Johns Hopkins University

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HONORS ANALYSIS I

Lecture Notes

"Per aspera ad astra"

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Part I

The Real and Complex Number Systems

1 Lecture 1: Foundations

1.1 The Incompleteness of the Rational Numbers

Example 1.1 (The Irrationality of $\sqrt{2}$)**.** There is no $x \in \mathbb{Q}$ such that $x^2 = 2$. □

Proof. We argue by contradiction. Assume there exists such an $x \in \mathbb{Q}$. We can write $x = p/q$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$. Then $x^2 = p^2/q^2 = 2 \implies p^2 = 2q^2$. This means p^2 is even, which implies p must also be even. So, we can write $p = 2k$ for some $k \in \mathbb{Z}$. Substituting this back gives $(2k)^2 = 2q^2 \implies 4k^2 = 2q^2 \implies 2k^2 = q^2$. This means q^2 is also even, which implies q is even. If both p and q are even, then $\gcd(p, q) \neq 1$, which contradicts our initial assumption. Thus, no such rational number exists. □

Example 1.2 (The "Gap" in $\mathbb{Q}$)**.** To see why the rational numbers are insufficient, consider the sets

$$A = \{a \in \mathbb{Q} \mid a^2 < 2\} \quad \text{and} \quad B = \{b \in \mathbb{Q} \mid b^2 > 2\}$$

The set A is bounded above (by 2, for instance), but it contains no largest element. Similarly, the set B is bounded below, but it contains no smallest element. The real numbers are constructed to "fill in" these gaps. □

1.2 Ordered Sets and Completeness

Definition 1.3 (Ordered Set)**.** An **order** on a set S is a relation, $<$, such that for any $x, y, z \in S$:

1. Exactly one of the statements $x < y$, $x = y$, or $y < x$ is true (Trichotomy).
2. If $x < y$ and $y < z$, then $x < z$ (Transitivity).

Definition 1.4 (Supremum and Infimum)**.** Let S be an ordered set and $E \subset S$.

- If there exists a $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, we say E is **bounded above**, and β is an **upper bound**.
- The **supremum** of E, denoted $\sup E$, is the least upper bound of E.
- The **infimum** of E, denoted $\inf E$, is the greatest lower bound of E.

Principle 1.5 (Least Upper Bound Property)**.** An ordered set S has the **least upper bound property** if every non-empty subset of S that is bounded above has a supremum which exists in S.

Theorem 1.6 (LUB Property $\implies$ GLB Property)**.** Let S be an ordered set with the least upper bound property. If $B \subset S$ is non-empty and bounded below, then $\inf B$ exists in S.

Proof. Let L be the set of all lower bounds for B. Since B is bounded below, L is not empty. Furthermore, since every element of B is an upper bound for L, the set L is bounded above. By the least upper bound property, $\sup L$ exists in S. Let $\alpha = \sup L$. We claim $\alpha = \inf B$.

1. For any $x \in B$, x is an upper bound for L. By definition of the supremum, $\alpha \leq x$. This shows α is a lower bound for B.

2. Let γ be any lower bound for B. Then $\gamma \in L$. By definition of supremum, $\gamma \leq \alpha$.

This shows that α is the greatest lower bound of B. $\square$

2 Lecture 2: Fields and The Real Numbers

2.1 Field Axioms

Definition 2.1 (Field)**.** A **field** is a set F with two operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the field axioms (closure, commutativity, associativity, distributivity, identity elements, and inverse elements).

Remark 2.2. The sets $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields with the usual operations. The set of integers $\mathbb{Z}$ is not a field because not every element has a multiplicative inverse. $\square$

Definition 2.3 (Ordered Field)**.** An **ordered field** is a field F which is also an ordered set, such that for all $x, y, z \in F$:

1. If $y < z$, then $x + y < x + z$.

2. If $x > 0$ and $y > 0$, then $xy > 0$.

2.2 The Real Numbers as a Complete Ordered Field

Definition 2.4 (The Real Numbers)**.** There exists a unique ordered field, $\mathbb{R}$, which has the least upper bound property. We call its elements **real numbers**.

Remark 2.5. The least upper bound property states that every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$. This property distinguishes $\mathbb{R}$ from $\mathbb{Q}$. $\square$

2.3 Fundamental Properties of the Real Numbers

Theorem 2.6 (Archimedean Property)**.** If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that $nx > y$.

Theorem 2.7 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** If $x, y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

Theorem 2.8 (Existence of n-th Roots)**.** For every real number $x > 0$ and every integer $n > 0$, there exists a unique positive real number y such that $y^n = x$. This number is denoted by $\sqrt[n]{x}$ or $x^{1/n}$.

Proof. Uniqueness is clear: if $y_1^n = y_2^n = x$ with $y_1, y_2 > 0$, then if $y_1 < y_2$, we would have $y_1^n < y_2^n$, a contradiction.

For existence, let $E = \{t \in \mathbb{R} \mid t > 0 \text{ and } t^n < x\}$. The set E is non-empty, since $t = x/(x+1) \in E$. (If $x > 1$, $0 < t < 1 \implies t^n < t < x$. If $x \leq 1$, $x + 1 > 1 \implies t < x \leq 1 \implies t^n < t < x$). The set E is bounded above by $1 + x$. (If $t > 1 + x$, then $t > 1$ and $t > x$, so $t^n > t > x$, thus $t \notin E$).

By the least upper bound property, $y = \sup E$ exists. We will show $y^n = x$ by ruling out the other two possibilities. We will use the identity $b^n - a^n = (b - a)(b^{n-1} + \cdots + a^{n-1})$. For $0 < a < b$, this gives $b^n - a^n < (b - a)nb^{n-1}$.

Case 1: $y^n < x$**.** Choose an h such that $0 < h < \frac{x-y^n}{n(y+1)^{n-1}}$ and $h < 1$. Then $(y + h)^n - y^n < h \cdot n(y + h)^{n-1} < h \cdot n(y + 1)^{n-1} < x - y^n$. This implies $(y + h)^n < x$, so $y + h \in E$. But $y + h > y$, which contradicts that y is an upper bound for E. Thus $y^n < x$ is false.

Case 2: $y^n > x$**.** Let $k = \frac{y^n - x}{ny^{n-1}}$. For $0 < k < y$, consider $t = y - k$. Then $t > 0$. We have $y^n - (y - k)^n < (y - (y - k))ny^{n-1} = kny^{n-1} = y^n - x$. This implies $y^n - (y - k)^n < y^n - x$, which simplifies to $x < (y - k)^n$. This means that for any $t \geq y - k$, we have $t^n \geq (y - k)^n > x$, so $t \notin E$. This implies $y - k$ is an upper bound for E. But $y - k < y$, which contradicts that y is the *least* upper bound. Thus $y^n > x$ is false.

Since $y^n \not< x$ and $y^n \not> x$, we must have $y^n = x$. $\square$

3 Lecture 3: Complex Numbers and $\mathbb{R}^n$

3.1 The Extended Real Number System

Definition 3.1 (Extended Real Numbers)**.** The extended real number system $\bar{\mathbb{R}}$ consists of $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We define an order by declaring that for any $x \in \mathbb{R}$, $-\infty < x < +\infty$.

Remark 3.2. In $\bar{\mathbb{R}}$, every subset has a supremum and an infimum. However, $\bar{\mathbb{R}}$ is not a field because $\pm\infty$ do not have additive or multiplicative inverses. $\square$

3.2 The Complex Numbers

Definition 3.3 (Complex Numbers)**.** The **complex numbers**, $\mathbb{C}$, is the set of all ordered pairs (a, b) where $a, b \in \mathbb{R}$. For $x = (a, b)$ and $y = (c, d)$, we define:

- **Addition**: $x + y = (a + c, b + d)$
- **Multiplication**: $x \cdot y = (ac - bd, ad + bc)$

Remark 3.4. We view $\mathbb{R} \subset \mathbb{C}$ by identifying a real number a with the pair $(a, 0)$. By defining $i = (0, 1)$, we find $i^2 = (-1, 0)$, which corresponds to -1. This allows us to write any complex number (a, b) as $a + bi$. The real part is $\text{Re}(z) = a$ and the imaginary part is $\text{Im}(z) = b$. $\square$

Definition 3.5 (Modulus and Conjugate)**.** For a complex number $z = a + bi \in \mathbb{C}$:

- The **conjugate** of z is $\bar{z} = a - bi$.
- The **modulus** (or absolute value) of z is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$.

Theorem 3.6 (Properties of Complex Numbers)**.** Let $z, w \in \mathbb{C}$. Then:

1. $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.
2. $z + \bar{z} = 2\text{Re}(z)$ and $z - \bar{z} = 2i\text{Im}(z)$.
3. $|z| \geq 0$, and $|z| = 0 \iff z = 0$.
4. $|zw| = |z||w|$.
5. **Triangle Inequality:** $|z + w| \leq |z| + |w|$.

Proof of the Triangle Inequality. We start by examining the square of the modulus:

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + 2\mathrm{Re}(z\bar{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\bar{w}| + |w|^2 \quad \text{(since } \mathrm{Re}(x) \leq |x|\text{)} \\
&= |z|^2 + 2|z||\bar{w}| + |w|^2 \\
&= (|z|+|w|)^2
\end{aligned}$$

Taking the square root of both sides yields the desired inequality. □

Theorem 3.7 (Cauchy-Schwarz Inequality)**.** Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n \in \mathbb{C}$. Then

$$\left|\sum_{i=1}^{n} a_i \bar{b_i}\right|^2 \leq \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right)$$

Proof. Let $A = \sum |a_i|^2$, $B = \sum |b_i|^2$, and $C = \sum a_i \bar{b_i}$. We want to show $|C|^2 \leq AB$. If $B = 0$, then all $b_i = 0$, so $C = 0$ and the inequality holds. Assume $B > 0$.

$$\begin{aligned}
0 &\leq \sum_{i=1}^{n} |Ba_i - Cb_i|^2 \\
&= \sum_{i=1}^{n} (Ba_i - Cb_i)(B\bar{a_i} - \bar{C}\bar{b_i}) \\
&= B^2 \sum |a_i|^2 - B\bar{C} \sum a_i \bar{b_i} - BC \sum \bar{a_i} b_i + |C|^2 \sum |b_i|^2 \\
&= B^2 A - B\bar{C}(C) - BC(\bar{C}) + |C|^2 B \\
&= B^2 A - 2B|C|^2 + B|C|^2 = B^2 A - B|C|^2 = B(AB - |C|^2)
\end{aligned}$$

Since $B > 0$, we must have $AB - |C|^2 \geq 0$, which implies $|C|^2 \leq AB$. □

3.3 Euclidean Spaces ($\mathbb{R}^n$)

Definition 3.8 (Euclidean Space)**.** For $n \in \mathbb{N}$, the **n-dimensional Euclidean space**, $\mathbb{R}^n$, is the set of all ordered n-tuples $x = (x_1, \ldots, x_n)$ where each coordinate $x_i \in \mathbb{R}$.

Remark 3.9. For $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we define:

- **Vector Addition:** $x + y = (x_1 + y_1, \ldots, x_n + y_n)$.
- **Scalar Multiplication:** $\alpha x = (\alpha x_1, \ldots, \alpha x_n)$.
- **Inner Product:** $x \cdot y = \sum_{i=1}^{n} x_i y_i$.
- **Norm:** $|x| = \sqrt{x \cdot x} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$.

□

Theorem 3.10 (Properties of the Norm)**.** For $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$:

1. $|x| \geq 0$, and $|x| = 0 \iff x = 0$.
2. $|\alpha x| = |\alpha||x|$.
3. **Cauchy-Schwarz Inequality:** $|x \cdot y| \leq |x||y|$.
4. **Triangle Inequality:** $|x + y| \leq |x| + |y|$.

Proof. Properties (1) and (2) follow from the definition. Property (3) follows from the complex version of the Cauchy-Schwarz inequality. The proof for (4) is analogous to the one for complex numbers. □

Part II

Metric Spaces and Topology

4 Lecture 4: Functions and Cardinality

4.1 Functions and Mappings

Definition 4.1 (Function)**.** A **function** $f : A \to B$ is a rule that assigns to each element $x \in A$ a unique element $f(x) \in B$.

- A is the **domain**.
- B is the **codomain**.
- The set $f(A) = \{f(x) | x \in A\}$ is the **range** of f.

Definition 4.2 (Types of Functions)**.** Let $f : A \to B$ be a function.

- f is **injective** (one-to-one) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- f is **surjective** (onto) if its range is the entire codomain ($f(A) = B$).
- f is **bijective** if it is both injective and surjective.

4.2 Cardinality and Countability

Definition 4.3 (Cardinality)**.** Two sets A and B have the same **cardinality**, written $A \sim B$, if there exists a bijection from A to B.

- A set is **finite** if it is empty or has the same cardinality as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
- A set is **infinite** if it is not finite.
- A set is **countable** if it has the same cardinality as $\mathbb{N}$.
- A set is **uncountable** if it is not countable.

5 Lecture 5: Uncountability and Metric Spaces

5.1 Countability and Uncountability

Theorem 5.1. A countable union of countable sets is countable.

Theorem 5.2. Every subset of a countable set is at most countable

Proof Sketch. Let A be a countable set, so we can list its elements $A = \{x_1, x_2, x_3, \ldots\}$. Let $E \subset A$ be an infinite subset. Let n_1 be the smallest subscript such that $x_{n_1} \in E$. Let n_2 be the smallest subscript greater than n_1 such that $x_{n_2} \in E$. Continuing this process, we can construct a sequence from the elements of E. This establishes a bijection $f(k) = x_{n_k}$ from $\mathbb{N}$ to E, so E is countable. □

Theorem 5.3. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of countable sets, and put $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable.

Proof Sketch. Since each E_n is countable, we can list its elements: $E_n = \{x_{n1}, x_{n2}, x_{n3}, \ldots\}$. We can arrange all the elements of S in an infinite array and then list them by tracing diagonals, starting with x_{11}, then x_{12}, x_{21}, then x_{13}, x_{22}, x_{31}, and so on. This process ensures every element is reached, creating a single sequence that can be mapped to $\mathbb{N}$. □

Theorem 5.4 (Cantor's Diagonalization)**.** Let A be the set of all sequences whose elements are 0 or 1. The set A is uncountable.

Proof. We argue by contradiction. Suppose A is countable. Then we can list all its elements (which are sequences themselves) in a sequence: $s_1, s_2, s_3, \ldots$.

We construct a new sequence s from the list. Let $(s_k)_n$ denote the n-th term of the sequence s_k. We define the k-th term of our new sequence s, denoted $(s)_k$, as follows:

$$(s)_k = 1 - (s_k)_k$$

The sequence s is an element of A, but it cannot be in our list, because for any k, s differs from s_k in the k-th position. This contradicts our assumption that we could list all sequences in A. Therefore, A is uncountable. □

Remark 5.5. The uncountability of $\mathbb{R}$ can be shown using a similar diagonalization argument with decimal or binary expansions. □

5.2 Metric Spaces

Definition 5.6 (Metric Space)**.** A **metric space** is an ordered pair (X, d) where X is a set and d is a **metric** (or distance function) $d : X \times X \to [0, \infty)$ satisfying for all $x, y, z \in X$:

1. $d(x, y) \geq 0$, and $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$ (Symmetry).
3. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality).

Example 5.7. $\mathbb{R}^n$ with the Euclidean distance $d(x, y) = |x - y|$ is a metric space. □

Definition 5.8 (Open Ball)**.** In a metric space (X, d), an **open ball** with center $p \in X$ and radius $r > 0$ is the set

$$B_r(p) = \{x \in X \mid d(p, x) < r\}$$

An open ball is often called a **neighborhood** of p.

Definition 5.9 (Interior Point and Open Set)**.** Let $E \subset X$.

- A point $p \in E$ is an **interior point** of E if there exists $r > 0$ such that $B_r(p) \subset E$.
- The set E is **open** if every one of its points is an interior point.

Definition 5.10 (Limit Point and Closed Set)**.** Let $E \subset X$.

- A point $p \in X$ is a **limit point** of E if every neighborhood of p contains a point $q \in E$ such that $q \neq p$.
- The set E is **closed** if it contains all of its limit points.

Theorem 5.11. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Proof. Suppose there is a neighborhood $B_r(p)$ which contains only a finite number of points of E, say $\{q_1, \ldots, q_n\}$, all distinct from p. Let $r_{min} = \min\{d(p, q_i) \mid i = 1, \ldots, n\}$. Since all $q_i \neq p$, we have $r_{min} > 0$. The neighborhood $B_{r_{min}}(p)$ contains no point $q \in E$ with $q \neq p$. This contradicts the assumption that p is a limit point of E. □

Remark 5.12. A direct consequence of this theorem is that finite sets have no limit points. □

6 Lecture 6: Open Sets, Closed Sets, and Closures

6.1 Set Operations: Complements and De Morgan's Laws

Definition 6.1 (Complement of a Set)**.** Let (X, d) be a metric space and $E \subset X$. The **complement** of E is the set

$$E^c = X \setminus E = \{y \in X \mid y \notin E\}$$

Example 6.2. In $(\mathbb{R}, |\cdot|)$, the complement of $[-1, 1]$ is $[-1, 1]^c = (-\infty, -1) \cup (1, \infty)$. However, in the metric space $(\{x \in \mathbb{R} \mid x \geq 0\}, |\cdot|)$, the complement is $[-1, 1]^c = (1, \infty)$. □

Theorem 6.3 (De Morgan's Laws)**.** Let (X, d) be a metric space and let $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c) \quad \text{and} \quad \left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha (E_\alpha^c)$$

Proof of the first identity. Let $x \in (\bigcup_\alpha E_\alpha)^c$. This holds if and only if $x \notin \bigcup_\alpha E_\alpha$. $\iff$ $x \notin E_\alpha$ for all $\alpha \in A$. $\iff$ $x \in E_\alpha^c$ for all $\alpha \in A$. $\iff$ $x \in \bigcap_\alpha (E_\alpha^c)$. $\square$

6.2 Fundamental Topological Definitions

Definition 6.4 (Limit Point)**.** Let (X, d) be a metric space and $E \subset X$. A point $p \in X$ is a **limit point** of E if every ball centered at p contains at least one point of E other than p. That is, for every $r > 0$,

$$(B_r(p) \setminus \{p\}) \cap E \neq \emptyset$$

The set of all limit points of E is denoted by E'.

Definition 6.5 (Interior Point)**.** Let (X, d) be a metric space and $E \subset X$. A point $p \in E$ is an **interior point** of E if there exists a ball centered at p that is completely contained within E. That is, there exists some $r > 0$ such that

$$B_r(p) \subset E$$

The set of all interior points of E is called the **interior** of E and is denoted by E^0.

Definition 6.6 (Open and Closed Sets)**.** A set E is **open** if every point of E is an interior point. That is, for every $y \in E$, there exists some $r > 0$ such that $B_r(y) \subset E$. A set E is **closed** if its complement, E^c, is open.

6.3 Properties of Open and Closed Sets

Theorem 6.7. Let (X, d) be a metric space. A set $E \subset X$ is open if and only if its complement E^c is closed.

Proof. $\implies$: Assume E is open. Let x be a limit point of E^c. By definition, this means every ball around x contains a point of E^c. Therefore, no ball around x is fully contained in E. Since E is open, this implies x cannot be in E. Thus, $x \in E^c$. Since every limit point of E^c is in E^c, the set E^c is closed.

$\impliedby$: Assume E^c is closed. Let $x \in E$. Then x is not in E^c. Since E^c contains all its limit points, x cannot be a limit point of E^c. This means there exists some $r > 0$ such that the ball $B_r(x)$ contains no points of E^c. Thus, $B_r(x) \subset E$, which makes x an interior point of E. Since x was an arbitrary point in E, E is open. $\square$

Theorem 6.8 (Unions and Intersections).
1. The union of any collection of open sets is open.
2. The intersection of any collection of closed sets is closed.
3. The intersection of a finite number of open sets is open.
4. The union of a finite number of closed sets is closed.

Proof. We prove (1) and (3). The other two follow from De Morgan's laws.

For (1), let $\{E_\alpha\}_{\alpha \in A}$ be a collection of open sets. If $x \in \bigcup_\alpha E_\alpha$, then $x \in E_\beta$ for some index $\beta \in A$. Since E_β is open, there exists an $r > 0$ such that $B_r(x) \subset E_\beta$. But $E_\beta \subset \bigcup_\alpha E_\alpha$, so $B_r(x) \subset \bigcup_\alpha E_\alpha$. Thus, the union is open.

For (3), let $F_1, \ldots, F_N$ be open sets. If $x \in \bigcap_{i=1}^N F_i$, then $x \in F_i$ for every $i \in \{1, \ldots, N\}$. Since each F_i is open, for each i there exists an $r_i > 0$ such that $B_{r_i}(x) \subset F_i$. Let $r = \min\{r_1, \ldots, r_N\}$. Since this is a finite set of positive numbers, $r > 0$. Then $B_r(x) \subset B_{r_i}(x) \subset F_i$ for all i. Therefore, $B_r(x) \subset \bigcap_{i=1}^N F_i$, so the finite intersection is open. □

Remark 6.9. An infinite intersection of open sets is not necessarily open. For example, in $\mathbb{R}$, $\bigcap_{n=1}^\infty (-1/n, 1/n) = \{0\}$, which is a closed set. □

6.4 Closure of a Set

Definition 6.10 (Closure). Let (X, d) be a metric space and $E \subset X$. The **closure** of E, denoted $\bar{E}$, is defined as $\bar{E} = E \cup E'$.

Theorem 6.11. For any set $E \subset X$:
1. $\bar{E}$ is a closed set.
2. $E = \bar{E}$ if and only if E is closed.
3. If F is any closed set containing E, then $\bar{E} \subset F$. (This means $\bar{E}$ is the smallest closed set containing E).

Proof. **For (1):** We show that $(\bar{E})^c$ is open. Let $x \in (\bar{E})^c$. This means $x \notin E$ and $x \notin E'$. Since x is not a limit point of E, there exists an $r > 0$ such that $B_r(x) \cap E = \emptyset$. We claim this ball also contains no limit points of E. Let $y \in B_r(x)$. Since $B_r(x)$ is open, there is a smaller ball $B_s(y) \subset B_r(x)$. This smaller ball $B_s(y)$ also has no intersection with E, so y cannot be a limit point of E. Thus, $B_r(x) \subset (E')^c$. Since $B_r(x) \subset E^c$ and $B_r(x) \subset (E')^c$, we have $B_r(x) \subset (E \cup E')^c = (\bar{E})^c$. Thus, $(\bar{E})^c$ is open, which means $\bar{E}$ is closed.

