forms, as a coordinate free definition that encompasses the main properties of calculus in Euclidean space.

5.4 Stokes' theorem and its applications

We are now ready to combine the notions introduced in the previous three subsections in order to establish the celebrated generalised Stokes' theorem, providing a deep connection between the notions of boundary and derivative with profound applications:

Theorem 12. *(Stokes' theorem) Given an oriented n-manifold with boundary, M, and a compactly supported $\omega \in \Omega^{n-1}(M)$ then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

While the theorem is concise to state and, as we will see shortly, not too difficult to prove, this is only because of the large amount of preliminaries required to make it so. Even the statement warrants some clarifications:

Remark 66. *It is implicitly understood that ∂M is equipped with the Stokes/induced orientation from M (hence the name) and that ω on the right hand side integral is really its restriction to ∂M (or $\iota_{\partial M}^*\omega$ where $\iota_{\partial M} : \partial M \to M$ is the inclusion/embedding of ∂M into M. Whenever $\partial M = \emptyset$ (i.e. if M is an oriented n-manifold) then the right hand side is interpreted as zero. Finally, if M is of dimension one then the right hand side integral is a sum; in particular, this implies the fundamental theorem of calculus for line integrals as we will see later.*

Proof. We will proceed by first showing that the conclusion of Stokes' theorem holds whenever ω is supported in the domain of a chart, and then use a partition of unity to patch up and obtain the conclusion globally. To this end we fix an atlas, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$, on M with smooth transition maps and a partition of unity, $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$, subordinate to the cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$.

Now suppose that $\varphi : U \to V$ is a chart on M, so that $V \subset \overline{\mathbb{H}}^n$ is open, and suppose that ω is compactly supported in U. Since then $\omega \in \Omega^{n-1}(U)$ we may express

$$\omega = \sum_{i=1}^n \omega_i dx^1 \wedge \dots dx^{i-1} \wedge \widetilde{dx^i} \wedge dx^{i+1} \wedge \dots \wedge dx^n,$$

for $\omega_i \in C^\infty(U)$ with $\mathrm{supp}(\omega_i) \subset U$ for each $i = 1, \dots, n$ and where $\widetilde{dx^i}$ means we omit this term in the wedge product. By definition of the integral, and denoting $\widetilde{dx_i}$ to omit the integral in the ith coordinate, we then have that

$$\int_{\partial M\cap U} \omega = (-1)^n \sum_{i=1}^n \int_{\partial\overline{\mathbb{H}}^n\cap V} (\omega_i \circ \varphi^{-1})(x_1, \dots, x_{n-1}, 0) dx_1 \dots dx_{i-1}\widetilde{dx_i}dx_{i+1} \dots dx_n;$$

where the factor of $(-1)^n$ appears from the induced orientation on $\partial\overline{\mathbb{H}}^n$. Since x_n vanishes identically on $\partial\overline{\mathbb{H}}^n$ the only potentially non-zero term in the above sum appears when we omit dx_n (since integrating with respect to the nth coordinate direction thus gives zero), and therefore we have that

$$\int_{\partial M\cap U} \omega = (-1)^n \int_{\partial\overline{\mathbb{H}}^n\cap V} (\omega_n \circ \varphi^{-1})(x_1, \dots, x_{n-1}, 0) dx_1 \dots dx_{n-1};$$

we will now compute the other integral term appearing in the conclusion of Stokes' theorem and show that it coincides with the above.

We compute the exterior of derivative of ω, using the definition of the partial derivative on a manifold, to be

$$d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} dx^1 \wedge \dots \wedge dx^n = \sum_{i=1}^n (-1)^{i-1} \left(\frac{\partial(\omega_i \circ \varphi^{-1})}{\partial x_i} \circ \varphi \right) dx^1 \wedge \dots \wedge dx^n;$$

by the definition of the integral we thus have that

$$\int_U d\omega = \sum_{i=1}^n (-1)^{i-1} \int_V \frac{\partial(\omega_i \circ \varphi^{-1})}{\partial x_i}(x_1, \dots, x_n)\, dx_1 \dots dx_n.$$

Since the ω_i have compact support in U the functions $\omega_i \circ \varphi^{-1}$ have compact support in V and hence there exists some $R > 0$ such that the support of $\omega \circ \varphi^{-1}$ is contained in $[-R, R]^{n-1} \times [0, R] \subset \overline{\mathbb{H}}^n$; in particular $\omega_i \circ \varphi^{-1}$ vanishes at the boundary of this rectangle except potentially on $(-R, R)^{n-1} \times \{0\}$.

We then use the fundamental theorem of calculus to compute each term in the sum of integrals above as follows: if $i < n$ then by integrating with respect to the ith coordinate (using Fubini's theorem to change the order of integration) we have

$$\int_V \frac{\partial(\omega_i \circ \varphi^{-1})}{\partial x_i}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \int_0^R \cdots \int_{-R}^R [\omega_i \circ \varphi^{-1}]_{-R}^R dx_1 \ldots dx_{i-1} \widetilde{dx_i} dx_{i+1} \ldots dx_n = 0;$$

where in the second equality we used the fact that $\omega_i \circ \varphi^{-1}(x) = 0$ if $x_i = \pm R$! Similarly, if $i = n$ then we have

$$\begin{aligned}\int_V \frac{\partial(\omega_i \circ \varphi^{-1})}{\partial x_i}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n &= \int_{-R}^R \cdots \int_{-R}^R [\omega_i \circ \varphi^{-1}]_0^R dx_1 \ldots dx_{n-1} \\ &= -\int_{-R}^R \cdots \int_{-R}^R (\omega_i \circ \varphi^{-1})(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots dx_{n-1} \\ &= -\int_{\partial \overline{\mathbb{H}}^n \cap V} (\omega_i \circ \varphi^{-1})(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots dx_{n-1}\end{aligned}$$

where in the second equality we used the fact that $\omega_n \circ \varphi^{-1}(x) = 0$ if $x_n = R$! Combining both cases above we see that

$$\int_U d\omega = (-1)^n \int_{\partial \overline{\mathbb{H}}^n \cap V} (\omega_i \circ \varphi^{-1})(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots dx_{n-1} = \int_{\partial M \cap U} \omega;$$

thus Stokes' theorem holds for forms supported in the domains of charts!

