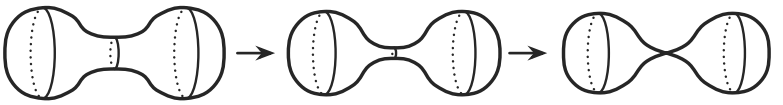
Weak Mean Curvature Flows

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Notes from Jacob Bernstein's Fall 2025 topics course



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Please email any typos or comments to `kmarsh34@jh.edu`.

PREFACE

These notes were typeset by Claude Opus 5.5 from my handwritten notes for Jacob Bernstein's Fall 2025 topics course on mean curvature flow and weak flows; Lecture 15, which I missed, is typeset from Hunter Stufflebeam's notes, and the final chapter covers the closing talks of the course, given by Kahnrad Braxton.

CHAPTER I

CLASSICAL MEAN CURVATURE FLOW

We begin with the classical (smooth) theory of mean curvature flow: its definition, some special solutions, and the basic tools (the avoidance principle and the monotonicity formulae) that we will later want to carry over to weak flows.

1.1 Parametrised and geometric flows

We write $g_{\mathbb{R}}$ and $\nabla^{\mathbb{R}}$ for the Euclidean metric and connection.

Definition 1. *(Classical mean curvature flow) A* ***classical*** *or* ***parametrised mean curvature flow*** *(MCF) is a smooth map*

$$F : [0, T) \times M^n \to \mathbb{R}^{n+k}$$

(generally $k = 1$), where M^n is an n-manifold, such that for each $t \in [0, T)$ the map $F(t, \cdot) : M^n \to \mathbb{R}^{n+k}$ is a (proper) embedding, and

$$\frac{\partial F}{\partial t}(t, p) = \vec{H}(t, p) = \Delta_{g_t} F(t, p), \tag{MCF}$$

where $\vec{H}(t, p) = \vec{H}_{\Sigma_t}(F(t, p))$ is the ***mean curvature vector*** *of $\Sigma_t = F(t, M)$ at $F(t, p)$. Properness is automatic if M is compact.*

Remark 1. *We recall two descriptions of the mean curvature vector:*

- *$\vec{H}_{\Sigma_t}(x_0) = \operatorname{tr}(A_{\Sigma_t}(x_0)) = \sum_{i=1}^n (\nabla^{\mathbb{R}}_{E_i} E_i)^{\perp}$, where A_{Σ_t} is the second fundamental form and $E_1, \dots, E_n$ is an orthonormal basis of $T_{x_0}\Sigma_t$, extended to orthonormal vector fields tangent to Σ_t near x_0.*
- *$\vec{H}(F(t, p)) = \Delta_{g_t} F(t, p)$ for the induced metric $g_t = F(t, \cdot)^* g_{\mathbb{R}}$.*

We can compute that for $R > 0$ the sphere of radius R,

$$\mathbb{S}^n(R) = \{x \in \mathbb{R}^{n+1} \,|\, |x|^2 = R^2\} \subset \mathbb{R}^{n+1},$$

is such that $\vec{H}_{\mathbb{S}^n(R)}(x_0) = -\frac{n}{R^2} x_0$ for any $x_0 \in \mathbb{S}^n(R)$.

Remark 2. *If $k = 1$ then $\vec{H}_\Sigma(p) = -H_\Sigma(p)\vec{n}_\Sigma(p)$, where $\vec{n}_\Sigma$ is the outward unit normal to Σ at p and $H_\Sigma(p)$ is the* ***scalar mean curvature*** *of Σ at p.*

Example 1. *(Shrinking spheres) Fix $M^n = \mathbb{S}^n(1) \subset \mathbb{R}^{n+1}$ and define $F : [0,T) \times M \to \mathbb{R}^{n+1}$ by setting $F(t,x) = \sqrt{R^2 - 2nt}\, x$ (for $R^2 > 2nT$). This is a MCF by a (calculus) computation, with $\Sigma_t = F(t,M) = \mathbb{S}^n(\sqrt{R^2 - 2nt})$.*

Remark 3. *This example cannot be extended as a MCF past time $\frac{R^2}{2n}$, nor can F even be extended as a smooth map up to this time (due to the factor $\sqrt{R^2 - 2nt} \to 0$). We call this the* ***formation of a singularity****.*