For (2): $E = \bar{E} \iff E = E \cup E' \iff E' \subset E$. This is the definition of a closed set.

For (3): Let F be a closed set with $E \subset F$. Let $x \in E'$. Then every neighborhood of x contains points of E, and therefore contains points of F. So x is a limit point of F. Since F is closed, it contains all its limit points, so $x \in F$. This shows $E' \subset F$. Since we already have $E \subset F$, it follows that $E \cup E' \subset F$, which means $\bar{E} \subset F$. □

6.5 Application in $\mathbb{R}$: The Supremum Property

Theorem 6.12. If $E \subset \mathbb{R}$ is non-empty and bounded above, then $\sup E \in \bar{E}$. In particular, if E is also closed, then $\sup E \in E$.

Proof. Let $\alpha = \sup E$. For any $\epsilon > 0$, the number $\alpha - \epsilon$ is not an upper bound for E. Therefore, there must exist some point $x \in E$ such that $\alpha - \epsilon < x \leq \alpha$. This means that for every $\epsilon > 0$, the interval $(\alpha - \epsilon, \alpha + \epsilon) = B_\epsilon(\alpha)$ contains a point $x \in E$. If $\alpha \in E$, we are done. If $\alpha \notin E$, then every neighborhood of α still contains a point x from E where $x \neq \alpha$. This is precisely the definition of a limit point. Therefore, $\alpha \in E'$, which implies $\alpha \in E \cup E' = \bar{E}$. □

7 Lecture 7: Relative Topology and Compactness

7.1 Set Operations and Topology

We begin by formalizing the interaction between set operations (unions/intersections) and the metric topology.

Principle 7.1 (De Morgan's Laws)**.** Let $\{E_\alpha\}_{\alpha \in A}$ be a collection of sets. De Morgan's laws state how the complement interacts with unions and intersections:

1. The complement of a union is the intersection of the complements:

$$\left(\bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

2. The complement of an intersection is the union of the complements:

$$\left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

For two sets, this simplifies to the familiar forms: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Theorem 7.2 (Properties of Open and Closed Sets)**.** Let (X, d) be a metric space.

1. The union of **any** collection of open sets is open.
2. The intersection of **any** collection of closed sets is closed.
3. The intersection of a **finite** collection of open sets is open.
4. The union of a **finite** collection of closed sets is closed.

7.2 Relative Topology

Definition 7.3 (Relative Openness)**.** Let (X, d) be a metric space, and $E \subset Y \subset X$. We say that E is **open relative to** Y if E is open in the metric space (Y, d); i.e., for each $x \in E$ there exists $r > 0$ such that:

$$B_r^X(x) \cap Y \subset E$$

Example 7.4. $(a, b) \subset \mathbb{R} \subset \mathbb{R}^2$. □

Example 7.5. Viewing $\mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$, then $(a, b) \subset \mathbb{R}$ is open, but $(a, b) \times \{0\}$ is **not** open in $\mathbb{R}^2$ (it contains no open ball in the plane). However, $(a, b) \times \{0\} = \big((a, b) \times \mathbb{R}\big) \cap (\mathbb{R} \times \{0\})$. □

Theorem 7.6. Let (X, d) be a metric space, and $E \subset Y \subset X$, then E is open relative to $Y \iff E = F \cap Y$ for some open set $F \subset X$.

Proof. $\implies$: If E is open relative to Y, then for each $x \in E$, we have some $r_x > 0$ with $B_{r_x}^X(x) \cap Y \subset E$. Let

$$F = \bigcup_{x \in E} B_{r_x}^X(x) \subset X.$$

This union of open balls is open in X. We're done if we show that $F \cap Y = E$.

- $(\subset)$: If $x \in E$, then $x \in B_{r_x}^X(x) \subset F$. Since $E \subset Y$, $x \in Y$. Thus $x \in F \cap Y$.
- $(\supset)$: If $y \in F \cap Y$, then $y \in B_{r_x}^X(x)$ for some $x \in E$. Since $y \in Y$, we have $y \in B_{r_x}^X(x) \cap Y \subset E$.

$\impliedby$: If $E = F \cap Y$ where $F \subset X$ is open. Let $x \in E$. Then $x \in F$. As F is open, there is some $r > 0$ with $B_r^X(x) \subset F$. But as $x \in E \subset Y$, we consider the intersection with Y:

$$B_r^X(x) \cap Y \subset F \cap Y = E$$

This implies E is open relative to Y. □

7.3 Compactness

Openness depends on the space the set lies in (as seen in relative topology). Compactness, which is a notion of ”smallness” or finiteness, does not!

Definition 7.7 (Compactness)**.** Let (X, d) be a metric space, and $E \subset X$. We say that a collection $\{F_\alpha\}_{\alpha \in A}$ of open sets in X is an **open cover** of E if

$$E \subset \bigcup_{\alpha \in A} F_\alpha.$$

We say that E is **compact** (in X) if every open cover has a finite subcover; i.e., if $\{F_\alpha\}_{\alpha \in A}$ covers E, then there exist indices $\alpha_1, \ldots, \alpha_N \in A$ such that

$$E \subset \bigcup_{i=1}^{N} F_{\alpha_i}$$

Remark 7.8. Every finite subset of a metric space is compact. This concept is of **central importance** in analysis. □

Theorem 7.9. Let (X, d) be a metric space and $K \subset Y \subset X$. Then K is compact in $Y \iff K$ is compact in X.

Proof. $(\implies)$: Suppose K is compact in Y. Let $\{F_\alpha\}_{\alpha \in A}$ be open sets in X which cover K. By the relative open set theorem, $F_\alpha \cap Y$ are open in Y. Since $K \subset Y$,

$$K \subset \left(\bigcup_{\alpha \in A} F_\alpha \right) \cap Y = \bigcup_{\alpha \in A} (F_\alpha \cap Y).$$

The collection $\{F_\alpha \cap Y\}_{\alpha \in A}$ is an open cover of K in Y. As K is compact in Y, there exist $\alpha_1, \ldots, \alpha_N$ such that

$$K \subset \bigcup_{i=1}^{N} (F_{\alpha_i} \cap Y) \subset \bigcup_{i=1}^{N} F_{\alpha_i}.$$

Thus, we found a finite subcover in X, so K is compact in X.

$(\impliedby)$: Suppose K is compact in X. Let $\{U_\alpha\}_{\alpha \in A}$ be open sets in Y covering K. By the relative open set theorem, $U_\alpha = G_\alpha \cap Y$ for some open sets G_α in X. Then $\{G_\alpha\}$ is an open cover of K in X. Since K is compact in X, there exist $\alpha_1, \ldots, \alpha_N$ such that $K \subset \bigcup_{i=1}^{N} G_{\alpha_i}$. Since $K \subset Y$, we have:

$$K \subset \left(\bigcup_{i=1}^{N} G_{\alpha_i} \right) \cap Y = \bigcup_{i=1}^{N} (G_{\alpha_i} \cap Y) = \bigcup_{i=1}^{N} U_{\alpha_i}.$$

Thus K is compact in Y. □

Remark 7.10. We can now unambiguously say "Let K be a compact metric space" without specifying the ambient space it lives in. □

Theorem 7.11. Compact sets are closed.

Proof. Let $K \subset X$ be compact. We show K^c is open. Fix $x \in K^c$. For each $y \in K$, let $r_y = \frac{1}{2}d(x,y)$. The collection $\{B_{r_y}(y)\}_{y \in K}$ is an open cover of K. Since K is compact, there exist finitely many points $y_1, \ldots, y_N \in K$ such that

$$K \subset \bigcup_{i=1}^{N} B_{r_{y_i}}(y_i) = V.$$

Let $W = \bigcap_{i=1}^{N} B_{r_{y_i}}(x)$. Since this is a finite intersection of open balls, W is open. By the triangle inequality and our choice of radius ($r_y = \frac{1}{2}d(x,y)$), the balls $B_{r_{y_i}}(y_i)$ and $B_{r_{y_i}}(x)$ are disjoint. Therefore $V \cap W = \emptyset$. Since $K \subset V$, we have $W \cap K = \emptyset$, so $W \subset K^c$. Since $x \in W$ and W is open, x is an interior point of K^c. Thus K^c is open, and K is closed. ☐

Theorem 7.12. Closed subsets of compact sets are compact.

Proof. Let $F \subset K$ be closed, where K is compact. Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of F. Consider the collection $\Omega = \{V_\alpha\}_{\alpha \in A} \cup \{F^c\}$. Since F is closed, F^c is open. Since $\{V_\alpha\}$ covers F, and F^c covers the rest of X, Ω is an open cover of K. Since K is compact, there is a finite subcover of Ω. This subcover may or may not include F^c. If we remove F^c from the finite subcover, the remaining sets still cover F (since F^c covers nothing in F). These remaining sets form a finite subcollection of $\{V_\alpha\}$ that covers F. Thus F is compact. ☐

8 Lecture 8: The Heine-Borel Theorem

Theorem 8.1. Let (X,d) be a metric space and $\{K_\alpha\}_{\alpha \in A}$ be compact subsets with the property that **any** finite intersection is non-empty, i.e. $K_{\alpha_1} \cap \ldots \cap K_{\alpha_n} \neq \emptyset$ if $\alpha_1, \ldots, \alpha_n$. Then,

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

In particular if $\{K_n\}_{n \geq 1}$, $K_n \supset K_{n+1}$ and $K_n \neq \emptyset$ compact sets, then $\cap_{n \geq 1} K_n \neq \emptyset$.

Proof. If not, i.e. $\cap_{\alpha \in A} K_\alpha = \emptyset$, there is some $\alpha_0 \in A$, with **no** point of K_{α_0} in **all** of the K_α. This implies that each $x \in K_{\alpha_0}$ is such that $x \in K_\alpha^c$ for some $\alpha \in A$. Thus, $K_{\alpha_0} \subset \bigcup_{\alpha \in A, \alpha \neq \alpha_0} K_\alpha^c$, but K_α are compact, hence closed. So, $\{K_\alpha^c\}_{\alpha \in A}$ is an open cover of K_{α_0}. Thus there are $\alpha_1, \ldots, \alpha_n \in A$ with $K_{\alpha_0} \subset \bigcup_{i=1}^{n} K_{\alpha_i}^c$. Thus, $K_{\alpha_0} \cap K_{\alpha_1} \cap \ldots \cap K_{\alpha_n} = \emptyset$, which contradicts the assumption! ☐

Theorem 8.2. Let $E \subset K$ be an **infinite subset*** of a compact set, then E has a limit point in K.

Proof. If not, every $x \in K$ is such that there is some $r_x > 0$, with

$$B_{r_x}(x) \cap E \subset \{x\}$$

(i.e. nbhd of x contains at most **one** point of E

Then, $K \subset \bigcup_{x\in K} B_{r_x}(x)$, and so by compactness there are $x_1, \ldots, x_N \in K$ with $K \subset \bigcup_{i=1}^N B_{r_{x_i}}(x_i)$. Then

$$E = K \cap E \subset \bigcup_{i=1}^{N} \left(B_{r_{x_i}}(x_i) \cap E\right) \tag{1}$$

$$\subset \bigcup_{i=1}^{N} \{x_i\} \tag{2}$$

But E is infinite, so we have a contradiction □

Let's focus on $\mathbb{R}^n$ now.

Theorem 8.3. Let $\{I_n\}_{n\geq 1}$ be a sequence of closed intervals, i.e. $I_n = [a_n, b_n], (a_n \leq b_n)$, such that $I_n \supset I_{n+1}$ $(I_{n+1} \subset I_n)[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$.

Proof. Let $I_n = [a_n, b_n]$, with $a_n \leq b_n$ for $n \geq 1$. Set

$$E = \{a_n | n \geq 1\},$$

then E is bounded above by b_1 (because $I_n \supset I_{n+1} \implies b_{n+1} \leq b_n$).

Also, $E \neq \emptyset$ since $a_1 \in E$, so the Least Upper Bound property implies that $\sup E$ exists!

As $I_n \supset I_{n+1}$ we have $a_n \leq a_m \leq b_m \leq b_n$ for all $m \geq n$.

Hence, $a_m \leq \sup E \leq b_m$ (if not it contradicts the definition of a supremum) for all $m \geq 1$.

This implies $\sup E \in I_m$ for all m Thus

$$\sup E \in \cap_{n\geq 1} I_n$$

□

Let's generalize this to $\mathbb{R}^n$.

Definition 8.4. An **n-cell** in $\mathbb{R}^n$ is a set

$$I = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$$

Theorem 8.5. If $\{I_n\}_{n\geq 1}$ are n-cells with $I_n \supset I_{n+1}$ for $n \geq 1$, then $\cap_{n\geq 1} I_n \neq \emptyset$.

Proof. Let $I_m = [a_1^m, b_1^m] \times \ldots \times [a_n^m, b_n^m]$, and repeat the previous theorem in each factor □

8.1 Compactness in $\mathbb{R}^n$ and the Heine-Borel Theorem

Theorem 8.6 (n-cells are compact)**.** Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ be an n-cell. Its diameter is $\delta = \left(\sum_{i=1}^n (b_i - a_i)^2\right)^{1/2}$. For any $x, y \in I$, we have $|x - y| \leq \delta$.

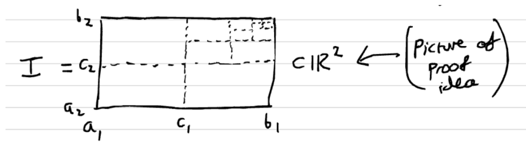


Figure 1: Illustration of the bisection method used in the proof of compactness. The original n-cell is repeatedly divided, and a sub-cell that has no finite subcover is chosen at each step.

Proof. We argue by contradiction. Assume there exists an open cover $\{F_\alpha\}_{\alpha \in A}$ of I that has no finite subcover. We will construct a sequence of nested n-cells to find a contradiction.

Step 1: Construct a sequence of n-cells. Let $I_0 = I$. We bisect each interval $[a_i, b_i]$ at its midpoint $c_i = (a_i + b_i)/2$. This partitions I_0 into 2^n sub-cells. Since I_0 cannot be covered by a finite number of sets from $\{F_\alpha\}$, at least one of these 2^n sub-cells must also not have a finite subcover. Let's choose one such sub-cell and call it I_1. We repeat this process. By induction, we obtain a sequence of n-cells $\{I_k\}_{k=1}^{\infty}$ with the following properties:

1. $I_0 \supset I_1 \supset I_2 \supset \ldots$
2. For each k, I_k cannot be covered by any finite subcollection of $\{F_\alpha\}$.
3. If $x, y \in I_k$, then $|x - y| \leq 2^{-k}\delta$.

Step 2: Find the contradiction. By the Nested n-cell Theorem (Theorem 8.5 from the previous lecture), the intersection of these non-empty, nested cells is non-empty. So, there exists a point $\tilde{x}$ such that $\tilde{x} \in \bigcap_{k=1}^{\infty} I_k$. Since $\tilde{x} \in I$, and $\{F_\alpha\}$ is an open cover of I, there must be some set F_{α_0} in the cover such that $\tilde{x} \in F_{\alpha_0}$. Because F_{α_0} is an open set, there exists an $r > 0$ such that the open ball $B_r(\tilde{x}) \subset F_{\alpha_0}$. By the Archimedean property, we can choose an integer k large enough so that the diameter of I_k, which is $2^{-k}\delta$, is smaller than r. Since $\tilde{x} \in I_k$ and the diameter of I_k is less than r, the entire cell I_k must be contained within the ball centered at $\tilde{x}$.

$$I_k \subset B_r(\tilde{x}) \subset F_{\alpha_0}$$

This shows that the cell I_k is covered by a single set, F_{α_0}. This is a finite subcover (of size one). However, this contradicts property (2) of our construction, which states that no I_k has a finite subcover. Thus, our initial assumption must be false, and every open cover of I must have a finite subcover. Therefore, every n-cell is compact. □

Below is the main theorem of this lecture:

Theorem 8.7. Let $E \subset \mathbb{R}^n$, then the following are equivalent:

1. E is compact
2. E is closed and bounded
3. Every infinite subset of E has a limit point in E

Remark 8.8. Bounded means that there is some constant $m > 0$ with $|x - y| \leq m$ if $x, y \in E$. This is equivalent of being able to put the whole set in a ball. □

Remark 8.9. (1) $\iff$ (2) is called the Heine-Borel Theorem $\square$

Remark 8.10. In general, (1) $\iff$ (3) in metric spaces, but (2) does not imply (1) and (2) does not imply (3) in general. $\square$

Proof. For (2) $\implies$ (1), if E is closed and bounded, $E \subset I$ for some $n - cell$, but n-cells are compact, so E being closed implies that E is compact.

For (1) $\implies$ (3), it follows by a previous result (that every compact set is such that if you take an infinite subset, a limit point lies inside).

For (3) $\implies$ (2), if every infinite subset of E has a limit point in E, then we first note that E must be bounded; if not there must exist some sequence $(x_n) \subset E$, with $|x_n| > n$ for each $n \geq 1$, this sequence is an infinite subset of E with no limit points.

Hence we see that E must be bounded, it remains to show that E is closed.

If E were not closed, then there must exist a limit point of E, $x \in E^c$. Since x is a limit point of E there must exist some sequence $(x_n) \subset E$ such that $|x - x_n| < \frac{1}{n}$ for $n \geq 1$.

As $(x_n) \subset E$ is an infinite subset of E there must exist some limit point in E. If $(x_n) \subset E$ had some limit point $y \in E$ such that $y \neq x$, then as

$$|x - y| = |x - x_n + x_n - y| \leq |x - n| + |x_n - y|,$$

by the triangle inequality. And so,

$$|x_n - y| \geq |x - y| - |x - x_n| > \frac{|x - y|}{2} > 0$$

for $n \geq 1$ sufficiently large (since $|x - x_n| < \frac{|x-y|}{2}$ for large n). But then y cannot be a limit point of (x_n); hence x is the only limit point of $(x_n) \implies x \in E$, contradicting our assumption that $x \in E^c$. Thus E is closed.

We thus have (2) $\implies$ (1) $\implies$ (3) $\implies$ (2) and we are done. $\square$

As $n-$cells are compact, we also have

Theorem 8.11 (Every bounded infinite subset of $\mathbb{R}^n$ has a limit point)**.**

Proof. If $E \subset \mathbb{R}^n$ is bounded then there is some $n-$cell, $I \subset \mathbb{R}^n$, such that $E \subset I$. As the $n-$cells are compact and E is infinite, then we saw that E must have a finite limit point in I. $\square$

9 Lecture 9: Perfect Sets, Cantor Set

9.1 Perfect Sets

Definition 9.1 (Perfect Set)**.** Let (X, d) be a metric space, we say that $P \subset X$ is perfect (in X) if $P = P' \iff P$ is closed and has no isolated points.

Remark 9.2. If P $(\neq \emptyset)$ is perfect, it implies that P cannot be finite. $\square$

Example 9.3.

1. $\emptyset$ is perfect!
2. $[a, b]$ with $a \neq b$ is perfect in $\mathbb{R}$
3. $\mathbb{R}$ itself is perfect in $\mathbb{R}$
4. $[0, 1] \cap \mathbb{Q} \subset \mathbb{Q}$ is perfect. This is because if you take any rational number between 0 and 1, it is in the set and also is a limit point of the set.
5. $[0, 1] \cap \mathbb{Q} \subset \mathbb{R}$ is **not** perfect. Take $\frac{1}{\pi}$, this can also be approximated by rationals, thus is a limit point, but it is not in the set.
6. **Cantor Set**: to be constructed later. But it is perfect in $\mathbb{R}$ and has no open intervals contained inside it.!

$\square$

Theorem 9.4. If $P \subset \mathbb{R}^n$ is perfect, then P is uncountable.

Proof. As $P \neq \emptyset \implies P$ is infinite. Assuming P were countable we can write $P = (x_n)$ with $x_n = x_m \iff n = m$. We construct a sequence of compact sets, (K_n), such that $K_{n+1} \subset K_n \subset P$ with $\bigcap_{n\geq 1} K_n \neq 0$ but $x_n \notin K_n$ for each $n \geq 1$.

For $x_1 \in P$, there is some $\tilde{x_1} \in P$ with $\tilde{x_1} \in (B_1(x_1)\backslash\{x_1\} \cap P$. Choose

$$r_1 = \frac{1}{2}\min\{|x_1 - \tilde{x_1}|, 1 - |x_1 - \tilde{x_1}|\} > 0$$

then $x_1 \notin B_r(\tilde{x_1})$ and $\overline{B_{r_1}(\tilde{x_1})} \subset B_1(x_1)$

Next, we have some $\tilde{x_2} \in \big(B_{r_1}(\tilde{x_1})\backslash\{\tilde{x_1}, \tilde{x_2}\} \cap P$. Then setting

$$r_2 = \frac{1}{2}\min\{|\tilde{x_1} - x_2|, r_1 - |\tilde{x_1} - \tilde{x_2}\} > 0,$$

with $\overline{B_{r_2}(\tilde{x_2})} \subset B_{r_1}(\tilde{x_1})$ and $x_2 \notin B_{r_2}(\tilde{x_2})$.

Above was the base case. Inductively,we get $r_n > 0$ and $\tilde{x_n} \in P$ with $x_n \notin B_{r_n}(\tilde{x_n})$ and $\overline{B_{r_n}(\tilde{x_n})} \subset B_{r_{n-1}}(\tilde{x_{n-1}})$ for $n \geq 2$. Also, $\overline{B_{r_n}(\tilde{x_n})}$ is closed and bounded, hence by Heine-Borel, this implies compact! Let us set $K_n = \overline{B_{r_n}(\tilde{x_n})} \cap P$ then $K_{n+1} \subset K_n, K_n$ is compact, and $x_n \notin K_n$ for all $n \geq 1$.