For the general case we note that we can write $\omega = \sum_{\alpha \in \mathcal{A}} \rho_\alpha \omega$ where $\rho_\alpha \omega$ is compactly supported in the domain of a chart U_α for each $\alpha \in \mathcal{A}$; hence we can apply Stokes' theorem to this form by the arguments above. We note also that by construction of the partition of unity and the fact that the differential of a constant is zero we have

$$\sum_{\alpha \in \mathcal{A}} d(\rho_\alpha \omega) = \sum_{\alpha \in \mathcal{A}} d\rho_\alpha \wedge \omega + \sum_{\alpha \in \alpha} \rho_\alpha d\omega = d\left(\sum_{\alpha \in \mathcal{A}} \rho_\alpha\right) \wedge \omega + d\omega = d(1) \wedge \omega + d\omega = d\omega.$$

Using the above and the fact that Stokes' theorem holds for forms compactly supported in charts we see that

$$\int_M d\omega = \int_M \sum_{\alpha \in \mathcal{A}} d(\rho_\alpha \omega) = \sum_{\alpha \in \mathcal{A}} \int_M d(\rho_\alpha \omega) = \sum_{\alpha \in \mathcal{A}} \int_{\partial M} \rho_\alpha \omega = \int_{\partial M} \sum_{\alpha \in \mathcal{A}} \rho_\alpha \omega = \int_{\partial M} \omega;$$

here we are using the linearity of the integral along with the fact that the sums are finite since the support of ω is compact. Thus Stokes' theorem holds for general compactly supported forms. $\square$

We have an immediate application:

Example 90. *If $\gamma : [a, b] \to M$ is a smooth curve which is also an embedding (so that $\gamma([a, b])$ is a 1-manifold with boundary in M), for each $f \in C^\infty(M)$ we have that for the line integral of df we can apply Stokes' theorem to get*

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_{\gamma([a,b])} df = \int_{\{\gamma(a)\} \cup \{\gamma(b)\}} f = f(\gamma(b)) - f(\gamma(a));$$

thus Stokes' theorem recovers the fundamental theorem of line integrals for embedded curves!

We also record some immediate corollaries of Stokes' theorem:

Corollary 7. *We have the following:*

1. *If M is a compact oriented n-manifold then the integral of every exact form is zero (i.e. $\int_M d\omega = 0$ for every $\omega \in \Omega^{n-1}(M)$).*

2. *If M is an oriented n-manifold with boundary the integral of closed compactly supported forms over the boundary vanish (i.e. $\int_{\partial M} \omega = \int_M d\omega = 0$ for every compactly supported $\omega \in \Omega^{n-1}(M)$ with $d\omega = 0$ on M).*

3. *If M is an oriented compact k-manifold embedded in a manifold with boundary N and $\omega \in \Omega^k(N)$ is closed (i.e. $d\omega = 0$) on N with $\int_M \omega \neq 0$ then we have that:*
   - *ω is not exact.*
   - *M is not the boundary of a compact oriented $(k+1)$-manifold with boundary in N.*

As an application of the corollaries we have the following:

Example 91. *We saw previously that the 1-form $\omega = \frac{xdy - ydx}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ has non-zero integral over S^1. By the third corollary we see that ω cannot be exact (which we knew already since it was not conservative) and that S^1 cannot be the boundary of any compact oriented 2-manifold in $\mathbb{R}^2 \setminus \{0\}$. Stokes' theorem is thus telling us about which submanifolds can be boundaries based on the existence of specific differential forms on the manifold!*

We now discuss some applications of Stokes' theorem, the first of which is to re-establish the Green, Gauss, and Kelvin–Stokes theorems that one first encounters in a course in multivariable calculus in the language of differential forms. The Green and Kelvin–Stokes theorem are treated on the quizzes/homework so we will only establish the Gauss theorem here:

Theorem 13. *(Gauss' theorem) Let $V \subset \mathbb{R}^3$ be an open set with compact closure and smooth boundary $S = \partial V$, and $X : \mathbb{R}^3 \to \mathbb{R}^3$ is smooth. Then*

$$\int_V \operatorname{div}(X)\, dV = \int_S (X \cdot \hat{n})\, dS,$$

where $\operatorname{div}(X) = \frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z}$ if $X = (X^1, X^2, X^3)$, $\hat{n}$ is the outward pointing unit normal to S, and dV and dS are the volume and area elements on V and S respectively.

Proof. In order to show that this theorem follows from the Stokes' theorem above we need to reinterpret each of the terms above in the language of manifolds. Precisely, we may interpret V as a compact oriented 3-manifold with boundary given by $\partial V = S$, X as a vector field restricted to V, $dV = dx \wedge dy \wedge dz$, dS as the 2-form $|\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}| du \wedge dv$ where $(u, v) \mapsto (S^1(u, v), S^2(u, v), S^3(u, v))$ provides a local parametrisation of S, and finally $\hat{n} = \frac{\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}}{|\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}|}$ in terms of this parametrisation.

By defining $\omega = X^1 dy \wedge dz + X^2 dz \wedge dx + X^3 dx \wedge dy$ we compute that its exterior derivative is precisely $d\omega = \operatorname{div}(X) dV$! We can thus establish Gauss' theorem by first showing that $\omega = (X \cdot \hat{n})\, dS$ and then applying Stokes' theorem. We compute that on S we have

$$dy \wedge dz = dS^2 \wedge dS^3 = \left(\frac{\partial S^2}{\partial u} \frac{\partial S^3}{\partial v} - \frac{\partial S^2}{\partial v} \frac{\partial S^3}{\partial u} \right) du \wedge dv = \hat{n}_1 \left| \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v} \right| du \wedge dv,$$

and similarly one can show that both

$$dz \wedge dx = \hat{n}_2 \left| \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v} \right| du \wedge dv \quad \text{and} \quad dx \wedge dy = \hat{n}_3 \left| \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v} \right| du \wedge dv.$$