We see that this is a general phenomenon when M is closed. If $F : [0,T) \times M^n \to \mathbb{R}^{n+k}$ is a MCF of a closed manifold M, consider the function

$$\rho(t,p) = 2nt + |F(t,p)|^2 \geq 2nt.$$

As M is closed, $\rho_0 = \sup_M \rho(0,\cdot) = R_0^2$ for some $R_0 > 0$ (so Σ_0 lies inside $\mathbb{S}^n(R_0)$), and one can check that

$$\frac{\partial \rho}{\partial t} = \Delta_{g_t} \rho$$

(the heat equation with a time varying metric). The parabolic maximum principle thus gives $\rho \leq \rho_0 = R_0^2$, which combined with $\rho \geq 2nt$ shows that

$$T \leq \frac{R_0^2}{2n}.$$

Remark 4. *This only tells us that the flow fails to be classical at* some *time! Flows can however cease to be classical* without *disappearing. For the 'dumbbell' $\Sigma^2 \subset \mathbb{R}^3$ of Figure 1.1, the mean curvature is small on the spherical lobes and large at the neck, so the neck 'pinches' off before the spherical regions disappear. Thus this flow fails to remain classical without disappearing (we make this precise in the next section)!*

Remark 5. *This fails in general if $n = 1$, by Grayson's theorem (Theorem 7). Here we are also implicitly assuming that M is connected; otherwise one can consider nested spheres as a precise example of a flow that fails to be classical without disappearing!*

Remark 6. *(Meta principle) MCF is a flow of submanifolds and* not *of parametrised submanifolds (which are useful from a computational/PDE point of view). One should really think of the image of the map F, i.e. of $\{\Sigma_t\}_{t\in[0,T)}$ or $t \in [0,T) \mapsto \Sigma_t \subset \mathbb{R}^{n+k}$.*

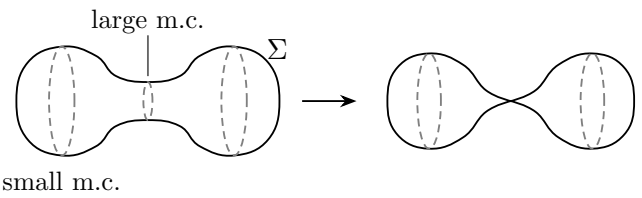


Figure 1.1. The dumbbell: the mean curvature is large at the neck and small on the lobes, so the neck pinches before the lobes disappear.

With this latter point of view, if we had some smooth $\varphi : [0, T) \times M^n \to M^n$ with $\varphi(t, \cdot) : M \to M$ a diffeomorphism for each t, then $G(t, p) = F(t, \varphi(t, p))$ satisfies

$$\frac{\partial G}{\partial t} = \frac{\partial F}{\partial t} \circ \varphi + DF\left(\frac{\partial \varphi}{\partial t}\right),$$

where the first term is normal to Σ_t and the second is tangent to Σ_t (and may be non-zero), and $\vec{H}(G) = \vec{H}(F \circ \varphi)$. Generally then G is *not* a MCF, as it may not satisfy $\frac{\partial G}{\partial t} = \vec{H}(G)$. However, noting that $DF(\frac{\partial \varphi}{\partial t})$ is tangent to Σ_t, we really should be considering

$$\left(\frac{\partial G}{\partial t}\right)^{\perp} = \vec{H}(G); \tag{$\circledast$}$$

then if G satisfies ($\circledast$), so does $G \circ \varphi$ for any such φ! This means that ($\circledast$) has *no* uniqueness and *no* good estimates (since φ was arbitrary). We say that ($\circledast$) is **degenerate parabolic**; the original MCF equation was not, however!

Exercise 1. *If G satisfies ($\circledast$) and M is closed then there is some φ as above such that $F = G \circ \varphi$ is a classical MCF (i.e. one can fix a gauge!). Later we will see that this is not a useful point of view.*

1.2 Special solutions

We now look at some special solutions, all in the case $k = 1$.