On the other hand, this implies that $\bigcap_{b\geq 1} K_n \neq \emptyset$, $\bigcap_{n\geq 1} K_n \subset P$, but on the other hand $x_m \notin \bigcap_{n\geq 1} K_n$ for any $m \geq 1 \implies \big(\bigcap_{n\geq 1} K_n\big) \cap P = \emptyset$

This is a contradiction! $\square$

9.2 Cantor Set

We will now construct a perfect (and nonempty) subset of $\mathbb{R}$ that contains no open interval. Taking inspiration from how we proved it above, we're going to take the intersection of compact sets.

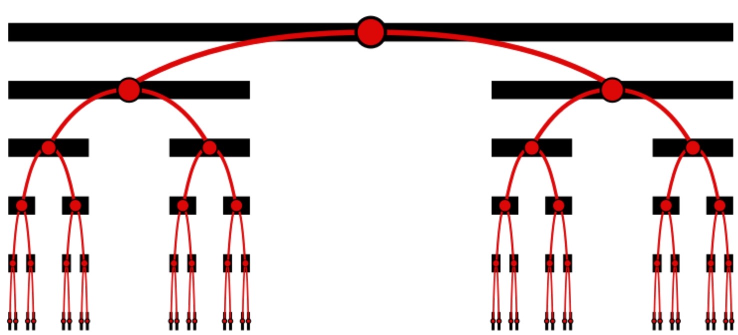


Figure 2: Cantor Set Expansion: each point in the set is represented here by a vertical line.

The cantor set is taking the union of the "dust" left over from this process.

Definition 9.5 ("Middle Third" Cantor Set Construction)**.** Let $K_0 = [0,1], K_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and inductively remove the middle third of each closed interval to obtain compact sets. $K_n \subset [0,1]$, where K_n consists of z^n disjoint closed intervals of length 3^{-n} with $K_{n+1} \subset K_n$.
Then

$$\mathcal{C}_{\frac{1}{3}} = \bigcap_{n \geq 1} K_n \neq \emptyset \text{ is the "middle third" Cantor Set.}$$

Proof. To see that $\mathcal{C}_{\frac{1}{3}}$ is perfect, if $x \in \mathcal{C}_{\frac{1}{3}}$, let $I_n \subset K_n$ be the interval of length 3^{-n} containing x. Let x_n be an endpoint of I_n **not** equal to x.

Then as $x_i x_n \in I_n \implies |x - x_n| \leq 3^{-n}$. Note that $x_n \in \mathcal{C}_{\frac{\infty}{3}}$ also!

Then we use the Archimedean Property; if $r > 0$, taking $n \geq 1$ such that $3^{-n} < r$, this implies $\big(B_r(x) \backslash \{x\} \cap \mathcal{C}_{\frac{1}{3}} \neq \emptyset$. So $x \in \mathcal{C}'_{\frac{1}{3}} \implies \mathcal{C}'_{\frac{1}{3}} = \mathcal{C}_{\frac{1}{3}}$. So $\mathcal{C}_{\frac{1}{3}}$ is perfect (and $\neq \emptyset$), which implies that it is uncountable and perfect.

If $(a,b) \subset \mathcal{C}_{\frac{1}{3}}$, and $x \in (a,b)$, by construction for some $n \geq 1$ we have $x \in I_n \subset (a,b)$ where $I_n \subset K_n$. But then I_{n+1} removes the middle third of I_n (i.e. if $I_n = [t_n, s_n] \implies I_{n+1} = [t_n, t_n + \frac{s_n - t_n}{3}] \cup [s_n - \frac{s_n - t_n}{3}, s_n]$ $(I_n \subset K_{n_1})$ but then $(t_n + \frac{s_n + t_n}{3}, s_n - \frac{s_n - t_n}{3} \not\subset (a,b)$ so $\mathcal{C}_{\frac{1}{3}}$ contains no open intervals! $\square$

10 Lecture 10: Connectedness

Definition 10.1. Let (X,d) be a metric space and $E \subset X$, then we say that E is **connected in** X if there do **not** exist open sets $A, B \subset X$ with $A \cap B = \emptyset$ such that $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$ and $E \subset A \cup B$ (if it is possible, we will say E is disconnected).

Definition 10.2 (Contrapositive)**.**

$$P \implies Q \iff \text{not } Q \implies \text{not } P$$

10.1 Properties of Connected Sets

Theorem 10.3. Let (X,d) be a metric space, $E \subset X$ is connected in $X \iff E$ is connected in E.

Proof. $(\Longleftarrow)$: If E is not connected in X, then there are open sets $A, B \subset X$ with $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$, $E \subset A \cup B, A \cap B = \emptyset$. Then, $A \cap E$ and $B \cap E$ are open in E,

$$\begin{aligned} E &= (A \cap E) \cup (B \cap E) \\ &= (A \cup B) \cap E \end{aligned}$$

So E is **not** connected in E.

($\implies$) : If E is not connected in E, then $E \subset$

If E is not connected in E, then $E \subset C \cup D$ (open sets), $C \cap E, D \cap E \neq \emptyset, C \cap D = \emptyset$. Since C, D are open, for $x \in C$ and $y \in D$, there are $r_x, r_y > 0$, with $B_{r_x}(x) \cap E \subset C, B_{r_y}(y) \cap E \subset D$. Since $C \cap D = \emptyset \implies d(x,y) \geq \max\{r_x, r_y\} > 0$.

(If not, i.e. $d(x,y) < r_y \implies x \in B_{r_y}(y) \cap E \subset E$, which is a contradiction.)

Then for each such $x \in C$, $y \in D$, we let $\tilde{r_x} = \frac{r_x}{2}, \tilde{r_y} = \frac{r_y}{2}$, so that if $z \in B_{\tilde{r_x}}(x) \cap B_{\tilde{r_y}}(y)$ then

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(y,z) \\ &< \tilde{r_x} + \tilde{r_y} = \frac{r_x}{2} + \frac{r_y}{2} \\ &\leq \max\{r_x, r_y\} \end{aligned}$$

This is a contradiction. So $B_{\tilde{r_x}}(x) \cap B_{\tilde{r_y}}(y) = \emptyset$! Setting

$$A = \bigcup_{x \in C} (B_{\tilde{r_x}}(x)) \subset X, B = \bigcup_{y \in D} B_{\tilde{r_y}}(y) \subset X$$

are open.

This implies that $A \cap B =, E \subset A \cup B, A \cap E, B \cap E \neq \emptyset$.

Thus, E is **not** connected in X. □

Remark 10.4. Connectedness is a **topological** property. □

Definition 10.5 (Clopen)**.** A set $E \subset X$ is **clopen** in X if it is both closed and open.

Remark 10.6. $\emptyset, X$ are open □

Remark 10.7. If $x = (0,1) \cup (2,3)$, then $(0,1)$ is clopen in X! □

Theorem 10.8. Let (X,d) be a metric space, X is connected if and only if $\emptyset, X$ are the only clopen sets in X.

Proof. ($\impliedby$) : If X is not connected then there are $A, B \neq \emptyset$ open in X with $A \cap B = \emptyset, X = A \cup B$.

Then $A^c = B, B^c = A \implies B, A$ are closed. Hence, A, B are clopen and $\neq \emptyset, X$

($\implies$) : If $A \subset X$ is non-empty, $\neq X$, and clopen, then A^c is clopen, then $X = A \cup A^c, A \cap A^c = \emptyset$. And A, A^c are open, so X is **not** connected. □

Theorem 10.9. $E \subset \mathbb{R}$ is connected $\iff$ whenever $x, y \in E$ and $x < y \implies (x, y) \subset E$, i.e. if $x \leq z \leq y \implies z \in E$. Thus E must be one of

$$\mathbb{R}, (-\infty, b], (-\infty, b), [a, +\infty), (a, \infty), [a, b], [a, b), (a, b], (a, b)$$

for $a \leq b$.

Proof. ($\implies$) : If $x, y \in E$, $x < y$ but for some $x < z < y$, $z \notin E$, then $E \subset (-\infty, z) \cup (z, \infty)$. This is open, disjoint, and not equal to the empty set. x is in the first one and y is in the second one. This is contradicts E being connected.

($\impliedby$) : If E is not connected, let $A, B \subset \mathbb{R}$ be open, $A \cap E, B \cap E \neq \emptyset$, $E \subset A \cup B$. Let $x \in A, y \in B$ with $x < y$, without loss of generality (since $A \cap B = \emptyset$)).

set $z = \sup(A \cap [x, y])$, which exists since $x \in (A \cap [x, y])$ and bounded above by y.

Now $z \notin B$, because otherwise if $z \in B$, it would imply that as B open $z - \epsilon$ for some $\epsilon > 0$).

As B is open, if $z \in B$ this implies that $z - \epsilon \in B$ for some $\epsilon > 0$, but as $A \cap B = \emptyset$, $z - \epsilon$ would be a **smaller** upper bound for $A \cap [x, y]$.

Hence $z < y$. Similarly, if $z \in A$, as A is open $z + \delta \in A \cap [x, y]$ for some $\delta > 0$. So z would **not** be an upper bound.

This implies that $x < z$ and $z \notin A$.

Thus, $z \in A^c \cap B^c = (A \cap B)^c \subset E^c$. ($E \subset A \cup B$) and ($(A \cup B)^c \subset E^c$). So $z \notin E \implies$ property does **not** hold in E.

$\square$

END EXAM 1 CONTENT

Part III

Convergence and Series

11 Lecture 11: Convergence in Metric Spaces

We now have built up the machinery to precisely define and study limits of Sequences in metric spaces. As a consequence we are also able to make sense of infinite sums, i.e. series.

11.1 Definition of Convergence

Definition 11.1. Let (X, d) be a metric space, then a Sequence $(x_n) \subset X$, is said to **converge** to a point $x \in X$ if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that if $n \geq N$ then $d(x_n, x) < \epsilon$. We then say that x is the **limit** of (x_n) and write

$$x_n \to x \quad \text{or} \quad \lim_{n\to\infty} x_n = x.$$

If (x_n) does not converge then it **diverges**.

Remark 11.2. $x_n \to x \iff$ for every $\epsilon > 0$, $x_n \in B_\epsilon(x)$ for all sufficiently large n. Sometimes we say **eventually** to mean there is some $N \in \mathbb{N}$ such that a property holds for all $n \geq N$; thus $x_n \to x \iff (x_n)$ eventually lies in every neighborhood of x. We must specify convergence in X since, for example, $(1/n)$ converges to 0 in $\mathbb{R}$ but does not converge in $\mathbb{R} \setminus \{0\}$. □

Example 11.3.

- $(1/n) \to 0$ in $\mathbb{R}$ as noted above.
- (n^2) diverges and is unbounded.
- $((-1)^n)$ and $((i)^n)$ diverge but are bounded.
- $(1 + (-1)^n/n) \to 1$ in both $\mathbb{Q}$ and $\mathbb{R}$.
- Any constant sequence converges.
- $(e^{i/n}) \to 1$ in both $\mathbb{C}$ and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

□

11.2 Uniqueness and Boundedness

Theorem 11.4. Let (X, d) be a metric space and $(x_n) \subset X$ a sequence. Then:

1. Limits are unique.
2. Convergent sequences are bounded.
3. $x_n \to x \iff$ Every neighbourhood of x contains all but finitely many of the terms of (x_n).
4. If $E \subset X$ and x is a limit point of E, then there is a sequence $(x_n) \subset E$ such that $x_n \to x$.

Proof. For (**1**), if there exist $x, \tilde{x} \in X$ such that both $x_n \to x$ and $x_n \to \tilde{x}$, then for each $\epsilon > 0$ there exist $N, \tilde{N} \in \mathbb{N}$ such that $d(x_n, x) < \epsilon/2$ for $n \geq N$ and $d(x_n, \tilde{x}) < \epsilon/2$ for $n \geq \tilde{N}$. Thus, for $n \geq \max(N, \tilde{N})$ we have $d(x, \tilde{x}) \leq d(x, x_n) + d(x_n, \tilde{x}) < \epsilon/2 + \epsilon/2 = \epsilon$. Since $d(x, \tilde{x}) < \epsilon$ for all $\epsilon > 0$, we must have $d(x, \tilde{x}) = 0$, so $x = \tilde{x}$.

For (**2**), if $x_n \to x$ for some $x \in X$, then there is some $N \in \mathbb{N}$ such that $d(x_n, x) \leq 1$ for all $n \geq N$. Setting $M = \max\{1, d(x_1, x), \ldots, d(x_{N-1}, x)\}$. We see that $d(x_n, x) \leq M$ for all $n \geq 1$, hence (x_n) is bounded.

For (**3**), if $x_n \to x$ and $B_r(x)$ for $r > 0$ is any neighborhood of x, then there is some $N \in \mathbb{N}$ such that we have $x_n \in B_r(x)$ for all $n \geq N$; hence at most $N - 1$ of the terms (x_n) are outside of $B_r(x)$. On the other hand, if for every $r > 0$ the ball $B_r(x)$ contains all but finitely many terms of (x_n), then for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$; hence $x_n \to x$.

For (**4**), if x is a limit point of E, then for each $n \in \mathbb{N}$, the set $(B_{1/n}(x) \setminus \{x\}) \cap E$ is non-empty. We may choose $x_n \in (B_{1/n}(x) \setminus \{x\}) \cap E$ for each $n \geq 1$. For any $\epsilon > 0$, by the Archimedean property there is some $N \in \mathbb{N}$ such that $1/N < \epsilon$. If $n \geq N$, we have $d(x_n, x) < 1/n \leq 1/N < \epsilon$. Hence the sequence (x_n) converges to x. □

11.3 The Reverse Triangle Inequality

Theorem 11.5 (Reverse Triangle Inequality)**.** For any $x, y \in \mathbb{C}$ (or $\mathbb{R}$), the following inequality holds:

$$||x| - |y|| \leq |x - y|$$

Proof. We begin with the standard triangle inequality, $|a + b| \leq |a| + |b|$.

First, let $a = x - y$ and $b = y$. Applying the triangle inequality, we get:

$$|x| = |(x - y) + y| \leq |x - y| + |y|$$

Rearranging this inequality by subtracting $|y|$ from both sides gives us our first result:

$$|x| - |y| \leq |x - y| \tag{3}$$

Next, we apply the same logic by swapping the roles of x and y. Let $a = y - x$ and $b = x$:

$$|y| = |(y - x) + x| \leq |y - x| + |x|$$

Rearranging this gives:

$$|y| - |x| \leq |y - x|$$

Since $|y - x| = |-(x - y)| = |x - y|$, we can rewrite this as:

$$-(|x| - |y|) \leq |x - y|$$

Multiplying both sides by -1 reverses the inequality sign, giving our second result:

$$|x| - |y| \geq -|x - y| \tag{4}$$

Combining the results from (3) and (4), we have shown that the quantity $|x| - |y|$ is bounded by both $|x - y|$ and its negative:

$$-|x - y| \leq |x| - |y| \leq |x - y|$$

This is precisely the definition of the absolute value. Therefore, we can conclude:

$$||x| - |y|| \leq |x - y|$$

This completes the proof. □ □

11.4 Algebraic Limit Theorems in $\mathbb{R}^k$

Since $\mathbb{R}, \mathbb{C}$, and $\mathbb{R}^k$ have algebraic operations, we can see how they relate to limits.

Theorem 11.6. Suppose $(x_n), (y_n)$ are sequences in $\mathbb{C}$ with $x_n \to x$ and $y_n \to y$. Then:

1. $(x_n + y_n) \to x + y$.
2. $(x_n y_n) \to xy$.
3. $(x_n / y_n) \to x/y$, provided $y_n \neq 0$ for all n and $y \neq 0$.

Proof. For (**1**), given $\epsilon > 0$, there are $N_1, N_2 \in \mathbb{N}$ such that $|x_n - x| < \epsilon/2$ for $n \geq N_1$ and $|y_n - y| < \epsilon/2$ for $n \geq N_2$. For $n \geq \max(N_1, N_2)$, we have $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $(x_n + y_n) \to x + y$.

For (**2**), we use the identity $x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$. Since $x_n \to x$ and $y_n \to y$, we have $(x_n - x) \to 0$ and $(y_n - y) \to 0$. This implies $(x_n - x)(y_n - y) \to 0$, $x(y_n - y) \to 0$, and $y(x_n - x) \to 0$. By property (1), the sum also converges to zero: $(x_n y_n - xy) \to 0$. Thus, $(x_n y_n) \to xy$.

For (**3**), it suffices to show that $1/y_n \to 1/y$ and then apply the product rule. We want to bound $|\frac{1}{y_n} - \frac{1}{y}| = \frac{|y_n - y|}{|y_n||y|}$. Since $y_n \to y$ and $y \neq 0$, there exists an $N_1 \in \mathbb{N}$ such that for

$n \geq N_1$, $|y_n - y| < \frac{|y|}{2}$. By the reverse triangle inequality, $||y_n| - |y|| \leq |y_n - y| < \frac{|y|}{2}$. This implies $|y_n| > \frac{|y|}{2}$ for $n \geq N_1$. Now, for any $\epsilon > 0$, there is an $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, $|y_n - y| < \frac{\epsilon |y|^2}{2}$. For $n \geq \max(N_1, N_2)$, we have:

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y - y_n|}{|y_n y|} < \frac{|y - y_n|}{|y|^2/2} < \frac{\epsilon |y|^2/2}{|y|^2/2} = \epsilon$$

So $1/y_n \to 1/y$. $\square$

Theorem 11.7. Suppose $(x_n), (y_n) \subset \mathbb{R}^k$ and $(c_n) \subset \mathbb{R}$ are sequences with $x_n \to x, y_n \to y$ in $\mathbb{R}^k$ and $c_n \to c$ in $\mathbb{R}$. Then:

1. Let $x_n = (a_1^{(n)}, \ldots, a_k^{(n)})$. Then $x_n \to x = (a_1, \ldots, a_k) \iff a_i^{(n)} \to a_i$ for each $i = 1, \ldots, k$.

2. $(x_n + y_n) \to x + y$ and $(c_n x_n) \to cx$.

Proof. Property (2) follows from property (1) and the previous theorem applied to each component. We prove (1). Let $x_n = (a_1^{(n)}, \ldots, a_k^{(n)})$ and $x = (a_1, \ldots, a_k)$. Note the inequalities:

$$|a_i^{(n)} - a_i| \leq |x_n - x| = \left(\sum_{j=1}^{k} |a_j^{(n)} - a_j|^2 \right)^{1/2}$$

($\implies$): If $x_n \to x$, then for any $\epsilon > 0$, $|x_n - x| < \epsilon$ for large n. From the inequality, $|a_i^{(n)} - a_i| \leq |x_n - x| < \epsilon$, so $a_i^{(n)} \to a_i$ for each i.

($\impliedby$): If $a_i^{(n)} \to a_i$ for each i, then for any $\epsilon > 0$, there exists N_i such that for $n \geq N_i$, $|a_i^{(n)} - a_i| < \epsilon/\sqrt{k}$. Let $N = \max(N_1, \ldots, N_k)$. For $n \geq N$, we have:

$$|x_n - x| = \left(\sum_{i=1}^{k} |a_i^{(n)} - a_i|^2 \right)^{1/2} < \left(\sum_{i=1}^{k} \left(\frac{\epsilon}{\sqrt{k}} \right)^2 \right)^{1/2} = \left(k \frac{\epsilon^2}{k} \right)^{1/2} = \epsilon$$

Hence $x_n \to x$. $\square$

12 Section 5: Topological Properties of Sets

12.1 Recap: Open and Closed Sets

Remark 12.1 (Closure and Interior)**.** The **closure** of a set E, denoted $\bar{E}$, is the smallest closed set containing E. The **interior** of a set E, denoted E^o, is the largest open set contained in E. $\square$

Remark 12.2 (Open and Closed Balls)**.** In a metric space (X, d), for a point $p \in X$ and radius $r > 0$:

1. The **open ball** $B_r(p) = \{q \in X \mid d(p, q) < r\}$ is an open set.
2. The **closed ball** $\bar{B}_r(p) = \{q \in X \mid d(p, q) \leq r\}$ is a closed set.

□

Theorem 12.3. An arbitrary union of closed sets is not necessarily closed.

Counterexamples. 1. Consider the infinite union of closed intervals in $\mathbb{R}$:

$$\bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = [0, 1)$$

The resulting set $[0, 1)$ is not closed because it does not contain its limit point 1.

2. Consider the set $S = \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\}$. Each singleton set $\left\{\frac{1}{n}\right\}$ is closed. However, the set S is not closed because 0 is a limit point of S, but $0 \notin S$.

□

12.2 Compactness

Example 12.4. Let $K \subset \mathbb{R}$ be the set defined by $K = \{0\} \cup \left\{\frac{1}{n} \mid n = 1, 2, 3, \ldots\right\}$. Prove that K is compact directly from the definition (i.e., without using the Heine-Borel theorem). □

Proof. Let $\{G_\alpha\}$ be an arbitrary open cover of K. We must show that there exists a finite subcover.

Since $0 \in K$ and $\{G_\alpha\}$ covers K, there must be some open set G_{α_0} in the collection such that $0 \in G_{\alpha_0}$.

Because G_{α_0} is open, there exists an $\epsilon > 0$ such that the open interval $(-\epsilon, \epsilon)$ is a subset of G_{α_0}.

The sequence $\{1/n\}$ converges to 0. Therefore, for the $\epsilon > 0$ above, there exists a positive integer N such that for all integers $n \geq N$, we have $0 < \frac{1}{n} < \epsilon$. This implies that all points $\frac{1}{n}$ for $n \geq N$ are contained in the interval $(-\epsilon, \epsilon)$, and thus are contained in the single open set G_{α_0}.

So, the set G_{α_0} covers the point 0 and all but a finite number of points of K. The points of K not necessarily covered by G_{α_0} are the finitely many points in the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N-1}\right\}$.

For each of these remaining $N - 1$ points, say $\frac{1}{k}$ for $k \in \{1, 2, \ldots, N - 1\}$, we can choose one open set G_{α_k} from the original cover $\{G_\alpha\}$ such that $\frac{1}{k} \in G_{\alpha_k}$.