We can thus rewrite

$$\omega = (X^1 \hat{n}_1 + X^2 \hat{n}_2 + X^3 \hat{n}_3) \left| \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v} \right| du \wedge dv = (X \cdot \hat{n}) dS,$$

and hence by Stokes' theorem we see that

$$\int_V \operatorname{div}(X)\, dV = \int_V d\omega = \int_{\partial V} \omega = \int_S (X \cdot \hat{n})\, dS,$$

which is Gauss' theorem as desired. □

Remark 67. *Each of the three theorems from multivariable calculus mentioned above have analogues in Riemannian manifolds; we refer to [Lee12, Chapter 16] for more on this.*

Remark 68. *One can suitably interpret Maxwell's equations for electromagnetism (and indeed many equations from physics) in terms of differential forms; for a treatment of this we refer to [Eas18].*

Another nice application of Stokes' theorem is that it provides one of many possible proofs of the following result (which also holds more generally for compact convex sets):

Theorem 14. *(Brouwer fixed point theorem) Every smooth map from the closed unit ball to itself has a fixed point. Precisely, if $f : \overline{B}^n \to \overline{B}^n$ is smooth, then there is an $x \in \overline{B}^n$ with $f(x) = x$.*

Proof. We argue by contradiction and suppose that there is some smooth function $f : \overline{B}^n \to \overline{B}^n$ with no fixed points. For $x \in \overline{B}^n$ there is thus a unique line from $f(x)$ to x that intersects $\partial\overline{B}^n = S^{n-1}$ at a unique point, let us call it $g(x)$. We note that since f is smooth, the assignment of these points is smooth and hence $g : \overline{B}^n \to S^{n-1}$ is smooth. Moreover, if $x \in S^{n-1}$ then we must have $g(x) = x$ and thus g is the identity on S^{n-1}.

We let ω be the induced orientation form on S^{n-1} from $\mathbb{R}^n$ (which we saw previously), which is in particular such that $\int_{S^{n-1}} \omega > 0$. By pulling back ω by g we see that $g^*\omega \in \Omega^{n-1}(\overline{B}^n)$ which is such that $g^*\omega = \omega$ on S^{n-1} (since g is the identity there). As $\omega \in \Omega^{n-1}(S^{n-1})$ we have that $d\omega = 0$ and thus $d(g^*\omega) = g^*(d\omega) = g^*0 = 0$ (since the exterior derivative commutes with pullbacks of forms). Combining the above we use Stokes' theorem to see that

$$0 < \int_{S^{n-1}} \omega = \int_{\partial\overline{B}^n} g^*\omega = \int_{\overline{B}^n} d(g^*\omega) = \int_{\overline{B}^n} 0 = 0,$$

which is a contradiction! Thus we see that f must have a fixed point as desired. □

Remark 69. *The above theorem is remarkably useful, perhaps most notably in its application to game theory, but can also be used to establish the Jordan curve theorem; see [Mae84].*

Remark 70. *We conclude this subsection by noting that in practice one often wants to apply Stokes' theorem to objects that are 'rougher' than manifolds with boundary (e.g. to squares, cubes, and other polygons). This can be done by considering objects called* ***manifolds with corners*** *which briefly are topological manifolds with boundary obtained by replacing the codomain for charts (which one can think of as the model space) with $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_i \geq 0\}$ (as opposed to $\overline{\mathbb{H}}^n$). Stokes' theorem holds for manifolds with corners (and indeed for much rougher objects than these, for which one needs the language of chains and or geometric measure theory to discuss further) and we refer to [Lee12, Chapter 16] for more detail on this.*

5.5 De Rham cohomology and the degree

For the final topic in the course we will examine how the behaviour of the exterior derivative on differential forms is influenced by the underlying topology of a manifolds. Due to lack of time, we will just state results in this subsection and provide references for their proofs. We first make a definition:

Definition 59. *Given an n-manifold with boundary, M, we call a differential form, ω* ***closed*** *if $d\omega = 0$ and* ***exact*** *if $\omega = d\eta$ for some differential form η. We denote the collections of closed and exact differential forms as:*

- $\mathcal{C}^k(M) = \{$*closed k-forms on M*$\} = \mathrm{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M))$.

- $\mathcal{E}^k(M) = \{$*exact k-forms on M*$\} = \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$.

Remark 71. *Since $\Omega^k(M) = 0$ for $k < 0$ and $k > n$ we have $\mathcal{C}^n(M) = \Omega^n(M)$ and $\mathcal{E}^0(M) = 0$. Moreover, as $d^2 = 0$ we see that $\mathcal{E}^k(M) \subset \mathcal{C}^k(M)$. Notice that in the above definition we are using the fact that we have the sequence of linear maps (each denoted by d) given by*

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

in order to interpret the closed and exact forms as kernels and images respectively.

In the case of a 1-form, ω, we saw that it was the differential of some smooth function, hence exact, if and only if its line integral over all smooth closed curves vanished. Moreover, since applying the exterior derivative twice to any form gives zero, applying the exterior derivative to any exact form must give zero; thus every is exact form is closed. One could ask whether all closed forms are necessarily exact, but as we have seen this is not the case:

Example 92. *The 1-form $\omega = \frac{xdy - ydx}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ was shown not to be exact since its integral over S^1 was non-zero! However, one can compute directly that $d\omega = 0$ (by the symmetry in x and y) and hence ω is in fact closed. As an aside, we also note that if we restrict ω to the open set $U = \{(x, y) \in \mathbb{R}^2 \,|\, x > 0\}$ then in fact $\omega = d(\arctan(\frac{y}{x})) = d\theta$ and hence ω is exact on U; this is a special case of a more general result we will describe shortly, stating that every closed form is locally exact. We also note for later that while θ is not a well define defined function on S^1, it is well defined on $S^1 \setminus \{pt\}$.*

The failure of closed forms to be exact (i.e. the failure of the inclusion $\mathcal{C}^k(M) \subset \mathcal{E}^k(M)$) on a given manifold is precisely related to the topology of the manifold itself (for example, from Stokes' theorem we saw that S^1 was not the boundary of a compact oriented 2-manifold with boundary in $\mathbb{R}^2 \setminus \{0\}$ by examining closed forms). We quantify this behaviour, which can be thought of as measuring the failure of the fundamental theorem of calculus holding on manifolds, through the following notion:

Definition 60. *Given an n-manifold with boundary, M, the* ***de Rham cohomology group of degree k on*** *M is given by*

$$H^k_{dR}(M) = \mathcal{C}^k(M)/\mathcal{E}^k(M)$$

namely the quotient of the closed k-forms by the exact k-forms. Given $\omega \in H^k_{dR}(M)$ we denote its equivalence class or the ***cohomology class of*** *ω by $[\omega]$ and say that each $\widetilde{\omega} \in [\omega]$ (so that $\omega - \widetilde{\omega}$ is exact) is* ***cohomologous to*** *ω.*

Remark 72. *We have that $H^k_{dR}(M) = 0$ for $k < 0$ and $k > n$ since in either case $\Omega^k(M) = 0$, while the definition implies that $H^k_{dR}(M) = 0$ for $0 \leq k \leq n$ if and only if every closed k-form is exact, i.e. precisely whenever $\mathcal{C}^k(M) = \mathcal{E}^k(M)$. Notice that since the closed and exact forms are a real vector space, the de Rham cohomology groups are real (quotient) vector spaces.*

We now consider some examples:

Example 93. *The existence of* $\omega = \frac{xdy - ydx}{x^2+y^2}$ *which was a closed but not exact* 1*-form on* $\mathbb{R}^2 \setminus \{0\}$ *implies that necessarily* $H^1_{dR}(\mathbb{R}^2 \setminus \{0\}) \neq 0$*!*

Example 94. *A smooth function is closed, i.e.* $f \in \mathcal{C}^0(M)$*, if and only if* $df = 0$ *which implies that* f *is constant on each connected component of* M*. Since we always have that* $\mathcal{E}^0(M) = 0$*, we see that* $H^0_{dR}(M) = \mathbb{R}^m$ *if and only if* M *has* m *connected components; and thus we already see that the de Rham cohomology groups depend on the topological structure of the manifold! Thus if* $M = \{pt\}$ *is a manifold consisting of a point we have* $H^0_{dR}(M) = \mathbb{R}$ *and* $H^k_{dR}(M) = 0$ *for any* $k \neq 0$*.*

Just as we can pullback forms by smooth maps, we have the following notion:

Proposition 16. *If* $F : M \to N$ *is a smooth map between manifolds with boundary then the pullback of forms gives the linear* ***induced cohomology map*** *denoted* $F^* : H^k_{dR}(N) \to H^k_{dR}(M)$ *defined for each* $\omega \in H^k_{dR}(N)$ *by setting* $F^*[\omega] = [F^*\omega]$*. In particular, if* $F : M \to N$ *is a diffeomorphism then* F^* *is an isomorphism.*

Proof. See [Lee12, Proposition 17.2]. □

Remark 73. *This result tells us that the de Rham cohomology groups are diffeomorphism invariant and allows us to distinguish manifolds (i.e. show that they are not diffeomorphic) if their de Rham cohomology groups differ. In fact, the de Rham cohomology groups are invariant in an even stronger sense since they can be shown to be homotopy invariant (if you know what this means); e.g. see [Lee12, Proposition 7.11/Corollary 7.12]. This shows that even though the de Rham cohomology groups are defined using the smooth structure, they are topological invariants!*

The above proposition and remark give a number of examples:

Example 95. *Since one can find a homotopy from the identity to the zero map on* $\mathbb{R}^n$ *we have that* $H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$ *(since* $\mathbb{R}^n$ *is connected) and* $H^k_{dR}(\mathbb{R}^n) = 0$ *if* $k \neq 0$*! Similarly we have that* $H^k_{dR}(M \times \mathbb{R}^n) = H^k_{dR}(M)$ *for any* k *(since the* $\mathbb{R}^n$ *factor is contractible/homotopic to a point). The same reasoning gives the identical result for any star shaped region in* $\mathbb{R}^n$*; this is known as the* ***Poincaré lemma****, see [Lee12, Theorem 17.13/14] for a proof.*

Example 96. *Let* $f : \mathbb{R} \to S^1 \subset \mathbb{R}^2 \setminus \{0\}$ *be defined by* $f(\theta) = (\cos(\theta), \sin(\theta))$ *and* $\omega = \frac{xdy - ydx}{x^2+y^2}$*. We know that* ω *is not exact and hence* $[\omega] \neq 0$*, which implies that* $H^1_{dR}(S^1) \neq 0$*. Moreover, we computed that* $f^*\omega = d\theta$ *on* $\mathbb{R}$ *and so* $f^*[\omega] = [f^*\omega] = [d\theta] = 0$ *as* $d\theta$ *is clearly exact (noting here that* θ *is a well defined function on* $\mathbb{R}$*)! This tells us that the induced cohomology map* $f^* : H^1_{dR}(S^1) \to H^1_{dR}(\mathbb{R})$ *is not an isomorphism even though* f *is a local diffeomorphism.*

For top degree differential forms on compact orientable manifolds we can say more:

Example 97. *By Stokes' theorem we know that if* M *is a compact orientable* n*-manifold (with empty boundary) then any orientation form,* ω*, cannot be exact (else we would have* $\omega = d\eta$ *for some* η *and hence by Stokes' theorem we would have* $0 < \int_M \omega = \int_{\partial M} \eta = 0$ *since* $\partial M = 0$*). This tells us that we necessarily have* $H^n_{dR}(M) \neq 0$ *and if* $\widetilde{\omega} \in [\omega]$ *then since* $\widetilde{\omega} = \omega + d\eta$ *for some* $\eta \in \Omega^{n-1}(M)$ *we see that* $\int_M \widetilde{\omega} = \int_M \omega + d\eta = \int_M \omega$ *by Stokes' theorem. We thus see that cohomological* n*-forms give the same integral on compact orientable manifolds!*

The above example is a special instance of the following:

Theorem 15. *Given a compact connected orientable n-manifold, M, we have that $H^n_{dR}(M) = \mathbb{R}$ and given some orientation on M we can define this isomorphism by sending $\omega \in \Omega^n(M)$ to $\int_M \omega \in \mathbb{R}$.*

Proof. See [Lee12, Theorem 7.31]. $\square$

Remark 74. *This is a specific instance of a general result known as the **de Rham theorem**, which states that integration over chains defines an isomorphism between the de Rham cohomology group and the singular cohomology groups with real coefficients; see [Lee12, Chapter 18].*

The above result gives the following:

Example 98. *For S^1 we immediately have that $H^k_{dR}(S^1) = \mathbb{R}$ if $k = 0$ or $k = 1$ and are equal to zero otherwise! Similarly, one can use the theorem to show that $H^k_{dR}(S^n) = \mathbb{R}$ if $k = 0$ or $k = n$ and are equal to zero otherwise; for a proof see [Lot21, Theorem 7.7]. One can in fact use the de Rham cohomology groups of the spheres to give a proof of the hairy ball theorem; see [Lot21, Theorem 7.9].*

Another effective use of the above theorem is that it allows us to introduce the notion of the degree of a smooth map between compact connected orientable manifolds; capturing the idea of the number of times the map 'wraps' the domain around the codomain:

Theorem 16. *Given a smooth map $F : M \to N$ between compact connected orientable n-manifolds, there exists a unique integer, k, called the **degree of** F such that*

1. *For each $\omega \in \Omega^n(N)$ we have $\int_M F^*\omega = k \int_N \omega$.*
2. *If $y \in N$ is a regular value of F then $k = \sum_{x \in F^{-1}(y)} \operatorname{sgn}(x)$, where $\operatorname{sgn}(x) = \pm 1$ if $d_x F$ is orientation preserving/reversing.*

Proof. See [Lee12, Theorem 17.35]. $\square$

Remark 75. *One can show that the degree is also a homotopy invariant, see [Lee12, Proposition 17.36], which allows us to define the degree of continuous maps between compact connected orientable n-manifolds to be the degree of any smooth map it is homotopic to; this can always be arranged by the **Whitney approximation theorem** (see [Lee12, Theorem 6.26]). Again this shows that the degree is a topological property, even though it was initially defined based on a smooth structure!*

Let's look at some examples:

Example 99. *The identity map on any compact connected orientable manifold has degree 1 and any constant map has degree 0!*

Example 100. *One can compute explicitly that the antipodal map $-I_{n+1} : S^n \to S^n$ has degree $(-1)^{n+1}$ (since $\det(-I_{n+1}) = (-1)^{n+1}$).*

The notion of degree is a powerful tool in topology and can in fact be used to classify continuous maps from spheres to themselves, see [Hat01, Corollary 4.25], as well as establish alternate proofs of the Brouwer fixed point theorem and the fundamental theorem of algebra, see [Lot21, Section 7]. The degree of a map is also related to winding numbers in complex analysis, linking numbers of curves, and relates the so called index of a vector field with isolated zeros to the topology of the manifold itself via the Poincaré–Hopf theorem (in particular it tells us that if a compact connected orientable 2-manifold has a non-vanishing vector field then it must be a torus; providing yet another proof of the Hairy ball theorem).

A Homework

A.1 Sheet 1

1. [**5 points**] Show that $\mathbb{R}^n$ with the standard definition of open sets is a topological space.

2. [**5 points**] Show that $\mathbb{R}^n$ is second countable.

3. [**5 points**] Show that in a Hausdorff topological space every convergent sequence has a unique limit. Recall that we said a sequence converges in a topological space if it is eventually entirely contained in any open set which contains the limit.

4. [**5 points**] Show that a second countable topological space is separable; i.e. contains a countable dense subset, where dense means that it has non-empty intersection with every open set.

5. [**8 points**] Find diffeomorphisms between:

   (a) $(0, 1)$ and $\mathbb{R}$.

   (b) $(0, \infty)$ and $\mathbb{R}$

   (c) $\mathbb{R}^n$ and $\text{graph}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \,|\, y = f(x)\} \subset \mathbb{R}^{n+1}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is smooth.

   (d) $S^1 \times (0, \infty)$ and $\mathbb{R}^2 \setminus \{0\}$.

6. [**6 points**] Show that $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^m$ if and only if $n = m$ (hint: $f : \mathbb{R}^n \to \mathbb{R}^m$ is a diffeomorphism if and only if the derivative $D_x f$ is invertible at every point $x \in \mathbb{R}^n$). In particular, this provides an alternate way to show that the dimension of a smooth manifold is well defined.

7. [**6 points**] Show that if $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^n$ is smooth, and $D_x f$ is invertible for each $x \in U$, then $f(U)$ is open. Use this to show that if $M \subset \mathbb{R}^n$ is an n-manifold, then M must be open as a subset of $\mathbb{R}^n$. Deduce that there are no compact n-manifolds $M \subset \mathbb{R}^n$ (hint: which sets are both open and closed in $\mathbb{R}^n$?).

8. [**9 points**] Show that if M is an m-manifold and N is an n-manifold then $M \times N$ is an $(m+n)$-manifold (note: if $U \subset M$ and $V \subset N$ are open then $U \times V \subset M \times N$ is open). Deduce that the standard n-torus, $T^n = S^1 \times \cdots \times S^1$ (product of n circles), is an n-manifold.

9. [**5 points**] Write down an explicit diffeomorphism from the standard 2-torus, $T^2 = S^1 \times S^1$, to the donut torus, defined as $\{((2 + \cos(\theta)) \cos(\phi), (2 + \cos(\theta)) \sin(\phi), \sin(\theta)) \,|\, \theta, \phi \in \mathbb{R}\} \subset \mathbb{R}^2$. This shows that the donut torus is also a 2-manifold by the result of the previous problem.

10. [**6 points**] Recall that we defined the real projective space, $\mathbb{R}P^n$, to be the space of lines through the origin in $\mathbb{R}^{n+1}$. This is equivalent to saying that $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation, $\sim$, defined by setting $x \sim y$ if $x = \lambda y$ for some $\lambda \neq 0$. We often denote points in $\mathbb{R}P^n$ by equivalence classes, $[x]$, for points $x \in \mathbb{R}^{n+1} \setminus \{0\}$. A set in a quotient space is open if and only if its preimage under the quotient map is open.

    Show that $\mathbb{R}P^n$ is an n-manifold by verifying that for each $i = 1, \ldots n + 1$ the sets

    $$U_i = \{[(x_1, \ldots, x_{n+1})] \in \mathbb{R}P^n \,|\, x_i \neq 0\},$$

and maps $\varphi_i : U_i \to \mathbb{R}^n$ defined by setting

$$\varphi_i([(x_1, \ldots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right),$$

provide an atlas $\{\varphi_i : U_i \to \varphi_i(U_i)\}_{i=1}^{n+1}$ for $\mathbb{R}P^n$ with smooth transition maps.