Now, consider the collection of open sets:

$$\mathcal{C} = \{G_{\alpha_0}, G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_{N-1}}\}$$

This is a finite collection of sets from the original cover. By construction, it covers all points in K. Therefore, we have found a finite subcover. Since our choice of the initial open cover $\{G_\alpha\}$ was arbitrary, we conclude that K is compact. □

Theorem 12.5 (Cantor's Intersection Theorem)**.** Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of non-empty, compact sets in a metric space X such that they are nested, i.e., $K_1 \supset K_2 \supset K_3 \supset \cdots$. Then their intersection is non-empty:

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset$$

Remark 12.6. This is sometimes informally called the "Nested Interval Theorem" or "Onion Ring Theorem," particularly when dealing with closed and bounded intervals in $\mathbb{R}$, which are a special case of compact sets by the Heine-Borel theorem. □

12.3 Perfect Sets

Definition 12.7 (Perfect Set)**.** Let (X, d) be a metric space. A set $P \subset X$ is **perfect** if it is closed and every point of P is a limit point of P. Equivalently, a set is perfect if $P = P'$, where P' is the set of all limit points of P. This means a perfect set contains no isolated points.

Example 12.8 (Examples of Perfect Sets)**.**

- The empty set, $\emptyset$.
- Any closed interval $[a, b]$ in $\mathbb{R}$.
- The entire real line $\mathbb{R}$ or Euclidean space $\mathbb{R}^n$.
- The **Cantor set** is a classic example of a perfect set that is also totally disconnected, uncountable, and has measure zero.

□

Example 12.9 (Non-Example)**.** The set of rational numbers in $[0, 1]$, i.e., $[0, 1] \cap \mathbb{Q}$, is not perfect when considered as a subset of $\mathbb{R}$ because it is not closed. □

Theorem 12.10. Any non-empty perfect set in $\mathbb{R}^k$ is uncountable.

Remark 12.11. This is a powerful result. For instance, since the Cantor set is non-empty and perfect, this theorem immediately implies that the Cantor set is uncountable. □

13 Lecture 12: Subsequences and Cauchy Sequences

13.1 Subsequences

Note that the sequences $((-1)^n)$ and $((i)^n)$ do not converge, but we can look at parts of them that do. For example:

$$\begin{cases} (-1)^{2n} = 1 \to 1 \\ (-1)^{2n+1} = -1 \to -1 \end{cases}$$

and similarly for $(i)^n$, which has subsequences converging to $i, -1, -i$, and 1. This motivates the idea of a subsequence.

Definition 13.1 (Subsequence)**.** Let (X, d) be a metric space and $(x_n) \subset X$ a sequence. For any sequence of natural numbers $(n_k) \subset \mathbb{N}$, indexed by $k \geq 1$, such that $1 \leq n_k < n_{k+1}$ for all $k \geq 1$, we call

$$(x_{n_k}) = \{x_{n_k}\}_{k \geq 1}$$

a **subsequence** of (x_n). The limit of a convergent subsequence is called a **subsequential limit**. Note that the condition on the indices implies $n_k \geq k$ for all $k \geq 1$.

Theorem 13.2. A sequence (x_n) in a metric space converges to a point x if and only if every subsequence of (x_n) converges to x.

Proof. ($\Longleftarrow$): This direction is trivial. If every subsequence converges to x, we can choose the subsequence where $n_k = k$, which is the original sequence itself. Thus, (x_n) converges to x.

($\Longrightarrow$): Suppose $x_n \to x$. Let (x_{n_k}) be an arbitrary subsequence of (x_n). For any $\epsilon > 0$, since $x_n \to x$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, x) < \epsilon$. Since our indices are strictly increasing, we know that $n_k \geq k$. Therefore, if we take $k \geq N$, it follows that $n_k \geq N$. This means that for $k \geq N$, we have $d(x_{n_k}, x) < \epsilon$. Thus, the subsequence (x_{n_k}) also converges to x. □

13.2 The Bolzano-Weierstrass Theorem

Theorem 13.3 (Bolzano-Weierstrass Theorem)**.** Let (K, d) be a compact metric space and $(x_n) \subset K$ be a sequence. Then (x_n) has a convergent subsequence (with a limit in K). Furthermore, if $(x_n) \subset \mathbb{R}^k$ is any bounded sequence, it has a convergent subsequence.

Proof. Let $E = \{x_n \mid n \geq 1\}$ be the set of points in the sequence. There are two cases.

Case 1: E is a finite set. If the range of the sequence is finite, then by the Pigeonhole Principle, at least one point in E, say x, must be taken on infinitely many times. This means we can find a sequence of indices $n_1 < n_2 < n_3 < \dots$ such that $x_{n_k} = x$ for all k. This constant subsequence clearly converges to $x \in E \subset K$.

Case 2: E **is an infinite set.** Since E is an infinite subset of the compact set K, it must have a limit point in K; let's call it x. Since x is a limit point of E, we can construct a subsequence that converges to x. For each $k \in \mathbb{N}$, we choose an index $n_k > n_{k-1}$ such that $x_{n_k} \in B_{1/k}(x)$. The resulting subsequence (x_{n_k}) converges to x.

For the second part of the theorem, if (x_n) is a bounded sequence in $\mathbb{R}^k$, then its range is contained in some large, closed k-cell, which is compact by the Heine-Borel theorem. The result then follows from the first part of the proof. □

Theorem 13.4. Let (X, d) be a metric space and $(x_n) \subset X$ a sequence. The set of all subsequential limits of (x_n) is a closed set.

Proof. Let E be the set of all subsequential limits of (x_n). We want to show that E is closed, which means we must show that $E' \subset E$. Let x be a limit point of E. We need to show that $x \in E$, meaning we must construct a subsequence of (x_n) that converges to x.

Since x is a limit point of E, for each $k \in \mathbb{N}$, we can find a point $z_k \in E$ such that $d(z_k, x) < 1/2^k$.

Each z_k is a subsequential limit itself. This means we can pick a term from the original sequence, let's call it x_{n_k}, that is very close to z_k. Specifically, we can choose an index n_k such that $d(x_{n_k}, z_k) < 1/2^k$ and $n_k > n_{k-1}$.

Using the triangle inequality:

$$d(x_{n_k}, x) \le d(x_{n_k}, z_k) + d(z_k, x) < \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}}$$

As $k \to \infty$, the distance $d(x_{n_k}, x) \to 0$. This shows that the subsequence (x_{n_k}) converges to x. Therefore, $x \in E$, and we conclude that E is closed. □

13.3 Cauchy Sequences

The idea of convergence, $x_n \to x$, requires us to know the limit x beforehand. However, if a sequence converges, its terms must eventually get closer and closer to each other. This concept can be formalized without reference to a limit point.

Definition 13.5 (Cauchy Sequence)**.** Let (X, d) be a metric space. A sequence $(x_n) \subset X$ is a **Cauchy sequence** if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \ge N$, we have $d(x_n, x_m) < \epsilon$.

Theorem 13.6. In any metric space, a convergent sequence is a Cauchy sequence.

Proof. Suppose $x_n \to x$. Given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \ge N$, $d(x_n, x) < \epsilon/2$. Now, for any $n, m \ge N$, we can use the triangle inequality:

$$d(x_n, x_m) \le d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, (x_n) is a Cauchy sequence. $\square$

Theorem 13.7. In any metric space, a Cauchy sequence is bounded.

Proof. Let (x_n) be a Cauchy sequence. Taking $\epsilon = 1$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x_N) < 1$. Let $M = \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}$. Then for any $n \in \mathbb{N}$, we have $d(x_n, x_N) \leq M$. This shows that the sequence is bounded. $\square$

Theorem 13.8. Let (X, d) be a metric space. If a Cauchy sequence (x_n) has a convergent subsequence (x_{n_k}) with limit x, then the entire sequence (x_n) converges to x.

Proof. Let (x_{n_k}) be a subsequence of (x_n) such that $x_{n_k} \to x$. Given $\epsilon > 0$, there exists an N such that for $n, m \geq N$, $d(x_n, x_m) < \epsilon/2$, and for k large enough (so $n_k \geq N$), $d(x_{n_k}, x) < \epsilon/2$. For any $n \geq N$, we have:

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, the sequence (x_n) converges to x. $\square$

Definition 13.9 (Complete Metric Space)**.** A metric space (X, d) is called **complete** if every Cauchy sequence in X converges to a point in X.

Example 13.10.

- The space $\mathbb{R}^k$ is complete. This is a fundamental property of the real numbers, often called the **Cauchy criterion for convergence**.
- The space $\mathbb{Q}$ of rational numbers is **not** complete. A sequence of rational approximations to $\sqrt{2}$ is Cauchy in $\mathbb{Q}$ but does not converge to a point within $\mathbb{Q}$. Completeness is what "fills the gaps" in the rational number line.

$\square$

Theorem 13.11. A metric space X is complete if and only if every closed and bounded subset of X is compact.

Remark 13.12. In general metric spaces, compactness is a stronger condition than being complete and bounded. However, for $\mathbb{R}^k$, the Heine-Borel theorem states that a set is compact if and only if it is closed and bounded. This directly implies that $\mathbb{R}^k$ is complete. $\square$

Theorem 13.13. Let (X, d) be a complete metric space, and let $E \subset X$ be a closed subset. Then (E, d) is also a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in E. Since $E \subset X$, (x_n) is also a Cauchy sequence in the complete space X. Therefore, the sequence converges to some limit $x \in X$. Since E

is closed, it contains all its limit points. As the limit of the sequence $(x_n) \subset E$, x must be in $\bar{E}$. Since E is closed, $\bar{E} = E$, so $x \in E$. Thus, E is complete. $\square$

14 Lecture 13: Monotone Sequences and limit superior/inferior

14.1 Monotone Sequences

We've seen that convergent sequences are bounded, but bounded sequences don't necessarily converge (e.g., $((-1)^n)$). In $\mathbb{R}$, however, adding the condition of monotonicity is sufficient to guarantee convergence.

Definition 14.1 (Monotone Sequence)**.** A sequence $(x_n) \subset \mathbb{R}$ is **monotone** if it is either increasing or decreasing.

- It is **increasing** if $x_n \leq x_{n+1}$ for all $n \geq 1$.
- It is **decreasing** if $x_n \geq x_{n+1}$ for all $n \geq 1$.

The sequence is **strictly** monotone if the inequalities are strict $(<, >)$.

Theorem 14.2 (Monotone Convergence Theorem)**.** A monotone sequence in $\mathbb{R}$ converges if and only if it is bounded.

Proof. ($\Longrightarrow$): This is straightforward, as we already know that every convergent sequence is bounded.

($\Longleftarrow$): Suppose (x_n) is a monotone and bounded sequence. Let's assume (x_n) is increasing (the decreasing case is analogous, converging to the infimum). Let $E = \{x_n \mid n \geq 1\}$ be the set of points in the sequence. Since (x_n) is bounded, the set E is non-empty and bounded above. By the Least Upper Bound Property of $\mathbb{R}$, the supremum $x = \sup E$ exists.

We claim that $\lim_{n\to\infty} x_n = x$. Let $\epsilon > 0$. By the definition of the supremum, $x - \epsilon$ is not an upper bound for E. Therefore, there must exist some term x_N in the sequence such that $x_N > x - \epsilon$. Because the sequence is increasing, for all $n \geq N$, we have $x_n \geq x_N$. Furthermore, since x is the supremum of the set, we know $x_n \leq x$ for all n. Combining these inequalities, for all $n \geq N$ we have:

$$x - \epsilon < x_N \leq x_n \leq x < x + \epsilon$$

This implies that $|x_n - x| < \epsilon$ for all $n \geq N$. Thus, the sequence (x_n) converges to its supremum. $\square$

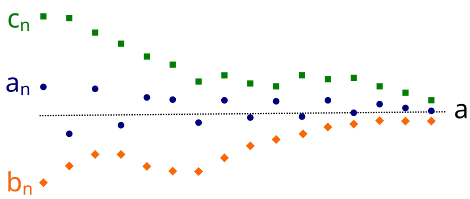


Figure 3: When a sequence lies between two other converging sequences with the same limit, it also converges to this limit.

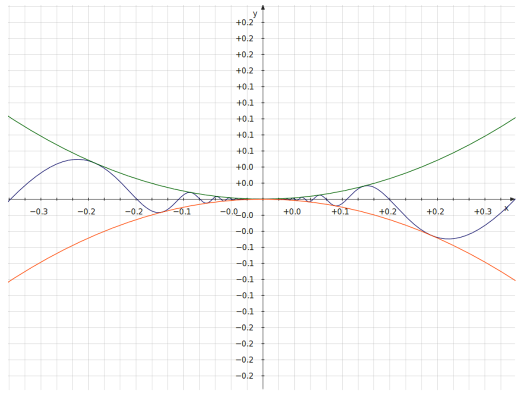


Figure 4: A functional visualization of the squeeze theorem.

14.2 Limit Superior and Limit Inferior

Theorem 14.3 (Squeeze Theorem)**.** Let $(x_n), (y_n), (z_n)$ be sequences in $\mathbb{R}$. If $x_n \leq y_n \leq z_n$ for all n greater than some N_0, and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then the sequence (y_n) also converges to L.

Definition 14.4 (Divergence to Infinity)**.** We say a sequence $(x_n) \subset \mathbb{R}$ **diverges to** $+\infty$ (written $x_n \to +\infty$) if for every $M > 0$, there exists an $N \in \mathbb{N}$ such that $x_n > M$ for all $n \geq N$. Similarly, $x_n \to -\infty$ if for every $M > 0$, there exists an $N \in \mathbb{N}$ such that $x_n < -M$ for all $n \geq N$.

Definition 14.5 (Limit Superior and Inferior)**.** Let (x_n) be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of (x_n) in the extended real numbers $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

- The **limit superior** of (x_n), denoted $\limsup_{n\to\infty} x_n$, is the supremum of E:

$$\limsup_{n\to\infty} x_n = \operatorname{Sup} E$$

- The **limit inferior** of (x_n), denoted $\liminf_{n\to\infty} x_n$, is the infimum of E:

$$\liminf_{n\to\infty} x_n = \operatorname{Inf} E$$

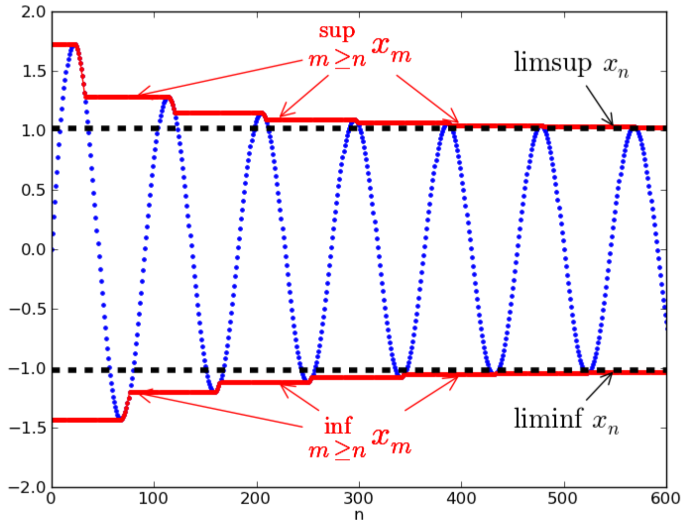


Figure 5: An illustration of limit superior and limit inferior. The sequence x_n (blue) accumulates around two values. The limsup (top dashed line) is the largest of these accumulation points, while the liminf (bottom dashed line) is the smallest.

Remark 14.6 (Alternative Definition)**.** The limit superior and inferior can also be defined equivalently as follows:

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$
- $\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$

$\square$

Theorem 14.7 (Properties of Lim Sup/Inf)**.** Let (x_n) be a sequence in $\mathbb{R}$.

1. A sequence (x_n) converges to $L \in \bar{\mathbb{R}}$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L$.
2. In general, $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.
3. $\limsup_{n\to\infty}(-x_n) = -\left(\liminf_{n\to\infty} x_n\right)$.
4. If $x_n \leq y_n$ for all sufficiently large n, then:

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n \quad \text{and} \quad \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$$

Example 14.8.

- If $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$, the set of subsequential limits is $E = \{-1, 1\}$. Therefore, $\limsup x_n = 1$ and $\liminf x_n = -1$.
- If (q_n) is a sequence that enumerates all rational numbers $\mathbb{Q}$, then every real number is a subsequential limit. Thus, $\limsup q_n = +\infty$ and $\liminf q_n = -\infty$.

□

15 Lecture 14: Series

15.1 Series Definition

Definition 15.1. Given a sequence $(x_n) \subset \mathbb{C}$ we write $S_n = \sum_{i=1}^n x_i = x_1 + \ldots + x_n \in \mathbb{C}$ for the $n-$**th partial sum** and we say that the **series** $\sum_{i=1}^\infty x_i$ **converges** if (S_n) converges.
i.e. if $S_n \to S \in \mathbb{C}$ we write $\sum_{i=1}^\infty x_i = S$. If (S_n) diverge we say that $\sum_{i=1}^\infty x_i$ **diverges**.

Remark 15.2. $(x_n) \subset \mathbb{C}$ a sequence $\iff$ $\sum_{k=1}^n (x_k - x_{k-1})$, for $x_0 = 0$. i.e. $S_n = x_n$ for $n \geq 1$. □

Theorem 15.3. $\sum_{n=1}^\infty x_n$ converges $\iff$ for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|\sum_{k=n}^m x_k| < \epsilon$ for $m, n \geq N$ (i.e. if (S_n) is Cauchy).

Proof. This follows since convergence is equivalent to the Cauchy property in $\mathbb{R}^2 \cong \mathbb{C}$. □

Remark 15.4. $\sum_{n=1}^\infty x_n$ converges implies that $x_n \to 0$, but $x_n \to 0$ does **not** imply $\sum_{n=1}^\infty x_n$ converges. □

Example 15.5. If $0 < x < 1$ then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

This follows since for the partial sum $S_n = \sum_{k=0}^n x^k = 1 + x + x^2 + \ldots + x^n$, we have $xS_n = x + x^2 + x^3 + \ldots + x^{n+1}$. Then $S_n(1-x) = 1 - x^{n+1}$, which implies that $S_n = \frac{1-x^{n+1}}{1-x}$. Since $0 < x < 1$, $x^{n+1} \to 0$ as $n \to \infty$, so $S_n \to \frac{1}{1-x}$. □

Definition 15.6 (e, Euler's Number)**.** We define

$$e = \sum_{k=0}^{\infty} \frac{1}{k!},$$

where $k! = k(k-1)\dots(2)(1)$ and $0! = 1$. Then,

$$\begin{aligned} 0 \le S_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}} \end{aligned}$$

This implies that $S_n < 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{1-\frac{1}{2}} = 3$. So we have that (S_n) is bounded and increasing (since $\frac{1}{k!} \ge 0$). In sum, this means that $S_n \to e$, so the series converges.

We can also define e as

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

This makes economists and actuaries happy.

Theorem 15.7. e is irrational.

Proof. Suppose for contradiction $e = \frac{p}{q}$ for $p, q \in \mathbb{N}$. We notice that

$$e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots < \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right)$$

The geometric series sums to $\frac{1}{1-\frac{1}{n+1}} = \frac{n+1}{n}$. Thus,

$$e - S_n < \frac{1}{(n+1)!}\frac{n+1}{n} = \frac{1}{n!n}$$

Now consider $n = q$. We have $0 < e - S_q < \frac{1}{q!q}$. Multiplying by $q!$ gives $0 < q!(e - S_q) < \frac{1}{q}$. If $e = p/q$, then $q!e = q!\frac{p}{q} = (q-1)!p \in \mathbb{Z}$. Also, $q!S_q = q!\left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!}\right) \in \mathbb{Z}$. This implies their difference, $q!(e - S_q)$, must be an integer. However, we have shown $0 < q!(e - S_q) < \frac{1}{q}$. For $q \ge 2$, this is a contradiction, as there is no integer between 0 and a number less than 1. (If $q = 1$, e would be an integer, but $2 < e < 3$). $\square$

Example 15.8 (Harmonic Series)**.** An important example of a divergent series is the **harmonic series**, $\sum \frac{1}{n}$. To see why this diverges (even though $\frac{1}{n} \to 0$), we can group the terms:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>4\cdot\frac{1}{8}=\frac{1}{2}} + \dots$$

This implies $S_{2^k} \geq 1 + \frac{k}{2}$. Since the right side is unbounded, the sequence of partial sums S_n diverges to ∞. $\square$

Remark 15.9. We note however that the **alternating harmonic series** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ does converge. Visually, the partial sums oscillate around the limit. It can be shown that this series converges to $\log(2)$. $\square$

15.2 Rearrangement of Series

Notice that if we rearrange the terms of the alternating harmonic series, we can get a different sum:

$$\begin{aligned}
\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \dots &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right) \\
&= \frac{1}{2}\log(2)
\end{aligned}$$

So the rearranged sum is different! This surprising fact is true for any conditionally convergent series.

Theorem 15.10 (Riemann Rearrangement Theorem)**.** Suppose $\sum x_n$ converges but $\sum |x_n|$ diverges (i.e., the series is **conditionally convergent**). Then for each $M \in \mathbb{R}$, there is some bijection (rearrangement) $\sigma : \mathbb{N} \to \mathbb{N}$ such that the rearranged series converges to M:

$$\sum_{n=1}^{\infty} x_{\sigma(n)} = M$$

Proof Idea. Let $A = \{x_n : x_n > 0\}$ and $B = \{x_n : x_n < 0\}$ be the sets of positive and negative terms. Since $\sum x_n$ converges and $\sum |x_n|$ diverges, it must be that the sum of the positive terms and the sum of the negative terms both diverge to ∞ and $-\infty$, respectively.

To get a rearranged series that sums to M, we define the rearrangement by taking just enough positive terms so that their sum is greater than M. Then, we take just enough negative terms so that the sum is less than M. We repeat this procedure, alternating between over- and under-approximating M. The deviation from M at each step is bounded by the magnitude of the last term added. Since the original series converges, its terms must go to zero ($x_n \to 0$). Therefore, the rearranged partial sums will converge to M. $\square$

Remark 15.11. By adapting the proof, one can rearrange any conditionally convergent series to diverge to $+\infty$ or $-\infty$, or to oscillate without approaching any limit. $\square$

15.3 Epsilon - Delta Definition of the Limit

Time to formalize some notions from calculus.