A.2 Sheet 2

1. Given a manifold, M, we say that two atlases $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{\varphi_\beta : U_\beta \to V_\beta\}_{\beta\in\mathcal{B}}$ with smooth transition maps for M are **equivalent** if their union $\{\varphi_\gamma : U_\gamma \to V_\gamma\}_{\gamma\in\mathcal{A}\cup\mathcal{B}}$ is also an atlas with smooth transition maps for M; an equivalence class of such atlases is called a **smooth structure** on M.

   (a) [**3 points**] Show that this is indeed defines an equivalence relation for atlases.

   (b) [**3 points**] Show that two atlases are equivalent if and only if they define the same set of smooth functions on M (i.e. $f : M \to \mathbb{R}$ is smooth with respect to the first atlas if and only if it is smooth with respect to the second atlas); a smooth structure thus determines which functions are declared to be smooth on a topological manifold.

   (c) [**3 points**] Show that each smooth structure on M contains a unique **maximal atlas** that contains any other atlas as a subset.

   With this in hand one could alternatively define a smooth manifold to be a topological manifold equipped with a smooth structure (or equivalently a maximal atlas).

2. Let the discrete group $\mathbb{Z}_2 = \{\pm 1\}$ act on $\mathbb{R}^n$ by diffeomorphisms by setting $f_{\pm 1} = \pm I_{\mathbb{R}^n}$.

   (a) [**6 points**] Show that $\mathbb{Z}_2$ acts on $\mathbb{R}^n \setminus \{0\}$ (which is an n-manifold as it is open in $\mathbb{R}^n$) by diffeomorphisms freely and properly discontinuously; and thus also on any submanifold $M \subset \mathbb{R}^n \setminus \{0\}$ provided that $M = -M$ (i.e. which is invariant under $-I_{\mathbb{R}^n}$).

   (b) [**3 points**] Deduce using the quotient manifold theorem that $S^n/\mathbb{Z}_2$ is a manifold, make a guess as to what manifold it is diffeomorphic to.

   (c) [**3 points**] Deduce using the quotient manifold theorem that the donut torus in $\mathbb{R}^3$ quotient by $\mathbb{Z}_2$ is a manifold, make a guess as to what manifold it is diffeomorphic to.

   (d) [**3 points**] Deduce using the quotient manifold theorem that if $C = S^1 \times \mathbb{R} \subset \mathbb{R}^3$ is a cylinder then $C/\mathbb{Z}_2$ is a manifold, make a guess as to what manifold it is diffeomorphic to.

3. [**3 points**] The *Smooth Urysohn Lemma* states that given disjoint closed sets $A, B \subset M$ of a smooth manifold, then there exists a smooth function on M taking values in $[0, 1]$ that is identically 1 on A and identically 0 on B. Prove this lemma.

4. [**6 points**] Show that every tangent vector arises from a smooth curve: i.e. if $v \in T_xM$ for a point, x, in some manifold, M, then there is a smooth curve $\gamma : I \to M$ such that $v = \gamma'(0)$, where $\gamma'(0)$ is the **velocity** of the curve γ at 0 defined by $\gamma'(0) = d_0\gamma\left(\frac{d}{dt}\big|_{t=0}\right)$ (so that by the definition of the differential $\gamma'(0)(f) = \frac{d}{dt}\big|_{t=0}(f \circ \gamma) = (f \circ \gamma)'(0)$ for each $f \in C^\infty(M)$).

5. **[4 points]** For a submanifold $M \subset \mathbb{R}^n \times [0, \infty) \subset \mathbb{R}^{n+1}$ with $0 \in M$, show that $T_0 M \subset \mathbb{R}^n \times \{0\}$ (hint: draw a picture and use the previous question). When does $T_0 M = \mathbb{R}^n \times \{0\}$?

6. **[4 points]** Suppose that we have two charts, $\varphi : U \to V$ and $\phi : \widetilde{U} \to \widetilde{V}$, on a manifold, M, with $x \in U \cap \widetilde{U}$. If we denote by $(x_1, \ldots, x_n) \in V$ and $(y_1, \ldots, y_n) \in \widetilde{V}$ the coordinates in V and $\widetilde{V}$ respectively, find an expression relating the coordinate vectors $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n$ to the coordinate vectors $\{\frac{\partial}{\partial y_i}\big|_x\}_{i=1}^n$ (hint: it may help to consult subsection 3.2 of the notes).

7. **[6 points]** By using the regular value theorem, show that the standard n-torus
$$T^n = S^1 \times \cdots \times S^1,$$
the product of n circles, is an n-manifold and compute its tangent space at each point.

8. **[3 points]** By using the regular value theorem, show that if $t \neq 0$ then
$$H_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = t\}$$
is a submanifold of $\mathbb{R}^3$. What is its dimension?

9. **[2 points]** Show that the cone
$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 0\}$$
is not a submanifold of $\mathbb{R}^3$ (hint: consider what the tangent space at the origin would be).

10. On homework 1 we that if M is an m-manifold and N is an n-manifold then $M \times N$ is an $(m+n)$-manifold, from which we deduce that $T_{(x,y)}(M \times N)$ is isomorphic to $T_x M \oplus T_y N$ for each $(x, y) \in M \times N$ (as their dimensions agree).

    (a) **[2 points]** Use this fact to verify your answer above for the tangent space to T^n.

    (b) **[6 points]** Show that the canonical isomorphism between $T_{(x,y)}(M \times N)$ and $T_x M \oplus T_y N$ is provided by the linear map $\alpha : T_{(x,y)}(M \times N) \to T_x M \oplus T_y N$ defined for each $v \in T_{(x,y)}(M \times N)$ by setting
    $$\alpha(v) = (d_{(x,y)}\pi_M(v), d_{(x,y)}\pi_N(v));$$
    where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the smooth projection maps to each factor (hint: define a specific linear map, say $\beta : T_x M \oplus T_y N \to T_{(x,y)}(M \times N)$, using the differential of the smooth inclusion maps into $M \times N$ for each factor and then apply the chain rule for differentials to show that $\alpha \circ \beta$ is the identity).