Definition 15.12 ($\epsilon - \delta$ Definition of a Limit)**.** Let $(X, d_x), (Y, d_y)$ be metric spaces, $E \subset X$ and $f : E \to Y$, and z a limit point of E. We write $f(x) \to y$ as $x \to z$ or $\lim_{x\to z} f(x) = y$ if there is $y \in Y$ so that for $\epsilon > 0$ there is some $\delta > 0$ such that

$$d_x(x, z) < \delta \implies d_y(f(x), y) < \epsilon$$

where $x \neq z$.

Example 15.13.

1. $\lim_{x\to 1}(\frac{1}{x}) = 1$
2. $\lim_{x\to a}(x^2) = a^2$
3. $\lim_{x\to 0}(\frac{1}{x})$ does not exist

$\square$

15.4 Function Operations

Now some basic definitions for messing with functions in $\mathbb{C}$ (and similarly in $\mathbb{R}^k$).

Definition 15.14. Let (X, d) be a metric space, $E \subset x$, and $f, g : E \to \mathbb{C}$, we define for $x \in E$,

1. $(f + g) : E \to \mathbb{C}$, $(f + g)(x) = f(x) + g(x)$
2. $(fg) : E \to \mathbb{C}$, $(fg)(x) = f(x)g(x)$
3. $(\frac{f}{g}) : E\backslash\{g = 0\} \to \mathbb{C}$, $(f/g)(x) = \frac{f(x)}{g(x)}$
4. $f \geq g$ if $f(y) \geq g(y)$ for all $y \in E$ and $f, g : E \to \mathbb{R}$.

And similarly for $\mathbb{R}^k$ valued functions.

Remark 15.15. The limit laws we expect then follow by the above. $\square$

15.5 Point Continuity

Definition 15.16 ($\epsilon - \delta$ point continuity)**.** Let $(X, d_x), (Y, d_y)$ be metric spaces, $E \subset X$ and $z \in E$, and $f : E \to Y$, then we say that f is **continuous** at z if for each $\epsilon > 0$ there is some $\delta > 0$ with $d_x(x, z) < \delta \implies d_y(f(x), f(z)) < \epsilon$. And, we say if f is continuous on E if it is continuous at all $z \in E$.

Remark 15.17. If z is a limit point of E then f is **continuous** at z if and only if $\lim_{x \to z} f(x) = f(z) = f(\lim_{x \to z} x)$
Also, if z is an isolated point in E, then f is continuous at z
Since some ball around $z \in E$ only contains z! □

Theorem 15.18. Let $(X, d_x), (Y, d_y)$ be metric spaces, then $f : X \to Y$ is continuous (**topological continuity**) if and only if $f^{-1}(U) \subset X$ is open in X whenever $U \subset Y$ is open in Y.

Proof. ($\implies$): If $U \subset Y$ is open in Y. Let $x \in f^{-1}(U) \implies f(x) \in U$, but since U is open there is some $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$.

As f is continuous at x for this $\epsilon > 0$ there is $\delta > 0$ such that $y \in B_\delta(x) \implies f(y) \in B_\epsilon(f(x)) \subset U$.

But then $y \in f^{-1}(B_z(f(x))$ for each $y \in B_\delta(x)$

$$\implies B_\delta(x) \subset f^{-1}(B_\epsilon(f(x))) \subset f^{-1}(U).$$

So $f^{-1}(U)$ is open in X.

($\impliedby$) : If $\epsilon > 0$ and $x \in X$, then the set $B_\epsilon(f(x)) \subset Y$ is then open in Y so

$$f^{-1}(B(\epsilon(f(x)) \subset X$$

is open in X.

Also, as $x \in f^{-1}(\{f(x)\}) \subset f^{-1}(B_z(f(x)))$, there is some $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$.

But then, if $y \in B_\delta(x) \implies f(y) \in B_\epsilon(f(x))$. This implies that f is continuous at x. So f is continuous on X. □

Remark 15.19. As C is closed in X if and only if C^c is open, f is continuous if and only if $f^{-1}(C)$ is closed for all closed sets C. □

15.6 Composition of Continuous Functions

Theorem 15.20 (Compositions of continuous functions are continuous)**.** $f : X \to Y$, $g : Y \to Z$ are continuous, this implies that $(g \circ f) : x \to z$ is continuous.

Proof. If $U \subset Z$ is open in Z as g is continuous, this implies that $g^{-1}(U)$ is open in Y.

Bu then by the continuity of f, this means that $f^{-1}(g^{-1}(U))$ is open in X.
But this holds if and only if $(g \circ f)^{-1}(U)$ is open in X. $\square$

Theorem 15.21.
1. If $f, g : X \to \mathbb{C}$ are continuous, $f+g, fg, f/g$ (where defined) are continuous.
2. If $h : X \to \mathbb{R}^k$ is such that $h = (h_1, \ldots, h_k$, then h is continuous if and only if $h_1, \ldots, h_n$ are continuous from $X \to \mathbb{R}$,
3. (1) holds for $\mathbb{R}^k$ valued functions (appropriately).

Proof. (1) follows by limit laws.
(2) follows by noting that

$$|h_j(x) - h_j(y)| \leq \big(\sum_{i=1}^{K} |h_1(x) - h_i(y)|^2\big)^2 \tag{5}$$

$$= |h(x) - h(y)| \tag{6}$$

($\Longleftarrow$) take $\epsilon/\sqrt{k}$
(3) follows (1) + (2). $\square$

Example 15.22.
1. Let $\phi_i : \mathbb{R}^k \to \mathbb{R}$ be defined by setting $\phi_i(x) = x_i$. This is **continuous** for each $i = 1, \ldots, k$.

   (Proof: $|\phi_i(x) - \phi_i(y)| = |x_i - y_i| \leq \|x - y\|$. So we can choose $\delta = \epsilon$).
2. Inductively, $P(x) = \sum c_{n_1,\ldots,n_k} x_1^{n_1} \ldots x_k^{n_k}$ is **continuous** by item (1) in the previous theorem, where $P : \mathbb{R}^k \to \mathbb{C}$. (Since polynomials are built from sums and products of the continuous coordinate maps ϕ_i and constant functions).
3. Rational functions (ratios of polynomials) are also **continuous** (where defined, i.e., denominator $\neq 0$).
4. The map $x \mapsto |x|$ is **continuous**. Why? If $x, y \in \mathbb{C}$ (or $\mathbb{R}^k$), by the Reverse Triangle Inequality:

$$\big||x| - |y|\big| \leq |x - y|$$

   So given $\epsilon > 0$, we can choose $\delta = \epsilon$. If $|x - y| < \delta$, then $\big||x| - |y|\big| < \epsilon$.

$\square$

16 Lecture 16: Continuity and Compactness

(i.e. continuous maps preserve compactness)

Theorem 16.1. Let $f : X \to Y$ be a continuous map from a compact metric space, then $f(X)$ is compact.

Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of $f(X)$, as f is continuous $\{f^{-1}(U_\alpha)\}_\alpha$ is an open cover of X. As X is compact there exists a finite subcover $\{f^{-1}(U_{\alpha_i})\}_{i=1}^n$ such that $X \subset \bigcup_{i=1}^n f^{-1}(U_{\alpha_i})$.

As $f(f^{-1}(U_{\alpha_i})) \subset U_{\alpha_i}$ for each $i = 1, \ldots, n$ we have

$$f(X) \subset f\left(\bigcup_{i=1}^n f^{-1}(U_{\alpha_i})\right) \subset \bigcup_{i=1}^n f(f^{-1}(U_{\alpha_i})) \subset \bigcup_{i=1}^n U_{\alpha_i}$$

Thus $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover for $f(X)$. □

Remark 16.2. We used the fact that $f(f^{-1}(U_{\alpha_i})) \subset U_{\alpha_i}$ which follows since if $x \in f^{-1}(U_{\alpha_i}) \implies f(x) \in U_{\alpha_i}$ and thus we conclude the inclusion. □

Theorem 16.3. Let $f : X \to Y$ be a continuous bijection from a compact metric space, then $f^{-1} : Y \to X$ (defined by $f^{-1}(f(x)) = x$) is continuous.

Proof. Let $U \subset X$ be open, we will show that $f(U)$ is open in Y which shows that f^{-1} is continuous as $(f^{-1})^{-1}(U) = f(U)$. We see that

$$U \text{ open} \iff U^c \text{ closed} \implies U^c \text{ compact (as } X \text{ is compact)}$$

$$\implies f(U^c) \text{ compact (as } f \text{ is continuous)} \implies f(U^c) \text{ closed.}$$

As f is a bijection we have that $Y \setminus f(U^c) = f(U)$ so that $f(U^c)^c = f(U)$ and hence as $f(U^c)$ is closed, $f(U^c)^c = f(U)$ is open. □

Remark 16.4. Both theorems fail if X is not compact, for example $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ shows $f(\mathbb{R})$ not compact and $g : [0, 2\pi) \to S^1 \subset \mathbb{C}$ given by $g(\theta) = e^{2\pi i\theta}$ shows that g^{-1} is defined but not continuous. □

In Euclidean Space

We have:

Theorem 16.5. If $f : X \to \mathbb{R}^k$ is a continuous map from a compact metric space, then $f(X)$ is closed and bounded.

Proof. By the first Theorem above, $f(X)$ is compact in $\mathbb{R}^k$, hence closed and bounded by the Heine-Borel Theorem. □

Specifically in $\mathbb{R}$ we have the so called Extreme Value Theorem:

Theorem 16.6 (Extreme Value Theorem)**.** If $f : X \to \mathbb{R}$ is a continuous map from a compact metric space, then there are $x, y \in X$ such that $f(x) = \text{Sup}\, f(X)$ and $f(y) = \text{Inf}\, f(X)$.

Proof. We may assume $X \neq \emptyset$ (or there is nothing to show). By the last result, $f(X)$ is bounded and so by the least upper bound property for $\mathbb{R}$, both $\text{Sup}\, f(X)$ and $\text{Inf}\, f(X)$ exist. Moreover, as $f(X)$ is closed we have $\overline{f(X)} = f(X)$ and so $\text{Sup}\, f(X), \text{Inf}\, f(X) \in f(X)$. $\square$

Remark 16.7. This is equivalent to saying that there are $x, y \in X$ such that $f(y) \leq f(z) \leq f(x)$ for all $z \in X$, i.e. f obtains its maximum at x and minimum at y. $\square$

16.1 Uniform Continuity

Notice in the definition of continuity, that it was specified at a given point; namely for $\epsilon > 0$ there was a $\delta > 0$ depending on the point chosen so that the definition held. If we can choose one $\delta > 0$ that works for all points we have:

Definition 16.8. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$. We say f is **Uniformly Continuous** on X if for each $\epsilon > 0$ there is $\delta > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$.

Remark 16.9. Uniform continuity $\implies$ continuity. Continuity $\not\Longrightarrow$ uniform continuity (e.g. $f(x) = x^2$). $\square$

Example 16.10. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax + b$ (linear function) is Uniformly Continuous on $\mathbb{R}$; to see this, if $\epsilon > 0$ then setting $\delta = \frac{\epsilon}{|a|+1}$ (this works, avoids division by 0 if $a = 0$). $\square$

Theorem 16.11. If $f : X \to Y$ is continuous and X is compact, then f is Uniformly continuous.

Proof. For $\epsilon > 0$, and since f is continuous, if $x \in X$ we can find $\delta_x > 0$ such that $d_X(x, y) < \delta_x \implies d_Y(f(x), f(y)) < \epsilon/2$. We then have an open cover $\{B_{\delta_x}(x)\}_{x \in X}$ of X; as X is compact there is a finite subcover $\{B_{\delta_{x_i}}(x_i)\}_{i=1}^n$ for some $\{x_i\}_{i=1}^n \subset X$. Set $\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_n}\} > 0$. So that if $x, y \in X$ with $d_X(x, y) < \delta$, then $x \in B_{\delta_{x_i}}(x_i)$ for some $i = 1, \ldots, n$, hence $d_X(x, x_i) < \delta_{x_i}$. And so $d_X(y, x_i) \leq d_X(y, x) + d_X(x, x_i) < \delta + \delta_{x_i} \leq \frac{\delta_{x_i}}{2} + \delta_{x_i} = \frac{3}{2}\delta_{x_i}$.

And so $d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_i)) + d_Y(f(y), f(x_i)) < \epsilon/2 + \epsilon/2 = \epsilon$.

(Note: The proof transcribed from the image is flawed. The triangle inequality $d_X(y, x_i) < \frac{3}{2}\delta_{x_i}$ does not imply $d_Y(f(y), f(x_i)) < \epsilon/2$. A correct proof typically uses an open cover of balls with radius $\delta_x/2$.) $\square$

Remark 16.12. Can also prove by contradiction (Check). □

We finally emphasize why Compactness is essential in these results:

Theorem 16.13. Let $E \subset \mathbb{R}$ be non-compact, then:

1. There is an unbounded continuous function on E.

2. There is a continuous bounded function on E with no maximum.

3. If E is bounded, there is a continuous function on E which is not uniformly continuous.

Proof. We first assume E is bounded and prove (1) and (3). Since E is bounded and non-compact (by Heine-Borel), it must not be closed.

For (1), there must exist a limit point y of E which is not in E. Define $f : E \to \mathbb{R}$ by $f(x) = \frac{1}{x-y}$ which is continuous but not bounded.

For (3), f defined above is not uniformly continuous. Since for any $\delta > 0$ we can find $x, t \in E$ (e.g., x, t close to y) with $|x - t| < \delta$ but

$$|f(x) - f(t)| = \left| \frac{1}{x-y} - \frac{1}{t-y} \right| = \left| \frac{t-x}{(x-y)(t-y)} \right|$$

which can be made arbitrarily large (e.g., ¿ 1) by choosing x and t sufficiently close to y.

For (2), using y as above, define $g : E \to \mathbb{R}$ by $g(x) = \frac{1}{1+(x-y)^2}$ which is continuous on E and bounded. We see that $\text{Sup}_{x \in E}\, g(x) = 1$ but $g(x) < 1$ if $x \in E$ (since $y \notin E$), so that g has no maximum!

Now if E is unbounded then $f(x) = x \implies$ (1) and $h(x) = \frac{x^2}{1+x^2} \implies$ (2). Since $\text{Sup}_{x \in E}\, h(x) = 1$ but $h(x) < 1$ for any $x \in E$. □

Remark 16.14. (3) fails if E is unbounded by considering any linear function on $\mathbb{N} \subset \mathbb{R}$ (every such function is Uniformly Continuous!). □

17 Lecture 17: Continuity, Connectedness, and Discontinuities

17.1 Continuity and Compactness

Theorem 17.1. If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Proof. If $f(X)$ is not connected then there are disjoint open nonempty sets $A, B \subset Y$ such that $f(X) \subset A \cup B$. As f is continuous, $f^{-1}(A), f^{-1}(B)$ are open nonempty sets in X; moreover they are disjoint as A and B are disjoint. If $x \in X$ then $f(x) \in A \cup B$ and hence $x \in f^{-1}(A) \cup f^{-1}(B)$ which implies $X \subset f^{-1}(A) \cup f^{-1}(B)$, a contradiction. □

This allows us to prove the **Intermediate Value Theorem**:

Theorem 17.2 (Intermediate Value Theorem)**.** If $f : [a, b] \to \mathbb{R}$ is continuous and C is a value between $f(a)$ and $f(b)$ (wlog $f(a) < C < f(b)$), then there is $x \in (a, b)$ such that $f(x) = C$.

Proof. By the last result $f([a, b])$ is connected as $[a, b]$ is. As $f(a) < C < f(b)$, the characterization for connected sets in $\mathbb{R}$ implies $C \in f([a, b])$; hence there is some $x \in [a, b]$ such that $f(x) = C$. Finally, we note that $x \neq a, b$ as $f(a) < C < f(b)$. □

17.2 Discontinuities

We will see that the converse of the IVT does not hold, i.e. there are functions taking every value between two given numbers that fail to be continuous.

Definition 17.3. If $f : X \to Y$ is not continuous at $x \in X$ we say that f has a **discontinuity** at x.

Example 17.4 (Heaviside Fn)**.** $H(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$, has a discontinuity at 0. □

The nature of discontinuity here can be classified:

Definition 17.5. Let $f : (a, b) \to Y$ and $a < x < b$. We write $f(x^+) = \lim_{t \to x^+} f(t)$ and $f(x^-) = \lim_{t \to x^-} f(t)$ for the **right and left hand limits** of f at x respectively, if $f(t_n) \to f(x^+)$ for all sequences $(t_n) \subset (x, b)$, $t_n \to x$. $f(t_n) \to f(x^-)$ for all sequences $(t_n) \subset (a, x)$, $t_n \to x$.

Remark 17.6. $\lim_{t \to x} f(t)$ exists $\implies$ $f(x^+) = f(x^-) = \lim_{t \to x} f(t)$. □

Example 17.7. $H(0^+) = 1$, $H(0^-) = 0$. So $\lim_{t \to 0} H(t)$ DNE. □

Definition 17.8. If $f : (a, b) \to Y$ is discontinuous at $x \in (a, b)$ then:

- x is a discontinuity of the **first kind** if $f(x^+), f(x^-)$ exist.
- x is a discontinuity of the **second kind** otherwise.

Example 17.9. 0 is a discontinuity of the first kind as $H(0^+) \neq H(0^-)$.

- $f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ has a discontinuity of the first kind at 0 even though $f(0^+) = f(0^-) = 0$ but $f(0) = 1$.

- $g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q} \end{cases}$ has a discontinuity of the second kind at every $x \in \mathbb{R}$ (as $g(x^\pm)$ DNE).

- $h(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$ is continuous at 0 (check) but has a discontinuity of the second kind at every $x \neq 0$.

$\square$

Example 17.10. Assuming $\sin(x)$ is well defined and continuous, we see that $\sin(1/x)$ is continuous on $\{x \neq 0\}$. Defining

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

then $f(0^\pm)$ DNE so 0 is a discontinuity of the second kind. Note that f attains every value in $[-1, 1]$ however, so the converse of IVT fails! $\square$

17.3 Monotone Functions

For our final topic in continuity, we study functions that never decrease or increase on $\mathbb{R}$.

Definition 17.11. We say that $f : (a, b) \to \mathbb{R}$ is **monotone** if it is either

- **increasing**, i.e. $f(x) \leq f(y)$ for $a < x \leq y < b$.
- **decreasing**, i.e. $f(x) \geq f(y)$ for $a < x \leq y < b$.

Remark 17.12. If f is both increasing and decreasing $\implies$ f is constant. $\square$

Theorem 17.13. Let $f : (a, b) \to \mathbb{R}$ be monotone increasing, then $f(x^+)$ and $f(x^-)$ exist for every $x \in (a, b)$. Moreover,

$$\text{Sup}_{a<t<x} f(t) = f(x^-) \leq f(x) \leq f(x^+) = \text{Inf}_{x<t<b} f(t).$$

And $f(x^+) \leq f(y^-)$ for $a < x < y < b$.

Proof. Note that $E = \{f(t) \mid a < t < x\}$ is bounded above by $f(x)$ and hence $\text{Sup}\, f(t)$

exists. For each $\epsilon > 0$ there must exist some $\delta > 0$ such that $\operatorname{Sup}_{a<t<x} f(t) - \epsilon < f(x - \delta) \leq \operatorname{Sup}_{a<t<x} f(t)$. (or it would not be the Sup). As f is monotone increasing, if $x - \delta < y < x$ then $f(x-\delta) \leq f(y) \leq \operatorname{Sup}_{a<t<x} f(t)$. So that $\operatorname{Sup}_{a<t<x} f(t) - \epsilon < f(y) \leq \operatorname{Sup}_{a<t<x} f(t) \implies \operatorname{Sup} f(t) = f(x^-)$.

Similarly we can see that $\operatorname{Inf}_{x<t<b} f(t) = f(x^+)$; thus if $a < x < y < b$ we have

$$f(x^+) = \operatorname{Inf}_{x<t<b} f(t) \leq \operatorname{Inf}_{x<z<y} f(z) \leq \operatorname{Sup}_{x<z<y} f(z) \leq \operatorname{Sup}_{a<t<y} f(t) = f(y^-).$$

□

Remark 17.14. Similar results hold for monotone decreasing functions; thus monotone functions only have discontinuities of the first kind. □

Theorem 17.15. Monotone functions have at most countably many discontinuities.

Proof. Without loss of generality let $f : (a,b) \to \mathbb{R}$ be monotone increasing and let $E = \{\text{discontinuity pts of } f\}$. If $x \in E$ then $f(x^-) < f(x^+)$ and to x we associate some $r_x \in \mathbb{Q}$ such that $f(x^-) < r_x < f(x^+)$. Now if $x \neq y$ and $y \in E$ then to y we associate $r_y \in \mathbb{Q}$ with $f(y^-) < r_y < f(y^+)$.

If $x < y$ (WLOG) then $f(x^+) \leq f(y^-)$ by the previous result.

$$f(x^-) < r_x < f(x^+) \leq f(y^-) < r_y < f(y^+)$$

and thus $g : E \to \mathbb{Q}$ defined by $g(x) = r_x$ is injective. We thus see that $g : E \to g(E)$ is a bijection from E to a subset of $\mathbb{Q}$, hence E is at most countable. □

We finish with a construction of a monotone function with prescribed discontinuities. Let $E \subset (a,b)$ be countable (e.g. $\mathbb{Q} \cap (a,b)$) and $(x_n) = E$ be an enumeration; then, for any sequence $(a_n) \subset \mathbb{R}$ of positive numbers with $\sum a_n$ converging let $f : (a,b) \to \mathbb{R}$ be

$$f(x) = \sum_{x_n \leq x} a_n.$$

One can check that f satisfies:

- f is monotone increasing
- $f(x_n^+) - f(x_n^-) = a_n > 0$
- f is continuous on $(a,b) \setminus E$.

In fact, $f(x^-) = f(x)$ so that f is continuous from the left. If we sum over $x_n < x$ instead we get $f(x^+) = f(x)$ i.e. continuous from the right.

Part IV

Continuity and Differentiation

18 Lecture 18: Normed Vector Space

Note that $\mathbb{R}, \mathbb{C}, \mathbb{R}^n$ have the structure of vector space that are also metric spaces.

Definition 18.1. A norm on a vector space, V, over $\mathbb{C}$ is a map $\|\cdot\| : V \to [0, \infty)$ satisfying the properties:

1. $\|x\| = 0 \iff x = 0$ (Positive Definite)
2. $\|\lambda x\| = |\lambda|\|x\|$ for any $\lambda \in \mathbb{C}$ (Absolute Homogeneity)
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (Triangle Inequality)

We call $(V, \|\cdot\|)$ a **normed** vector space.

Example 18.2. $\underbrace{(\mathbb{R}, |\cdot|)}_{\text{over } \mathbb{R}}, \underbrace{(\mathbb{C}, |\cdot|)}_{\text{over } \mathbb{C}}, \underbrace{(\mathbb{R}^k, |\cdot|)}_{\text{over } \mathbb{R}}$ are all normed vector spaces □

Example 18.3. Let $S = \{(x_n) \subset \mathbb{R}\}$ (a set of sequences). This is a vector space, on which we can define functions from $S \to [0, \infty]$.

$$\text{(Minkowski Inequality*)} \leftarrow \begin{cases} \|(x_n)\|_p & = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty \\ \|(x_n)\|_\infty & = \sup_{n \geq 1} |x_n| \quad \text{for } p = \infty \end{cases}$$

* this implies the triangle inequality. □

Definition 18.4.

$$\begin{cases} l^p & = \{(x_n) \in S \text{ such that } \|(x_n)\|_p < \infty\} \quad \text{for } 1 \leq p < \infty \\ l^\infty & = \{(x_n) \in S \text{ such that } \|(x_n)\|_\infty < \infty\} \quad \text{for } p = \infty \end{cases}$$

These are then **normed** vector spaces.

Example 18.5. If X is any metric space

$$\mathcal{C}(X) = \{f : X \to \mathbb{C} | f \text{ is continuous and bounded}\}$$

this is a vector space and we can define the **supremum norm**:

$$\|f\| = \sup_{x \in X} |f(x)|.$$

□

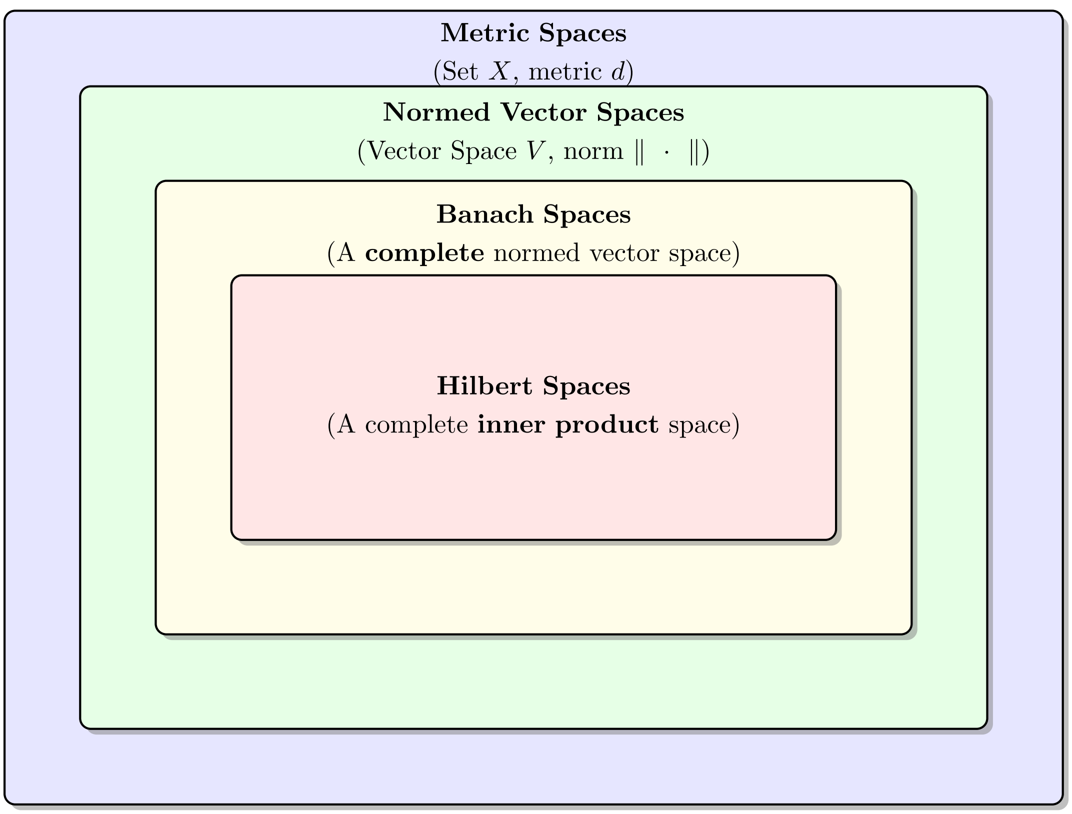
Remark 18.6. If X is compact $\|f\| < \infty$ for any continuous function by the extreme value theorem. □

Remark 18.7. Any norm defines a metric space since we can set $d : V \times V \to \mathbb{R}$ by setting

$$d(x, y) = \|x - y\|$$

for $x, y \in V$. □

Definition 18.8 (Banach Space)**.** If $(V, \|\cdot\|)$ is a normed vector space which is **complete** (as a metric space) with respect to the induced metric. Also called a **Banach Space**



As a simple line of text, the relationship is:

Hilbert $\subset$ **Banach** $\subset$ **Normed** $\subset$ **Metric**

Example 18.9 (Favorite Banach Spaces)**.** $(\mathbb{R}, |\cdot|), (\mathbb{C}, |\cdot|), (\mathbb{R}^k, |\cdot|)$ are Banach spaces! $\square$

A non-example is $(\mathbb{Q}, |\cdot|)$, which is **not** a Banach space.

18.1 Convergence in Normed Vector Spaces

Definition 18.10. Let $(x_n) \subset V$ be a sequence in a normed vector space $(V, \|\cdot\|)$, then the series $\sum x_n \in V$ is said to converge if (s_n) the sequence of partial sums converge where

$$S_N = \sum_{n=1}^{N} x_n \in V.$$

In other words there is $x \in V$ such that

$$\lim_{N\to\infty} \underbrace{\|x - \sum_{n=1}^{N} x_n\|}_{d(x,S_N)} = 0$$

Theorem 18.11 (Banach Series Criterion)**.** A normed vector space $(V, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series in V converges.
That is, $(V, \|\cdot\|)$ is complete if and only if for every sequence $(x_n) \subset V$:

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \implies \sum_{n=1}^{\infty} x_n \text{ converges in } V.$$

Proof. $(\implies)$: Suppose $(V, \|\cdot\|)$ is a Banach space. Let $(x_n) \subset V$ be a sequence such that $\sum_{n=1}^{\infty} \|x_n\|$ converges.

Let

$$S_N = \sum_{n=1}^{N} x_n, \quad \text{and} \quad T_N = \sum_{n=1}^{N} \|x_n\|.$$

For $N \geq M \geq 1$, by the Triangle Inequality:

$$\|S_N - S_M\| = \left\| \sum_{n=M+1}^{N} x_n \right\| \leq \sum_{n=M+1}^{N} \|x_n\| = |T_N - T_M|.$$

Since $\sum \|x_n\|$ converges in $\mathbb{R}$, the sequence of partial sums (T_N) is Cauchy in $\mathbb{R}$. The inequality above implies that (S_N) is Cauchy in V. Since V is Banach (complete), (S_N) converges in V. Thus $\sum x_n$ converges.

$(\Longleftarrow)$: Conversely, suppose that absolute convergence implies convergence in V. Let $(y_n) \subset V$ be a Cauchy sequence. We want to show (y_n) converges.

Since (y_n) is Cauchy, for each $k \geq 1$ we can choose an index n_k such that for all $n, m \geq n_k$:

$$\|y_n - y_m\| < 2^{-k}.$$

We can ensure $n_{k+1} > n_k$ strictly increasing. Set $x_1 = y_{n_1}$ and for $k \geq 1$ let $x_{k+1} = y_{n_{k+1}} - y_{n_k}$.

Then:

$$\sum_{j=1}^{k} x_{j+1} = \sum_{j=1}^{k} (y_{n_{j+1}} - y_{n_j}) = y_{n_{k+1}} - y_{n_1}.$$

So the partial sums of $\sum x_j$ recover the subsequence (y_{n_k}).

Consider the series of norms:

$$\sum_{k=1}^{\infty} \|x_{k+1}\| = \sum_{k=1}^{\infty} \|y_{n_{k+1}} - y_{n_k}\| < \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Since $\sum \|x_j\|$ converges in $\mathbb{R}$ (it is absolutely convergent), by our hypothesis, $\sum x_j$ converges in V. This means the subsequence (y_{n_k}) converges to some limit $y \in V$.

Since (y_n) is a Cauchy sequence and has a convergent subsequence $(y_{n_k}) \to y$, the entire sequence must converge to y.

(Standard argument: $\|y_n - y\| \leq \|y_n - y_{n_k}\| + \|y_{n_k} - y\| \to 0$).

Thus every Cauchy sequence in V converges, so V is **complete** (Banach). $\square$

Definition 18.12. Two norms $\|\cdot\|_1, \|\cdot\|_2$ on a vector space V are equivalent if there exist $A, B > 0$ such that

$$A\|x\|_1 \leq \|x\|_2 \leq B\|x\|_1$$

for all $x \in V$

18.2 Finite dimensional normed vector spaces

Now we want to show that on a finite dimensional vector space, all norms are equivalent.

Theorem 18.13 (All norms on a finite dimensional vector space are equivalent.)**.**

Proof. Equivalence of norms is an equivalence relation, so let V be finite dimensional and pick $\{e_i\}_{i=a}^{n}$ a basis of V note $(\dim(v) = n)$. Then define

$$\|v\| = \|\sum_{i=1}^{n} a_i e_i\| = \sum_{i=1}^{n} |a_i|, \|e_i\| = 1 \text{ in particular.}$$

If $x = 0$, we are done, so if we knew

$$A\|U\|_1 \leq \|U\|_2 \leq B\|U\| \quad \text{(call this } \star\text{)},$$

for $u \in V$, $\|U\|_1 = 1$, the we would have

$$A\|x\|_1 \leq \|x\|_2 \leq B\|x\|_1$$

for general $x \in V\backslash\{0\}$ by considering $U = \frac{x}{\|x\|_1}$ (multiply $\star$ by $\|x\|_1$).

We prove $\star$ for $\|\cdot\|_1 = \|\cdot\|$. By the reverse triangle inequality, for $x, y \in V$,

$$|\|x\|_2 - \|y\|_2| \leq \|x - y\|_2$$

if $x = \sum_{i=1}^n a_i e_i, \quad y = \sum_{i=1}^n b_i e_i$, then

$$|\|x\|_2 - \|y\|_2| \leq \|x - y\|_2 \leq \|\sum_{i=1}^n (a_i - b_i)e_i\|_2 = \sum_{i=1}^n |a_i - b_i|\|e_i\|_2.$$

This implies that

$$|\|x\|_2 - \|y\|_2| \leq \|x - y\|_2 \leq \|x - y\| \max_{i=1,\ldots,n}\{\|e_i\|_2\}$$

for $\epsilon > 0$, choose $\delta = \frac{\epsilon}{\max_{i=1,\ldots,n}\{\|e_i\|_2\}}$

then $\|x - y\| < \delta \implies |\|x\|_2 - \|y\|_2| < \epsilon$. This implies that $\|\cdot\|_2 : (V, \|\cdot\|) \to [0, \infty)$ is continuous.

Finally,

$S = \{x \in V \text{ such that } \|x\| - 1\}$ will be shown to be compact. By the extreme value theorem, $\|\cdot\|_2$ has a max and a min on S. This implies that $A \leq \|x\|_2 \leq B$ for some $A, B > 0$ for $x \in S \iff \star$.

Consider $T = \{(a_1, \ldots, a_n) \in \mathbb{R}^n | \sum_{i=1}^n |a_i| = 1\}$ and the map $f : T \to S$ by sending $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i e_i$.

T is closed and bounded in $\mathbb{R}^n$, so Heine-Borel implies that T is compact.

So it suffices to show that f is continuous for $\epsilon > 0$. Let $\delta = \frac{\epsilon}{\sqrt{n}}$.

This implies that

$$|(a_1, \ldots, a_n) - (b_1, \ldots, b_n)| < \frac{\epsilon}{\sqrt{n}}$$

This implies by Cauchy-Schwartz that

$$\|\sum_{i=1}^n (a_i - b_i)e_i\| = \sum_{i=1}^n |a_i - b_i| \leq n^{\frac{1}{2}} \left(\sum_{i=1}^n |a_i - b_i|^2\right)^{\frac{1}{2}} < \epsilon.$$

This implies that f is continuous. $\square$

19 Lecture 19: Differentiation

We are now able to recover familiar theory from Calculus for One-Variable functions using the precise formulation of limits:

Definition 19.1 (Derivative)**.** We say that $f : [a, b] \to \mathbb{R}$ is **differentiable** at $x \in [a, b]$ if

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} \text{ exists,}$$

and we call $f'(x)$ the **derivative** of f at x. If f is differentiable at every $x \in E \subset [a, b]$, we say that f is differentiable on E, and let $f' : E \to \mathbb{R}$ be the derivative as a function.

With this we recover all of the main results from Calculus I:

Theorem 19.2. If $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$ then it is continuous at x.

Proof. If $t \neq x$ then

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x}(t - x) \to f'(x) \cdot 0 = 0$$

as $t \to x$; hence f is continuous at x. □

Remark 19.3. $|x|$ is continuous at 0 but not differentiable at 0. □

Theorem 19.4. If $f, g : [a, b] \to \mathbb{R}$ are differentiable at x then:

1. $(f + g)'(x) = f'(x) + g'(x)$.
2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. (Product rule)

Proof. (1) follows from the limit laws. For (2), we note

$$f(t)g(t) - f(x)g(x) = f(t)(g(t) - g(x)) + g(x)(f(t) - f(x))$$

So that dividing by $(t - x)$ and letting $t \to x$ we are done. □

Example 19.5. Using (2), $(x^n)' = nx^{n-1}$ for $n \in \mathbb{Z}$ (with $x \neq 0$ for $n < 0$). Hence Polynomials, rational functions are differentiable. □

Theorem 19.6 (Chain Rule)**.** If $f : [a, b] \to \mathbb{R}$ is continuous, $f'(x)$ exists for some $x \in [a, b]$ and $g : I \to \mathbb{R}$ is differentiable at $f(x) \in I$ then

$$(g \circ f)'(x) = f'(x) \cdot g'(f(x))$$

Proof. By definition there are $u(t), v(s) \to 0$ as $t \to x$ and $s \to f(x)$ so that

$$\begin{cases} f(t) - f(x) = (t - x)(f'(x) + u(t)), \\ g(s) - g(f(x)) = (s - f(x))(g'(f(x)) + v(s)), \end{cases}$$

thus setting $s = f(t)$ we have

$$\begin{aligned} g(f(t)) - g(f(x)) &= (f(t) - f(x))(g'(f(x)) + v(f(t))) \\ &= (t - x)(f'(x) + u(t))(g'(f(x)) + v(f(t))). \end{aligned}$$

Dividing both sides by $(t - x)$ and letting $t \to x$ we are done. $\square$

Example 19.7 (Quotient Rule)**.** We derive this using the product rule and chain rule on $f \cdot g^{-1}$:

$$\begin{aligned} \left(\frac{f}{g}\right)' &= (f \cdot g^{-1})' \\ &= f'(g^{-1}) + f \cdot (g^{-1})' \\ &= f'g^{-1} + f \cdot (-g^{-2} \cdot g') \\ &= \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2} \end{aligned}$$

$\square$

Example 19.8. Let

$$f(x) = \begin{cases} -x^2 \sin(\frac{1}{x}), & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases}$$

then for $x \neq 0$ we have

$$f'(x) = -2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}).$$

At $x = 0$ we see that if $t \neq 0$ then

$$\left| \frac{f(t) - f(0)}{t} \right| = \left| t \sin(\frac{1}{t}) \right| \leq |t| \implies f'(0) = 0;$$

hence f is differentiable on $\mathbb{R}$! Note however that f' is not continuous as $\lim_{x \to 0}(\cos(\frac{1}{x}))$ DNE! $\square$

Definition 19.9 (Local Extrema)**.** We say that $f : X \to \mathbb{R}$ has a **local maximum** at $x \in X$ if there is some $\delta > 0$ such that $d(x, y) < \delta \implies f(y) \leq f(x)$. **Minima** are defined analogously. **Extrema** = max/minima.

Theorem 19.10 (Fermat's Theorem)**.** If $f : [a, b] \to \mathbb{R}$ has a local extrema at $x \in (a, b)$ then if $f'(x)$ exists, $f'(x) = 0$.

Proof. Suppose (w.l.o.g.) x is a local maximum, thus there is some $\delta > 0$ such that $y \in$

$(x-\delta, x+\delta) \implies f(y) \le f(x)$. Hence,

$$\frac{f(t)-f(x)}{t-x} \le 0 \quad \text{if } t \in (x, x+\delta),$$

and

$$\frac{f(t)-f(x)}{t-x} \ge 0 \quad \text{if } t \in (x-\delta, x),$$

$f'(x)$ if it exists, must equal zero. $\square$

Theorem 19.11 (Generalized Mean Value Theorem)**.** If $f, g : [a, b] \to \mathbb{R}$ are continuous functions which are differentiable on (a, b), then there is some $x \in (a, b)$ such that

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

In particular, if $g(x) = x$ then

$$f'(x) = \frac{f(b) - f(a)}{b - a} \quad \text{(Mean Value Theorem)}.$$

Proof. Let

$$h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$$

so that h is continuous on $[a, b]$ and differentiable on (a, b). If h is constant then $h'(x) = 0$ for all $x \in (a, b)$, so we're done. If h is not constant, then as $h(a) = h(b) = f(b)g(a) - g(b)f(a)$ there must be some local extrema $x \in (a, b)$ so that by the previous result $h'(x) = 0$. $\square$

Remark 19.12. This shows that

- $f' > 0 \implies f$ increasing.
- $f' = 0 \implies f$ constant.
- $f' < 0 \implies f$ decreasing.

$\square$

Theorem 19.13 (L'Hospital's Rule)**.** Let $f, g : (a, b) \to \mathbb{R}$ be differentiable, $g'(x) \neq 0$ for $x \in (a, b)$. If

$$\frac{f'(x)}{g'(x)} \to A \in \mathbb{R} \cup \{\pm\infty\}$$

as $x \to a$ (or $x \to b$) and either

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or
- $g(x) \to \pm\infty$ as $x \to a$,

then

$$\frac{f(x)}{g(x)} \to A \in \mathbb{R} \cup \{\pm\infty\} \quad \text{as } x \to a \text{ (or } x \to b).$$

Proof. We consider the cases $A < +\infty$ and $x \to a$ first; the case $x \to b$ is easy to adapt. Fix some $B \in \mathbb{R}$ with $A < B$ and choose some $r \in (A, B)$. We then have that for some $c \in (a, b)$ $x \in (a, c) \implies \frac{f'(x)}{g'(x)} < r$. Hence if $a < x < y < c$ the MVT implies there is $t \in (x, y)$ with

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r. \quad (+)$$

If $f(x), g(x) \to 0$ as $x \to a$, then $\frac{f(x)}{g(x)} \leq r < B$ for $x \in (a, c)$. Similarly, if $g(x) \to +\infty$ ($-\infty$ similar) then for smaller choice of c, $x \in (a, c)$ we have $g(x) > 0$ and $g(x) > g(y)$. Multiply $(+)$ by $(g(x) - g(y))/g(x)$ to get

$$\frac{f(x) - f(y)}{g(x)} < r\left(\frac{g(x) - g(y)}{g(x)}\right) \implies \frac{f(x)}{g(x)} < B$$

for $x \in (a, c)$ sufficiently close to a. Thus we have, in either case, that $\frac{f(x)}{g(x)} < B$ whenever $x \in (a, \hat{c})$ for some $\hat{c}$. If $A = -\infty$ we are then done. If $-\infty < A \leq +\infty$ then we can similarly find $\hat{c}$ such that if $\tilde{B} \in \mathbb{R}$ with $\tilde{B} < A$ $\frac{f(x)}{g(x)} > \tilde{B}$ for $x \in (a, \hat{c})$; then the remaining cases then follow. □

20 Lecture 20: Continuity of Derivatives

We saw already that f being differentiable does not imply that f' is continuous. However we can see that f' satisfies the conclusions of the IVT:

Theorem 20.1 (Darboux's Theorem)**.** If $f : [a, b] \to \mathbb{R}$ is differentiable and $c \in (f'(a), f'(b))$ (or $(f'(b), f'(a))$), then there is some $x \in (a, b)$ such that $f'(x) = c$.

Proof. Let $g(x) = f(x) - ct$ so that both

$$\begin{cases} g'(a) = f'(a) - c < 0, \\ g'(b) = f'(b) - c > 0, \end{cases}$$

and thus a, b are not extrema for g. Hence, by the EVT g has a local extrema $x \in (a, b)$ which by Fermat's theorem $\implies g'(x) = 0 \implies f'(x) = c$. □

Remark 20.2. This tells us that the derivative of a function cannot have any discontinuities of the first kind (i.e. no jumps). □

20.1 Polynomial Approximation

We can use MVT to approximate a function by its tangent: $f(y) = f(x) + f'(c)(y - x)$. More generally:

Theorem 20.3 (Taylor's Theorem)**.** Let $f : [a, b] \to \mathbb{R}$, $f^{(n-1)}$ be continuous on $[a, b]$, and $f^{(n)}$ defined on (a, b). Then if $\alpha, \beta \in [a, b]$ with $\alpha \neq \beta$,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

for some $x \in (\alpha, \beta)$. We call $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$ the $(n-1)$th order **Taylor Polynomial** of f at α.