A.3 Sheet 3

1. Explain whether the following smooth maps are immersions, submersions, or embeddings (hint: it may help to sketch the second and third maps below):

    (a) **[5 points]** The Hopf fibration, $f : S^3 \to S^2$, given by
    $$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2x_1x_4 + 2x_2x_3, 2x_2x_4 - 2x_1x_3).$$

(b) **[5 points]** The four-petal rose, $g : S^1 \to \mathbb{R}^2$, given by

$$g(\cos(\theta), \sin(\theta)) = (\sin(2\theta)\cos(\theta), \sin(2\theta)\sin(\theta)).$$

(c) **[5 points]** The figure-eight, $h : (-\pi, \pi) \to \mathbb{R}^2$, given by

$$h(t) = (\sin(2t), \sin(t)).$$

2. **[4 points]** Suppose that we have two charts, $\varphi : U \to V$ and $\phi : \widetilde{U} \to \widetilde{V}$, on a manifold, M, with $x \in U \cap \widetilde{U}$. If we denote by $(x_1, \ldots, x_n) \in V$ and $(y_1, \ldots, y_n) \in \widetilde{V}$ the coordinates in V and $\widetilde{V}$ respectively, homework 2 question 6 showed that if $v \in T_xM$ then the components, v_i, with respect to $\{\frac{\partial}{\partial x_i}\big|_x\}_{i=1}^n$ were related to the components, $\widetilde{v}_i$, with respect to $\{\frac{\partial}{\partial y_i}\big|_x\}_{i=1}^n$ by the formula
$$\widetilde{v}_j = \sum_{i=1}^n \frac{\partial(\phi \circ \varphi^{-1})^j}{\partial x_i}(\varphi(x))v_i.$$
Find an analogous formula relating the components of some $\omega \in T_x^*M$ in each set of coordinates; i.e. relate the components, ω_i, with respect to $\{dx_i|_x\}_{i=1}^n$ to the components, $\widetilde{\omega}_i$, with respect to $\{dy_i|_x\}_{i=1}^n$ (hint: examine the solution of homework 2 question 6). This question motivates why we call tangent vectors **covariant**, since their components transform in the same way as the coordinate partial derivatives, while we call cotangent vectors **contravariant**, since their components transform in the oppose way to the coordinate partial derivatives.

3. **[3 points]** Show that given a manifold, M, and an exact $\omega \in \Gamma(T^*M)$, then any two potentials for ω locally differ by a constant; i.e. if $\omega = df = dg$ for $f, g \in C^\infty(M)$ then $f - g$ is a constant.

4. **[6 points]** The length, $L(\gamma)$, of a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ is defined by setting

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt;$$

where this is integration in the usual one variable sense. Show that there is no $\omega \in \Gamma(T^*M)$ such that $L(\gamma) = \int_\gamma \omega$ for every smooth curve $\gamma : [a, b] \to \mathbb{R}^n$.

5. The purpose of this question is to relate the line integral for 1-forms in class to the usual line integral of a vector field in multivariable calculus. Given an open $U \subset \mathbb{R}^n$ and $X \in \Gamma(TU)$ and a smooth curve $\gamma : [a, b] \to U$, we define the **line integral** of X along γ to be

$$\int_\gamma X = \int_a^b X_{\gamma(t)} \cdot \gamma'(t)\, dt.$$

(a) **[5 points]** Show that for each $X \in \Gamma(TU)$ there exists some $\omega \in \Gamma(T^*U)$ such that

$$\int_\gamma X = \int_\gamma \omega$$

where the latter denotes the line integral of a 1-form as introduced in class (hint: use the dot product to define ω in terms of X).

(b) **[3 points]** We say that X is **conservative** if

$$\int_\gamma X = 0$$

for every smooth closed curve $\gamma : [a, b] \to U$. Show that X is conservative if and only if $X = \nabla f$ for some $f \in C^\infty(U)$ (hint: 1-forms are conservative if and only if they are exact).

(c) **[5 points]** Show that if $n = 3$, $X \in \Gamma(TU)$, and we define the **curl** of X by setting

$$\mathrm{curl}(X) = \left(\frac{\partial X^3}{\partial x_2} - \frac{\partial X^2}{\partial x_3}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial X^1}{\partial x_3} - \frac{\partial X^3}{\partial x_1}\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial X^2}{\partial x_1} - \frac{\partial X^1}{\partial x_2}\right)\frac{\partial}{\partial x_3},$$

then $\mathrm{curl}(X) = 0$ if X is conservative (hint: use (b)). Find an example that shows that the converse is not true if $U = \mathbb{R}^3 \setminus \{x = y = 0\}$ (hint: use an example from the notes).

6. **[3 points]** Given an m-manifold, $M \subset \mathbb{R}^n$, we define the **normal bundle** of M in $\mathbb{R}^n$, which we denote by $\nu(M; \mathbb{R}^n)$ by defining

$$\nu(M; \mathbb{R}^n) = \bigsqcup_{x \in M} (T_x M)^\perp,$$

which comes with a natural projection, $\pi : \nu(M; \mathbb{R}^n) \to M$, defined by $\pi(x, v) = x$ for $v \in (T_x M)^\perp$. Here we write $(T_x M)^\perp$ for the orthogonal complement (with respect to the dot product in $\mathbb{R}^n$) of $T_x M$ in $\mathbb{R}^n$. By the vector bundle chart lemma in the notes, this defines a rank $n - m$ vector bundle and thus $\nu(M; \mathbb{R}^n)$ has dimension $n = m + (n - m)$ as a manifold. Sections of $\nu(M; \mathbb{R}^n)$ are called **normal fields** to M in $\mathbb{R}^n$. Show that $\nu(S^m; \mathbb{R}^{m+1})$ is trivial by constructing a nowhere vanishing normal field, hence $\nu(S^m; \mathbb{R}^n)$ is diffeomorphic to $S^m \times \mathbb{R}$.

7. Given vector bundles $\pi_E : E \to M$ of rank k and $\pi_F : F \to M$ of rank l over a manifold M, we can form their **Whitney sum**, $E \oplus F$, defined by $E \oplus F = \bigsqcup_{x \in M} E_x \oplus F_x$ with projection $\pi : E \oplus F \to M$ (setting $\pi(x, v, w) = x$ for $v \in E_x$ and $w \in F_x$). By the vector bundle chart lemma in the notes, $\pi : E \oplus F \to M$ is a vector bundle of rank $k + l$ over M.