Remark 20.4. This is saying that f can be approximated by a polynomial of degree $(n - 1)$, namely $P(t)$, with the error of this approximation controlled by the nth derivative of f, $f^{(n)}$. $\square$

Proof. First we let $M \in \mathbb{R}$ be such that

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n,$$

and define $g(t) = f(t) - P(t) - M(t - \alpha)^n$ on $[a, b]$; we will show that $M = f^{(n)}(x)/n!$ for some $x \in (\alpha, \beta)$. We compute that

$$g^{(n)}(t) = f^{(n)}(t) - P^{(n)}(t) - M \cdot n! = f^{(n)}(t) - M \cdot n!$$

and hence we are done if we find $x \in (\alpha, \beta)$ with $g^{(n)}(x) = 0$. By construction, $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \ldots, n-1$ and hence $g^{(k)}(\alpha) = f^{(k)}(\alpha) - P^{(k)}(\alpha) = 0$ for such k; moreover, by the choice of M we have $g(\alpha) = g(\beta) = 0$. We now iteratively apply the MVT to produce $x_{k+1} \in (\alpha, x_k)$ such that $g^{(k)}(x_k) = 0$ for each $k = 0, \ldots, n$; hence we have some $x \in (\alpha, x_{n-1}) \subset (\alpha, \beta)$ such that $g^{(n)}(x) = 0$ and thus we have $M = f^{(n)}(x)/n!$ as desired. $\square$

Definition 20.5 (Real Analytic)**.** We say that $f : [a, b] \to \mathbb{R}$ is **real analytic** if $f^{(n)}$ exists on (a, b) for every $n \geq 1$ (i.e. smooth) and such that for each $y \in (a, b)$ we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!}(x - y)^k$$

Example 20.6. Polynomials, e^x, trigonometric functions, logarithms are! $|x|$ is not as it is not differentiable. $\square$

Example 20.7 (Smooth but not Analytic)**.** Let

$$f(x) = \begin{cases} e^{-1/x}, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0, \end{cases}$$

then $f^{(n)}$ exists on $\mathbb{R}$ for every n (Check), and in fact $f^{(n)}(0) = 0$. Hence for each $n \geq 1$, any Taylor Polynomial of f at 0 is 0; but $f(x) > 0$ for all small $\epsilon > 0$ so we see that f is not analytic! $\square$

20.2 Multivariable Differentiation

The definition of the derivative for functions $f : [a, b] \to \mathbb{R}^n$ or $\mathbb{C}$ makes sense provided we interpret norms and points correctly. All the rules (Sum, Product, Chain, diff'ble $\implies$ cts) hold with correct interpretation (e.g. if $f, g : [a, b] \to \mathbb{R}^n$ then $f \cdot g$ dot product for Product rule).

Remark 20.8. If $f : [a, b] \to \mathbb{C}$ we can write $f = f_1 + if_2$ for $f_1, f_2 : [a, b] \to \mathbb{R}$ thus $f' = f_1' + if_2'$ so that f is diff'ble $\iff$ f_1, f_2 are. Similarly, if $f : [a, b] \to \mathbb{R}^n$ then $f = (f_1, \ldots, f_n)$ for $f_i : [a, b] \to \mathbb{R}$ thus $f' = (f_1', \ldots, f_n')$ so that f is diff'ble $\iff$ $f_1, \ldots, f_n$ are. $\square$

We now see that MVT and its consequences fail however:

Example 20.9 (MVT Fails for $\mathbb{C}$)**.** If $f : \mathbb{R} \to \mathbb{C}$ is defined by $f(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$ then $f(2\pi) = f(0) = 1$ for each k but we have $f'(\theta) = -\sin(\theta) + i\cos(\theta) \implies |f'(\theta)| = 1$ for any $\theta \in \mathbb{R}$. Hence $f(2\pi) - f(0) \neq 2\pi f'(\theta)$ for any $\theta \in (0, 2\pi)$; so MVT fails! $\square$

Example 20.10 (L'Hospital's Fails for $\mathbb{C}$)**.** If $f : \mathbb{R} \to \mathbb{C}$ is defined by $f(x) = x$ then defined by $g(x) = x + x^2 e^{i/x^2}$ and $\lim_{x \to 0} \frac{f(x)}{g(x)} = 1$, but we have $g'(x) = 1 + (2x - \frac{2i}{x})e^{i/x^2}$ so that

$$|g'(x)| \geq -1 + |2x - \frac{2i}{x}| \geq -1 + \frac{2}{x}(\text{as } x \in (0, 1)).$$

Thus

$$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2 - x} \implies \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

We therefore see that L'Hospital's rule fails too! $\square$

Note that the MVT shows that for diff'ble $f : [a, b] \to \mathbb{R}$ $|f(b) - f(a)| \leq (b - a) \sup_{a<x<b} |f'(x)|$; even though it does not hold for multivariable functions we have an analogue of the above:

Theorem 20.11 (MVT for Vector-Valued Functions)**.** If $f : [a, b] \to \mathbb{R}^k$ is a continuous function and differentiable on (a, b) then there is $x \in (a, b)$ such that

$$|f(b) - f(a)| \leq (b - a)|f'(x)|.$$

Proof. Let $z = f(b) - f(a)$ and define $g : [a, b] \to \mathbb{R}$ by setting $g(t) = z \cdot f(t)$; g is then a continuous function which is diff'ble on (a, b) so the MVT implies

$$g(b) - g(a) = (b - a)g'(x) = (b - a)(z \cdot f'(x))$$

for some $x \in (a, b)$. Note that

$$g(b) - g(a) = z \cdot (f(b) - f(a)) = z \cdot z = |z|^2;$$

and thus combining the above we see that $|z|^2 = (b - a)(z \cdot f'(x))$. By the Cauchy-Schwarz inequality we see that

$$|z \cdot f'(x)| \leq |z||f'(x)|$$

and so

$$|z|^2 \leq (b - a)|z \cdot f'(x)| \leq (b - a)|z||f'(x)|;$$

thus we have (even if $z = 0$)

$$|z| \leq (b - a)|f'(x)|$$

or $|f(b) - f(a)| \leq (b - a)|f'(x)|$ for some $x \in (a, b)$, as desired. $\square$

Part V

Sequences of Functions

21 Lecture 21: Pointwise vs. Uniform Convergence

21.1 Pointwise Convergence and its Failures

Definition 21.1 (Pointwise Convergence)**.** Given a sequence (f_n) of $\mathbb{C}$ valued functions on a metric space (X, d) such that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in X$ we define the limit, $f : X \to \mathbb{C}$, by setting

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for each } x \in X.$$

We then say that (f_n) **converges pointwise** to f. Similarly, if $\sum_n f_n(x)$ converges for every $x \in X$ we define the sum, $f : X \to \mathbb{C}$, by setting

$$f(x) = \sum_n f_n(x) \quad \text{for each } x \in X.$$

We want to understand whether limits/sums of functions preserve the properties of the sequence; e.g. if (f_n) is a sequence of continuous/differentiable functions, is the limit/sum continuous/differentiable? Moreover, can we relate (f_n') to f'? Recall that f is continuous at $x \iff f(x) = \lim_{t\to x} f(t)$, and thus asking whether the limit of continuous functions is continuous is asking if

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t);$$

Namely, can we 'swap' limits? We see now that pointwise convergence is not sufficient:

Example 21.2 (Swapping Limits Fails)**.** Let $S_{m,n} = \frac{m}{m+n}$ for each $m, n \geq 1$. Then,

$$\lim_{m\to\infty} S_{m,n} = 1 \implies \lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} = 1,$$

$$\lim_{n\to\infty} S_{m,n} = 0 \implies \lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} = 0.$$

So pointwise convergence of functions is not enough to guarantee continuity of limits!

□

21.1.1 Failure to preserve continuity

Example 21.3 (Limit of Cts is not Cts)**.** For $x \in \mathbb{R}$ and $n \geq 0$ let $f_n(x) = \frac{x^2}{(1+x^2)^n}$. Set

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

As $f_n(0) = 0$ we have $f(0) = 0$. If $x \neq 0$ then we have

$$f(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots = x^2 \left(\frac{1}{1 - \frac{1}{1+x^2}} \right) = 1 + x^2.$$

Hence

$$f(x) = \begin{cases} 0, & \text{for } x = 0 \\ 1 + x^2, & \text{for } x \neq 0 \end{cases}$$

Thus f is not continuous! □

21.1.2 Failure to preserve derivatives

Example 21.4 (Limit of Derivatives)**.** For $x \in \mathbb{R}$ and $n \geq 1$ let

$$g_n(x) = \frac{\sin(nx)}{\sqrt{n}}$$

so that $g(x) = \lim_{n\to\infty} g_n(x) = 0$ but $g_n'(x) = \sqrt{n}\cos(nx)$ is such that $\lim_{n\to\infty} g_n'(x)$ DNE so that g_n' does not converge pointwise to $g' = 0$. □

21.2 Uniform Convergence

We introduce a stronger notion of convergence for functions:

Definition 21.5 (Uniform Convergence)**.** We say that a sequence (f_n) of $\mathbb{C}$ valued functions converges **uniformly** to f on $E \subset X$, for a metric space (X, d), if for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon \quad \text{for every } x \in E.$$

We say that $\sum f_n$ **converges uniformly** to f if the partial sums $S_N = \sum_{n=1}^{N} f_n$ converge uniformly.

Remark 21.6. (f_n) converges uniformly to $f \implies (f_n)$ converges pointwise to f. We sometimes write $f_n \rightrightarrows f$ or $f_n \xrightarrow{unif} f$ to abbreviate. □

21.3 The Uniform Cauchy Criterion

Theorem 21.7 (Uniform Cauchy Criterion)**.** A sequence (f_n) of $\mathbb{C}$ valued functions on $E \subset X$, for a metric space (X, d), converges uniformly on E if and only if it is **uniformly Cauchy**; namely for each $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$m, n \geq N \implies |f_m(x) - f_n(x)| < \epsilon \quad \text{for all } x \in E.$$

Proof. ($\implies$): If $f_n \rightrightarrows f$ and $\epsilon > 0$ then there is some $N \in \mathbb{N}$ such that $n \geq N \implies |f_n(x) - f(x)| < \epsilon/2$ for all $x \in E$. Hence if $m, n \geq N$ then

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $x \in E$; thus (f_n) is uniformly Cauchy.

($\impliedby$): If (f_n) is uniformly Cauchy and $\epsilon > 0$ then there is some $N \in \mathbb{N}$ such that $m, n \geq N \implies |f_m(x) - f_n(x)| < \epsilon/2$ for all $x \in E$. Noting that the sequence $(f_n(x))$ is Cauchy in $\mathbb{C}$ for each $x \in E \implies f(x) = \lim_{m\to\infty} f_m(x)$ exists for each $x \in E$. Combining these facts, if $m, n \geq N$ then for each $x \in E$

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} < f_m(x) - f_n(x) < \frac{\epsilon}{2},$$

so that $-\epsilon/2 \leq \lim_{m\to\infty}(f_m(x) - f_n(x)) = f(x) - f_n(x) \leq \epsilon/2$ and so $|f(x) - f_n(x)| \leq \epsilon/2 < \epsilon$ for all $x \in E$; thus $f_n \rightrightarrows f$. □

Theorem 21.8 (Sup-Norm Convergence)**.** Suppose that $f_n \xrightarrow{ptwise} f$ on $E \subset X$, for a metric space (X, d). Then $f_n \rightrightarrows f$ on E if and only if

$$M_n = \sup_{x\in E} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

Proof. If $f_n \rightrightarrows f$ then $M_n \to 0$ by definition. If $M_n \to 0$ then for each $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that $M_n < \epsilon$ for $n \geq N$. Hence for $n \geq N$ we have

$$|f_n(x) - f(x)| \leq \sup_{y \in E} |f_n(y) - f(y)| = M_n < \epsilon \quad \text{for all } x \in E,$$

so $f_n \rightrightarrows f$. □

Example 21.9. The functions $f_n(x) = \frac{1}{nx+1}$ on $(0,1) \subset \mathbb{R}$ for $n \geq 1$ are such that $f_n \xrightarrow{ptwise} 0$ but for each n we have

$$\sup_{x \in (0,1)} |0 - f_n(x)| = \sup_{x \in (0,1)} \left| \frac{1}{nx+1} \right| = 1$$

So we see that $\sup_{x \in (0,1)} |f_n(x)| \not\to 0$ so $f_n \not\rightrightarrows 0$. □

21.4 The Weierstrass M-Test

Theorem 21.10 (Weierstrass M-test)**.** If (f_n) is a sequence of $\mathbb{C}$ valued functions on $E \subset X$, for a metric space (X, d), with $|f_n(x)| \leq M_n$ for each $x \in E$, then $\sum f_n$ converges uniformly if $\sum_n M_n$ converges.

Proof. As $\sum M_n$ converges in $\mathbb{R}$, its partial sums, (S_N), are Cauchy in $\mathbb{R}$. Hence for each $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that $n \geq m \geq N$ implies by the Δ inequality that

$$\left| \sum_{i=m}^{n} f_i(x) \right| \leq \sum_{i=m}^{n} |f_i(x)| \leq \sum_{i=m}^{n} M_i < \epsilon$$

so that the partial sums of $\sum f_n$ are uniformly Cauchy, and hence uniformly convergent by the first result above. □

Remark 21.11 (Converse of M-test)**.** The converse statement fails in general! To see this we can choose the constant functions on $\mathbb{R}$ defined by

$$f_n(x) = \frac{(-1)^{n+1}}{n} \quad \text{for } n \geq 1;$$

then $|f_n(x)| = \frac{1}{n}$ so that $\sum_n f_n(x) = \log(2)$ for all $x \in \mathbb{R}$ but $\sum_n \frac{1}{n}$ diverges! Hence $\sum_n f_n$ converges uniformly but $\sum_n M_n$ does not converge, so converse fails! One could also consider "sliding hump" functions

$$f_n = \frac{1}{n} \chi_{(n,n+1)} = \begin{cases} 1/n, & x \in (n, n+1) \\ 0, & \text{o/w} \end{cases}$$

□

22 Lecture 22: Uniform Convergence and Continuity

We now see how uniform convergence guarantees continuity of limits.

Theorem 22.1 (Uniform Convergence and Limits)**.** Suppose that $f_n \rightrightarrows f$ on $E \subset X$, for a metric space (X, d), x is a limit point of E and $\lim_{t \to x} f_n(t) = A_n$. Then (A_n) converges and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

Moreover, if the (f_n) are continuous on E then f is continuous on E.

Proof. As $f_n \rightrightarrows f$ the sequence (f_n) is uniformly Cauchy; hence for each $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that $n, m \geq N \implies |f_n(t) - f_m(t)| < \epsilon$ for each $t \in E$. We then can send $t \to x$ to see that $\implies |A_n - A_m| \leq \epsilon$, so that (A_n) is Cauchy in $\mathbb{C}$ and thus converges to some $A \in \mathbb{C}$.

Note that for each $n \geq 1$ and $t \in E$ we have

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|,$$

by applying Δ ineq. twice. For $\epsilon > 0$ we first choose $n \geq 1$ such that both

$$|f(t) - f_n(t)| < \frac{\epsilon}{3} \quad \text{for all } t \in E \text{ (since } f_n \rightrightarrows f\text{)},$$

and

$$|A_n - A| < \frac{\epsilon}{3} \quad \text{(since } A_n \to A \text{ as } n \to \infty\text{)}.$$

Finally, we choose some neighborhood, U, of x in X such that $|f_n(t) - A_n| < \epsilon/3$ for all $t \in (E \cap U) \setminus \{x\}$ (since $\lim_{t \to x} f_n(t) = A_n$). Combining the above we have that $|f(t) - A| < \epsilon$ for all $t \in (E \cap U) \setminus \{x\}$, thus we have the desired conclusions. □

Remark 22.2. Previous examples show that $f_n \xrightarrow{ptwise} f$ and f_n continuous do not guarantee f is continuous! □

We can however guarantee the converse on compact sets:

Theorem 22.3 (Dini's Theorem)**.** If $f_n \xrightarrow{ptwise} f$ on a compact set K, the $(f_n), f$ are continuous, and $f_n \geq f_{n+1}$ (or $f_n \leq f_{n+1}$) for each $n \geq 1$, then $f_n \rightrightarrows f$ on K.

Proof. Consider the functions $g_n = f_n - f$ for each $n \geq 1$, then the (g_n) are continuous with $g_n \xrightarrow{ptwise} 0$ and $g_n \geq g_{n+1}$. We show that $g_n \rightrightarrows 0$, which implies $f_n \rightrightarrows f$. For $\epsilon > 0$ let $K_n = g_n^{-1}([\epsilon, \infty))$ (closed as g_n is continuous) for each $n \geq 1$; as K is compact $\implies K_n$ compact also. If $x \in K_{n+1}$ then $\epsilon \leq g_{n+1}(x) \leq g_n(x)$, so $x \in K_n$ also, thus $K_{n+1} \subset K_n$ for each $n \geq 1$. Now for each $x \in K$ we have $g_n(x) \to 0$ so that $x \notin K_n$ for all n sufficiently large (since $g_n(x) < \epsilon$ eventually in n); thus we have $\bigcap_{n \geq 1} K_n = \emptyset$ and $K_{n+1} \subset K_n$ for $n \geq 1 \implies K_N = \emptyset$ for some $n \geq 1$. To conclude we must have some $N \in \mathbb{N}$ such that

$K_n = \emptyset$ for $n \geq N \implies g_n(x) < \epsilon$ for all $x \in K$ and $n \geq N$; as we must have $g_n \geq 0$ for every $n \geq 1$ (else $g_n(x) \to 0$ fails since $g_n \geq g_{n+1}$). This implies $g_n \rightrightarrows 0 \iff f_n \rightrightarrows f$. $\square$

Remark 22.4. We need compactness as the example $f_n(x) = \frac{1}{nx+1}$ on $(0,1)$ shows ($f_n \geq f_{n+1}$ holds). Similarly the monotone requirement is necessary. Consider the 'sliding hump' functions on $[0,1]$:

$$g_n(x) = \begin{cases} nx, & \text{for } 0 \leq x \leq 1/n \\ 2 - nx, & \text{for } 1/n \leq x \leq 2/n \\ 0, & \text{for } 2/n < x \end{cases}$$

These are not monotone, $g_n \xrightarrow{ptwise} 0$ and $g_n \not\rightrightarrows 0$ (as $g_n(1/n) = 1$ for all $n \geq 1$). $\square$

Recall, we defined $\mathcal{C}(X)$ to be the set of continuous bounded functions on a metric space (X, d), which becomes a normed vector space with the **supremum norm**, $||f|| = \sup_{x \in X} |f(x)|$. This norm induced a metric, $d(f,g) = ||f - g||$ on $\mathcal{C}(X)$. Since $f_n \rightrightarrows f \iff \sup_{x \in X} |f_n(x) - f(x)| \to 0$ we see that $f_n \to f$ in $\mathcal{C}(X) \iff f_n \rightrightarrows f$ on X. Moreover:

Theorem 22.5. $\mathcal{C}(X)$ is a complete metric space.

Proof. If $(f_n) \subset \mathcal{C}(X)$ is a Cauchy sequence it must be uniformly Cauchy. $\implies$ there is some $f : X \to \mathbb{C}$ such that $f_n \rightrightarrows f$ on X. As the (f_n) are continuous we have that f is also continuous. We also know that f is bounded since $f_n \rightrightarrows f \implies |f_n(x) - f(x)| < 1$ for all $x \in X$ and some n sufficiently large. Hence $f \in \mathcal{C}(X)$ and since $f_n \rightrightarrows f$ we have $d(f_n, f) \to 0$ as $n \to \infty$. $\square$

We now discuss how uniform convergence is related to differentiation. We have already seen that $g_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ on $\mathbb{R}$ s.t. $g_n \xrightarrow{ptwise} 0$ but g_n' does not converge. Similarly even if the derivatives converge, the limiting function may fail to be differentiable; for example consider $h_n(x) = x^n$ on $[0,1]$, then $h_n'(x) = nx^{n-1}$ so that $h_n \xrightarrow{ptwise} h(x) = \begin{cases} 0, & \text{for } 0 \leq x < 1 \\ 1, & \text{for } x = 1 \end{cases}$. (one could also consider $\sqrt{x^2 + 1/n} \to |x|$).

Theorem 22.6 (Uniform Convergence and Differentiation)**.** Let (f_n) be a sequence of $\mathbb{R}$ valued differentiable functions on $[a,b]$ such that $(f_n(x_0))$ converges for some $x_0 \in [a,b]$. If (f_n') converges uniformly on $[a,b]$, then (f_n) converges uniformly on $[a,b]$ to a function f and $f'(x) = \lim_{n \to \infty} f_n'(x)$ for all $x \in [a,b]$.

Proof. Let $\epsilon > 0$. Since $(f_n(x_0))$ converges it is Cauchy so there is some $N \in \mathbb{N}$ such that $m, n \geq N \implies |f_m(x_0) - f_n(x_0)| < \epsilon/2$ and since (f_n') is uniformly convergent it is uniformly Cauchy and so potentially taking N larger we also have $\implies |f_m'(t) - f_n'(t)| < \frac{\epsilon}{2(b-a)}$ for all $t \in [a,b]$. By MVT, for any $x, t \in [a,b]$ there is some $c \in (x,t)$ (or $c \in (t,x)$) such that if $m, n \geq N$ then

$$|(f_m(x) - f_n(x)) - (f_m(t) - f_n(t))| = |x - t||f_m'(c) - f_n'(c)|$$

$$\leq \frac{\epsilon|x-t|}{2(b-a)} \leq \frac{\epsilon}{2} \quad (+)$$

Hence for each $x \in [a,b]$ we have for $m,n \geq N$ that

$$\begin{aligned}|f_m(x) - f_n(x)| &\leq |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0))| \\ &\quad + |f_m(x_0) - f_n(x_0)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;\end{aligned}$$

so that (f_n) is uniformly Cauchy and hence uniformly convergent on $[a,b]$. Let the limit of (f_n) be f and, for $x \in [a,b]$ fixed, define for $t \in [a,b] \setminus \{x\}$ functions

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x} \quad \phi(t) = \frac{f(t) - f(x)}{t-x}$$

note then that $\lim_{t\to x} \phi_n(t) = f_n'(x)$ for each $n \geq 1$. By (+) we have

$$|\phi_m(t) - \phi_n(t)| \leq \frac{\epsilon}{2(b-a)} \quad \text{for } m,n \geq N$$

hence (ϕ_n) converges uniformly on $[a,b] \setminus \{x\}$ (as it is uniformly Cauchy). As $f_n \rightrightarrows f$ we have that $\phi_n \to \phi$ on $[a,b] \setminus \{x\}$, and as x is a limit point of $[a,b] \setminus \{x\}$ by an earlier result we know that

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{n\to\infty} f_n'(x),$$

so that $f'(x) = \lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$. □

Remark 22.7. If we don't have $(f_n(x_0))$ convergent for some $x_0 \in [a,b]$ then (f_n) may not even converge; e.g. Consider $f_n(x) = n$ on $[0,1]$, then $f_n' = 0$ for all n but f_n diverge! □

23 Lecture 23: A continuous but nowhere differentiable function

We will use the theory we have built up to see how wildly behaved continuous functions can be in general; Constructions of this type were said to belong to a 'gallery of monsters' by Poincaré (fractals).