   (a) **[2 points]** Given a submanifold $M \subset \mathbb{R}^n$, show that the Whitney sum of the tangent and normal bundle of M is trivial (where the normal bundle is defined as in question 6 above).

   (b) **[3 points]** Show that the Whitney sum of trivial bundles is trivial. Use this to show that T^n is parallelisable (hint: use homework 2 question 10).

   (c) **[2 points]** Give an example to show that the Whitney sum of a non-trivial bundle with a trivial bundle can be trivial (hint: S^2 is not parallelisable). Although not for credit but worth thinking about, the Whitney sum of two Möbius bundles is trivial (so the Whitney sum of non-trivial bundles can be trivial).

8. We define the matrix group $SU(2)$ by setting

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\},$$

where the bar denotes complex conjugation (i.e. if $\alpha = a + bi$ then $\overline{\alpha} = a - bi$ where $i = \sqrt{-1}$). You may assume throughout this question that $SU(2)$ is a Lie group (i.e. a smooth manifold with smooth maps for group operations).

(a) **[4 points]** Considering $\mathbb{C}^2$ diffeomorphic to $\mathbb{R}^4$ and the 2×2 matrices with entries in $\mathbb{C}$ diffeomorphic to $\mathbb{R}^8$, show that $SU(2)$ is the image of S^3 via an embedding, so that S^3 is diffeomorphic to $SU(2)$. Hence, state the dimension of $SU(2)$.

(b) **[2 points]** Explain why (a) shows that S^3 is parallelisable.

(c) **[3 points]** As an alternative to (b), explain why the existence of the vector fields

$$s_1(x) = (-x_2, x_1, x_4, -x_3), s_2(x) = (-x_3, -x_4, x_1, x_2), s_3(x) = (-x_4, x_3, -x_2, x_1)$$

also show that S^3 is parallelisable.

A.4 Sheet 4

1. **[4 points]** Given a real vector space, V, of dimension n, show that if $L : V \to V$ is linear, $\omega \in \Lambda^n V^*$, and $v_1, \dots, v_n \in V$ then

$$\omega(Lv_1, \dots, Lv_n) = \det(L)\omega(v_1, \dots, v_n).$$

This result was used several times in class (hint: prove it on basis elements first).

2. Let $B = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\} \subset \mathbb{R}^2$ and define a Riemannian metric, g, on B by setting

$$g = 4\frac{dx \otimes dx + dy \otimes dy}{(1 - (x^2 + y^2))^2};$$

which is the **Poincaré disk** model of Hyperbolic space. Answer the following:

(a) **[6 points]** Given a smooth curve $\gamma : [0, T] \to B$, find a general formula for the length, $L_g(\gamma)$, of γ with respect to g.

(b) **[4 points]** Find the length of the curve $\gamma(t) = (t, 0)$ for $t \in [0, T]$, where $T \in (0, 1)$.

(c) **[2 points]** What happens to the lengths in (b) as $T \to 1$? Explain this geometrically.

3. **[4 points]** A **Riemann surface** is a 2-manifold admitting an atlas whose transition maps are holomorphic (viewed as maps from $\mathbb{C}$ to itself). Show that every Riemann surface is orientable (hint: $f : \mathbb{C} \to \mathbb{C}$ with $f(x, y) = (u(x, y), v(x, y))$ is holomorphic if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$). Then, provide both an example and non-example of a Riemann surface with justification.

4. **[8 points]** Given manifolds M and N, show that $M \times N$ is orientable if and only if both M and N are orientable.

5. We say that a diffeomorphism $F : M \to N$ between oriented manifolds is **orientation preserving** if whenever we have orientation forms ω for M and η for N that $F^*\eta = f\omega$ for some strictly positive $f \in C^\infty(M)$ (i.e. if $F^*\eta$ gives the same orientation as ω). Answer the following:

(a) **[2 points]** Show that $F : M \to N$ is orientation preserving if and only if $\det(d_x F) > 0$ for each $x \in M$ (hint: use question 1 and recall the definition of the pullback of forms).

(b) **[2 points]** Deduce for which n the antipodal map, defined by minus the identity, on $\mathbb{R}^n$ is orientation preserving.

(c) **[12 points]** Using (a) and (b) or otherwise, show that real projective space, $\mathbb{R}P^n$, of dimension n is orientable if and only if n is odd.

Using the quotient constructions as in homework 2 question 2, the same arguments for (c) also show that both the Möbius band and Klein bottle are not orientable.

6. **[4 points]** First, using interior multiplication of forms find the Stokes orientation form, ω_{S^2}, induced on S^2 from $\overline{B}^3$ (with the standard orientation form from $\mathbb{R}^3$). Then, by defining the map $F : (0, \pi) \times (0, 2\pi) \to S^2$ by setting

$$F(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)),$$

determine the volume of S^2 with respect to ω_{S^2}.

7. **[6 points]** For a 1-form, ω, and vector fields X, Y show that

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]),$$

where we recall that $[X, Y]$ is the Lie bracket of the vector fields (hint: work locally and recall that $df(X) = Xf$ for smooth functions f). This provides a coordinate free way to compute the exterior derivative of a 1-form; more general formulae exist for k-forms where $k \geq 2$.

8. **[6 points]** Use Stokes' theorem to show that if $\Sigma \subset \mathbb{R}^3$ is a compact, oriented 2-manifold with boundary, $\partial\Sigma = \Gamma$, then for each vector field X we have

$$\int_\Sigma \operatorname{curl}(X) \cdot d\Sigma = \int_\Gamma X \cdot d\Gamma,$$

which is the Kelvin–Stokes Theorem. In the above we define

$$d\Sigma = (dy \wedge dz, dz \wedge dx, dx \wedge dy), d\Gamma = (dx, dy, dz),$$

and

$$\operatorname{curl}(X) = \left(\frac{\partial X^3}{\partial x_2} - \frac{\partial X^2}{\partial x_3}, \frac{\partial X^1}{\partial x_3} - \frac{\partial X^3}{\partial x_1}, \frac{\partial X^2}{\partial x_1} - \frac{\partial X^1}{\partial x_2}\right).$$

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