Theorem 23.1. There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere differentiable.

Proof. Let us extend $|x|$ on $[-1,1]$ to a periodic function, φ, on $\mathbb{R}$: (so that $\varphi(x+2) = \varphi(x)$ for any $x \in \mathbb{R}$) By the reverse Δ inequality we have (check) for $s,t \in \mathbb{R}$ that $|\varphi(x) - \varphi(y)| \leq |x-y| \implies \varphi$ is continuous on $\mathbb{R}$. We then define $f_n(x) = (\frac{3}{4})^n \varphi(4^n x)$ for $x \in \mathbb{R}$, so that $|f_n(x)| \leq (\frac{3}{4})^n$ for each $n \geq 1$ and $x \in \mathbb{R}$. By the Weierstrass M-test we have that $\sum f_n$

converges uniformly to

$$f(x) = \sum_n f_n(x) = \sum_n \left(\frac{3}{4}\right)^n \varphi(4^n x) \quad \text{since } \sum (\frac{3}{4})^n < \infty.$$

Moreover, since the partial sums are continuous we have from their uniform convergence that f is also continuous; we will show that it is nowhere differentiable.

For each $x \in \mathbb{R}$ and $m \geq 1$ let $\delta_m = \pm\frac{1}{2} \cdot 4^{-m}$ where $\pm$ is chosen so that the interval between $4^m x$ and $4^m(x + \delta_m)$ contains no integer (as $4^m|\delta_m| = 1/2$). We now set $\gamma_n = \frac{1}{\delta_m}(\varphi(4^n(x + \delta_m)) - \varphi(4^n x))$ for $n \in \mathbb{N}$, so that if $n > m$ we have $\gamma_n = 0$ (since then $4^n \delta_m$ is even) and if $0 \leq n < m$ then $|\gamma_n| \leq 4^n$ (since $|\varphi(x) - \varphi(y)| \leq |x - y|$). We also note that $|\gamma_m| = 4^m$ (since no integer is between $4^m x$ and $4^m(x+\delta_m)$ we have $|\varphi(4^m(x+\delta_m)) - \varphi(4^m x)| = |4^m(x + \delta_m) - 4^m x| = 4^m|\delta_m| = 1/2$).

Finally we have that f is not differentiable at x since

$$\begin{aligned}
\frac{f(x+\delta_m) - f(x)}{\delta_m} &= \sum_n \left(\frac{3}{4}\right)^n \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m} \\
&= \sum_n \left(\frac{3}{4}\right)^n \gamma_n = \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \\
&= \left(\frac{3}{4}\right)^m \gamma_m + \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n
\end{aligned}$$

and so by reverse Δ ineq. we have

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\left(\frac{3}{4}\right)^m \gamma_m - \left(-\sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n\right)\right|$$

$$\geq \left|\left(\frac{3}{4}\right)^m \gamma_m\right| - \left|\sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n\right|.$$

Recalling $|\gamma_m| = 4^m$ and $|\gamma_n| \leq 4^n$ for $0 \leq n < m$ we have

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = 3^m - \left(\frac{3^m - 1}{2}\right) = \frac{3^m + 1}{2},$$

but $3^m \to \infty$ as $m \to \infty \implies f'(x)$ DNE. As $x \in \mathbb{R}$ was arbitrary f not differentiable anywhere! $\square$

23.1 Equicontinuity

The Bolzano-Weierstrass theorem guarantees that bounded sequences in $\mathbb{R}^k$ have convergent subsequences. One could ask whether an analogous result holds for bounded sequences of functions:

Definition 23.2. We say that a sequence (f_n) of $\mathbb{C}$ valued functions on a metric space, (X, d), are:

- **Pointwise bounded** on X if there is some $\phi : X \to \mathbb{R}$ such that $|f_n(x)| \leq \phi(x)$ for all $n \geq 1$.
- **Uniformly bounded** on X if there is some M such that $|f_n(x)| \leq M$ for all $n \geq 1$ and $x \in X$.

Remark 23.3.

- Uniformly convergent $\implies$ Uniformly bounded.
- If X is countable one can use Cantor's diagonalization argument to find a subsequence converging pointwise on X if the sequence is pointwise bounded.
- If a sequence is uniformly bounded it does not necessarily contain a pointwise convergent subsequence. This is shown in the text using the dominated convergence theorem on the sequence $(\sin(nx))$ on $[0, 2\pi]$.

One could also ask if convergent uniformly bounded sequences of functions contain uniformly convergent subsequences; but this also fails: $\square$

Example 23.4. Let

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \quad \text{on } [0, 1] \text{ for } n \geq 1.$$

We then see that $|f_n(x)| \leq 1$ so that (f_n) is uniformly bounded, and moreover $f_n(x) \to 0$ for all $x \in [0, 1]$. However, we have $f_n(1/n) = 1$ for all $n \geq 1$; so no subsequence converges uniformly. $\square$

Let us address the first remark above:

Theorem 23.5. If (f_n) is a pointwise bounded sequence of $\mathbb{C}$ valued functions on a countable metric space, (X, d), then there is a subsequence (f_{n_k}) such that $(f_{n_k}(x))$ converges for every $x \in X$.

Proof. Let us enumerate $X = (x_i)$. As $(f_n(x_1))$ is bounded there is a subsequence (f_k^1) of (f_n) such that $(f_k^1(x_1))$ converges as $k \to \infty$. As $(f_k^1(x_2))$ is bounded there is a subsequence (f_k^2) of (f_k^1) such that $(f_k^2(x_2))$ converges as $k \to \infty$. We inductively continue this process, generating a subsequence (f_k^{l+1}) of (f_k^l) such that $(f_k^{l+1}(x_{l+1}))$ converges as $k \to \infty$. We then choose the diagonal subsequence (f_k^k) of (f_n) which is such that $(f_k^k(x_i))$ converges for each $i \geq 1$ (since $(f_k^k)_{k \geq i}$ is a subsequence of (f_k^i) for $k \geq i$ by construction). Relabeling $f_k^k = f_{n_k}$ for each $k \geq 1$ we are done. $\square$

We have seen that pointwise and uniform boundedness are usually not enough to extract well behaved subsequences. We thus introduce:

Definition 23.6 (Equicontinuous)**.** A collection, $\mathcal{F}$, of $\mathbb{C}$ valued functions on a set $E \subset X$, for a metric space (X, d), is said to be **equicontinuous** on E if for each $\epsilon > 0$ there is some $\delta > 0$ such that

$$x, y \in E, d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon \quad \text{for every } f \in \mathcal{F}.$$

Remark 23.7. If $\mathcal{F}$ is equicontinuous $\implies$ every $f \in \mathcal{F}$ is uniformly continuous. $\square$

Example 23.8.

- If $\mathcal{F} = (f_n)$ for differentiable functions on $[0,1]$ with (f_n') uniformly bounded $\implies \mathcal{F}$ is equicontinuous; if $|f_n'(x)| \leq M$ for all $n, x \in [0,1]$ and $\epsilon > 0$, set $\delta = \frac{\epsilon}{M+1}$ so that by MVT we have

$$|x - y| < \frac{\epsilon}{M+1} \implies |f_n(x) - f_n(y)| \leq |x - y| \sup_{t \in [0,1]} |f_n'(t)| < \frac{\epsilon M}{M+1} < \epsilon.$$

- We saw that $G = (\frac{x^2}{x^2 + (1-nx)^2})$ on $[0,1]$ was uniformly bdd, pointwise $\to 0$, but had no uniformly convergent subsequence. G is also not equicontinuous as $g_n(1/n) = 1$ but $g_n(0) = 0$ for every $n \geq 1$.

- $H = (\arctan(nx))$ is not equicontinuous since $\arctan(x) \to \pm\pi/2$ as $x \to \pm\infty$.

We will see that there are strong relations between equicontinuity and uniform convergence. $\square$

24 Lecture 24: The Arzela-Ascoli Theorem

Theorem 24.1. Let (K, d) be compact and $(f_n) \subset \mathcal{C}(K)$, then if (f_n) converges uniformly on K, (f_n) is equicontinuous on K.

Proof. Let $\epsilon > 0$ and note that since (f_n) converges uniformly, it is uniformly Cauchy, there is some $N \in \mathbb{N}$ such that $m, n \geq N \implies ||f_n - f_m|| < \epsilon/3$. Now as continuous functions on compact sets are uniformly continuous, there is some $\delta_i > 0$ for each $i = 1, \ldots, N$ such that $d(x,y) < \delta_i \implies |f_i(x) - f_i(y)| < \epsilon/3$. Combining the above, for $n \geq N$ and $\delta_N > 0$ we have

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{if } d(x,y) < \delta_N. \end{aligned}$$

Hence, if we set $\delta = \min\{\delta_1, \ldots, \delta_N\}$ we have $d(x,y) < \delta \implies |f_n(x) - f_n(y)| < \epsilon$ for all $n \geq 1$, so that (f_n) is equicontinuous. $\square$

Theorem 24.2 (Arzela-Ascoli Theorem)**.** Let (K, d) be compact and $(f_n) \subset \mathcal{C}(K)$, then if (f_n) is pointwise bounded and equicontinuous on K then both:

1. (f_n) is uniformly bounded on K.
2. (f_n) has a uniformly convergent subsequence.

Proof. For (1), we choose $\delta > 0$ from equicontinuity of (f_n) so that $d(x, y) < \delta \implies |f_n(x) - f_n(y)| < 1$ for all $n \geq 1$. By compactness $K \implies$ there are $x_1, \ldots, x_L \in K$ such that $K \subset \bigcup_{i=1}^{L} B_\delta(x_i)$ and since (f_n) is pointwise bounded there are $M_1, \ldots, M_L$ such that $|f_n(x_i)| \leq M_i$ for each $i = 1, \ldots, L$. We then have $|f_n(x)| \leq M + 1$ for all $x \in K$ where $M = \max\{M_1, \ldots, M_L\}$. and $n \geq 1$.

For (2), we first show that K contains an at most countable subset $E \subset K$ which is dense; i.e. if $U \subset K$ is open then $E \cap U \neq \emptyset$. Since K is compact, for each $n \geq 1$ there is a finite set $\{x_i^n\}_{i=1}^{L_n} \subset K$ such that $K \subset \bigcup_{i=1}^{L_n} B_{1/n}(x_i^n)$; We then set $E = \bigcup_{n \geq 1} \{x_i^n\}_{i=1}^{L_n}$, which is at most countable. If $U \subset K$ is open then for each $x \in U$ there is some $\delta > 0$ such that $B_\delta(x) \subset U$ and hence for $1/n < \delta$ (Archimedean property) there is some $x_i^n \in E$ such that $d(x, x_i^n) < 1/n \leq \delta$ so that $x_i^n \in U$ also; hence $E \cap U \neq \emptyset$ so E is dense in K.

Now as (f_n) is pointwise bounded on $E \subset K$ and E is at most countable, there is a pointwise convergent subsequence, (f_{n_k}), of (f_n) on E. Set $g_k = f_{n_k}$ for each $k \geq 1$, we will show that (g_k) is uniformly convergent. Let $\epsilon > 0$ and choose $\delta > 0$ from equicontinuity of (g_k) such that $d(x, y) < \delta \implies |g_k(x) - g_k(y)| < \epsilon/3$ for all $k \geq 1$. For $n \geq 1/\delta$ again we have $K \subset \bigcup_{i=1}^{L_n} B_\delta(x_i^n)$ for $\{x_i^n\}_{i=1}^{L_n} \subset E$ by construction of E. As (g_k) is pointwise convergent on E there is some $N \in \mathbb{N}$ such that $l, m \geq N \implies |g_l(x_i^n) - g_m(x_i^n)| < \epsilon/3$ for $i = 1, \ldots, L_n$. Now, if $x \in K$ then $x \in B_\delta(x_i^n)$ for some $i = 1, \ldots, L_n$ and so for $l, m \geq N$ we have

$$\begin{aligned} |g_l(x) - g_m(x)| &\leq |g_l(x) - g_l(x_i^n)| + |g_l(x_i^n) - g_m(x_i^n)| + |g_m(x_i^n) - g_m(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

thus (g_k) is uniformly Cauchy and hence uniformly convergent (as $\mathcal{C}(K)$ is complete). $\square$

Remark 24.3.
- Compactness of K is necessary; e.g. if $\mathcal{F} = \{f\}$ (a constant sequence) then equicontinuity is equivalent to uniform continuity, however every affine function $f(x) = ax + b$ on $\mathbb{R}$ is uniformly continuous but certainly not uniformly bounded!
- We saw equicontinuity is necessary as $G = (\frac{x^2}{x^2+(1-nx)^2})$ is uniformly, hence pointwise, bounded but has no uniformly convergent subsequence.
- Pointwise boundedness is necessary; e.g. $\mathcal{H} = (n)$ are equicontinuous on $[0,1]$ but not pointwise bounded (hence not uniformly bdd) and certainly has no convergent subsequence (let alone uniformly convergent)!

The Arzela-Ascoli theorem has deep implications in the study of differential equations and its proof involves a combination of almost all of the concepts we have seen in this course. As a concrete application we have the following: We saw that (f_n) diff'ble with (f_n') uniformly bdd on $[0,1]$ were equicontinuous by the MVT (as $|f(x)-f(y)| \leq |x-y|\sup_{z\in[a,b]}|f'(z)|$). If (f_n) are uniformly α-Hölder continuous on a compact metric space (X,d); i.e. if (f_n) are $\mathbb{C}$ valued and for some $\alpha \in (0,1]$ and M we have

$$|f_n(x) - f_n(y)| \leq M \cdot d(x,y)^\alpha \quad \text{for all } n, x, y,$$

then (f_n) is equicontinuous (if $\epsilon > 0$ let $\delta = (\epsilon/M)^{1/\alpha} \implies Md(x,y)^\alpha < \epsilon$) and so Arzela-Ascoli applies if (f_n) is pointwise bounded to guarantee that (f_n) is both uniformly bounded and contains a uniformly convergent subsequence! □

Part VI

Construction of $\mathbb{R}$

25 Lecture 25: Construction of $\mathbb{R}$

Early on we stated the following existence theorem:

Theorem 25.1. There exists an ordered field, $\mathbb{R}$, which has the least upper bound property. Moreover, $\mathbb{Q} \subset \mathbb{R}$.

We used this without proof throughout the course, we will now prove it using the so called **Cauchy Completion** of $\mathbb{Q}$; one can also equivalently construct $\mathbb{R}$ from $\mathbb{Q}$ by use of **Dedekind cuts** as is done in Rudin Chapter 1.

Proof (Cauchy Completion). We consider the set, C, of all Cauchy sequences in $\mathbb{Q}$; recall that every Cauchy sequence is bounded. The set C satisfies all of the field axioms, except for the existence of multiplicative inverses. Precisely, if $(x_n), (y_n) \in C$ then we define $+, \cdot$ on C by $(x_n) + (y_n) = (x_n + y_n)$ and $(x_n) \cdot (y_n) = (x_n \cdot y_n)$, the boundedness of $(x_n), (y_n) \implies (x_n \cdot y_n)$ is Cauchy in particular, with $0 = (0) \in C$, $1 = (1) \in C$ are the additive and multiplicative identities, and $(-x_n) = -(x_n)$. We do not have multiplicative inverses with this operation since for example $(1,0,\dots)\cdot(0,1,0,\dots)=(0)$. We resolve the lack of multiplicative inverses by identifying sequences whose terms' difference goes to zero; namely we say that $(x_n)\sim(0)$ ((x_n) equivalent to (0)) if $\lim_{n\to\infty}|x_n|=0$ and $(x_n)\sim(y_n)$ if $(x_n-y_n)\sim(0)$. The equivalence classes $[(x_n)]=\{(y_n)\in C \mid (x_n)\sim(y_n)\}$ can then be added/multiplied (as $(x_n)\sim(0)\implies (x_n)+(y_n)\sim(y_n)$ and $(x_n)(y_n)\sim(0)$). Also if $[(x_n)]\neq[(0)]$ then $\lim_{n\to\infty}|x_n|>0$ and so for some $N\in\mathbb{N}$ we have $|x_n|>0$ for $n\geq N$; setting $y_n=0$ for $n<N$ and $y_n=x_n^{-1}$ for $n\geq N$ we have $[(x_n)(y_n)]=[(1)]$. Noting that $[(1)]\neq[(0)]$ as $|1-0|=1>0$ the set of equivalence classes of Cauchy sequences forms a field; we define this to be $\mathbb{R}$.

We can view $\mathbb{Q}\subset\mathbb{R}$ by identifying $x\in\mathbb{Q}$ with the constant sequence $(x)\in C$ so that $[(x)]\in\mathbb{R}$. We need to show that $\mathbb{R}$ is ordered and satisfies the least upper bound property. We extend the absolute value, $|\cdot|$, on $\mathbb{Q}$ to $\mathbb{R}$ by setting $|[(x_n)]|=[(|x_n|)]$ for any $[(x_n)]\in\mathbb{R}$. One can then check that $\mathbb{R}_{>0}=\{x\in\mathbb{R}\backslash\{0\}\mid |x|=x\}$ contains $\mathbb{Q}_{>0}=\{x\in\mathbb{Q}\backslash\{0\}\mid|x|=x\}$ by the above map (similarly for <0) and hence $\mathbb{R}$ is ordered (and this order agrees with that of $\mathbb{Q}$); we say that $x>y$ for $x,y\in\mathbb{R}$ if $x-y\in\mathbb{R}_{>0}$. This also shows that $\mathbb{R}$ satisfies the Archimedean Property!

To see that $\mathbb{R}$ has the least upper bound property we will first show that $\mathbb{R}$ is **complete** w.r.t. $|\cdot|$; namely if (X_n) is Cauchy in $\mathbb{R}$ then $X_n\to X$ for some $X\in\mathbb{R}$. First we see that every Cauchy sequence $(x_n)\subset\mathbb{Q}$ converges to $X=[(x_n)]\in\mathbb{R}$ since if $\epsilon>0$ then for some $N\in\mathbb{N}$ we have $|x_n-x_m|<\epsilon$, and so by definition if $n\geq N$ then we have that $|X-x_N|=|[(x_n-x_N)]|=[|(x_n-x_N)|]<\epsilon$; hence $x_n\to X$. Next, we have that $\mathbb{Q}$ is **dense** in $\mathbb{R}$ since if $X\in\mathbb{R}$ and $\epsilon>0$ then $X=[(x_n)]$ and so there is some $N\in\mathbb{N}$ such that we have $|X-x_N|\leq\epsilon$ where $x_N\in\mathbb{Q}$ (same reasoning as above as (x_n) is Cauchy in $\mathbb{Q}$). Now, if $(X_n)\subset\mathbb{R}$ is Cauchy, by the density of $\mathbb{Q}$ in $\mathbb{R}$ we can find $(r_n)\subset\mathbb{Q}$ such that $|X_n-r_n|<1/n$ for all $n\geq1$. The sequence $(r_n)\subset\mathbb{Q}$ is then Cauchy by the Δ ineq. and hence by the reasoning above $r_n\to[(r_n)]\in\mathbb{R}$. By construction $\lim_{n\to\infty}|X_n-r_n|=0$ and so $(X_n-r_n)\sim(0)$, hence applying the Δ ineq. again we have that $X_n\to[(r_n)]$ also.

Finally, if $E\subset\mathbb{R}$ is nonempty and bounded above by $y_0\in\mathbb{R}$, let $x_0\in E$ be any non-upper bound for E. We then recursively define

$$y_{n+1}=\begin{cases}\frac{y_n+x_n}{2}, & \text{if } \frac{y_n+x_n}{2} \text{ is an upper bound for } E,\\ y_n, & \text{otherwise.}\end{cases}$$

and

$$x_{n+1}=\begin{cases}\frac{y_n+x_n}{2}, & \text{if } \frac{y_n+x_n}{2} \text{ is an upper bound for } E,\\ x_n, & \text{otherwise.}\end{cases}$$

We thus have two sequences $(x_n),(y_n)\subset\mathbb{R}$ where $x_n\leq x_{n+1}$, $y_n\geq y_{n+1}$, and $x_n\leq y_n$ for all $n\geq1$, and inductively $|y_n-x_n|\leq 2^{-n}|y_0-x_0|$. Moreover both are Cauchy as for $m>n$ we have

$$\begin{aligned}|y_m-y_n|&=|y_m-y_{m-1}+y_{m-1}-\cdots+y_{n+1}-y_n|\\&\leq\left|\frac{y_{m-1}-x_{m-1}}{2}\right|+\cdots+\left|\frac{y_n-x_n}{2}\right|\\&\leq(2^{-m}+\cdots+2^{-n-1})|y_0-x_0|\end{aligned}$$

$\leq 2^{-n}|y_0 - x_0|$; (similarly for (x_n)). $(x_n), (y_n)$ are Cauchy and $\mathbb{R}$ is complete, hence they converge to some limit, s (as $|y_n - x_n| \to 0$ their limits are the same). By construction each y_n is an upper bound for E and so $x \leq s$ for all $x \in E$ (else $x_n \to s$) and similarly as each x_n is not an upper bound for E we have that $x_n \leq U$ for any upper bound U of E, hence $s \leq U$. Therefore we see that s is the least upper bound for E; as E was arbitrary we see that $\mathbb{R}$ has the least upper bound property. $\square$

The statement that $\mathbb{R}$ has the least upper bound property is often called the **axiom of completeness** which in the proof we saw followed from the completeness of $\mathbb{R}$ (and Archimedean property). It is also equivalent to: monotone convergence theorem, nested intersection property (+AP), Bolzano-Weierstrass theorem, IVT, and the fact every infinite decimal sequence converges.