1.2.1 Graphical solutions

Suppose we have some $u : [0, T) \times \Omega \to \mathbb{R}$, for $\Omega \subset \mathbb{R}^n$, such that

$$\Sigma_t = \text{graph}(u(t, \cdot)) = \{(x, u(t, x)) \,|\, x \in \Omega\}.$$

If we define $G : (t, x) \mapsto (x, u(t, x))$ then

$$\left(\frac{\partial G}{\partial t}\right)^{\perp} = (0, u_t)^{\perp} = \frac{u_t}{\sqrt{1 + |Du|^2}}\, \vec{n}_{\Sigma_t},$$

but since

$$\vec{n}_{\Sigma_t} = \frac{(-Du(t,\cdot), 1)}{\sqrt{1 + |Du(t,\cdot)|^2}} \quad \text{and} \quad \vec{H}_{\Sigma_t}(G) = \operatorname{div}\left(\frac{Du(t,\cdot)}{\sqrt{1 + |Du(t,\cdot)|^2}}\right) \vec{n}_{\Sigma_t},$$

we see that G solves $(\circledast)$ if

$$u_t = \sqrt{1 + |Du|^2}\, \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right), \tag{GMCF}$$

which is a quasilinear parabolic PDE.

Example 2. *(Grim reaper) Explicitly, if we define* $\Omega = (-\frac{\pi}{2}, \frac{\pi}{2})$ *and*

$$u(t, x) = t - \log(\cos(x)),$$

then u *satisfies (GMCF) on* $(-\infty, \infty)$*; this is called the* ***grim reaper*** *(see Figure 1.2). By 'crossing' with* $\mathbb{R}^k$ *factors we get higher dimensional analogues.*

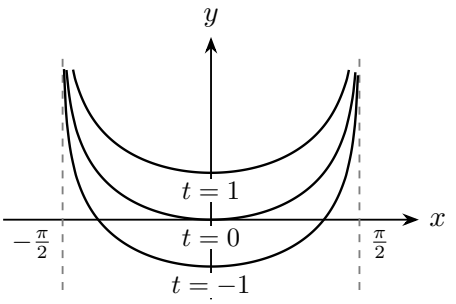


Figure 1.2. The grim reaper $u(t, x) = t - \log(\cos(x))$ at times $t = -1, 0, 1$; it translates upwards at unit speed.

1.2.2 Translating solutions

If we fix $v \in \mathbb{R}^{n+1}$ we can look for solutions of the form $F(t, \cdot) = tv + F_0$ for some fixed parametrisation $F_0 : M \to \mathbb{R}^{n+1}$; by $(\circledast)$ this is equivalent to asking that

$$\vec{H}(F_0) = v^{\perp} \tag{translator equation}$$

(and $v = 0$ gives the minimal surface equation!). This is a 'minimal surface equation style' equation, and turns out to be the Euler–Lagrange equation of some functional (so it is susceptible to variational methods).

Remark 7. *No translator can be closed (by finite extinction; c.f. there are no compact minimal surfaces in $\mathbb{R}^n$)!*

1.2.3 Shrinkers

If we assume $F(t,\cdot) = \sqrt{-t}\, F_0$ for $t < 0$ (so that $\Sigma_t = \sqrt{-t}\, \Sigma_{-1}$), then by $(\circledast)$ this is a solution of MCF if and only if

$$\vec{H}(F_0) = -\frac{F_0^\perp}{2} \qquad \text{(shrinker equation)}$$

(so $\vec{H}_{\Sigma_{-1}}(x) = -\frac{x^\perp}{2}$). This is a 'minimal surface like equation', and is also an Euler–Lagrange equation.

Remark 8. *There are closed shrinkers, e.g. $\mathbb{S}^n(\sqrt{2n})$!*

Also, the **Angenent doughnut** [Ang92], formed by asking for an $\mathbb{S}^1$ symmetry in $\mathbb{R}^3$, reduces the shrinker equation to a second order nonlinear ODE which has a periodic solution (a closed orbit of the ODE in the half-plane); this gives a shrinker $F_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$. (Uniqueness of this shrinker is open!)

Question 1. *It might be worth finding an example of some non-closed manifold M for which the equations (MCF) and $(\circledast)$ are not 'related'/'equivalent'; in other words, for which a solution of $(\circledast)$ cannot be turned into a solution of (MCF) by any choice of φ as before.*

1.2.4 Expanders

Changing the sign in the shrinker ansatz, we look for $\Sigma_t = \sqrt{t}\, \Sigma_1$ for $t > 0$, with $F_1 : M \to \Sigma_1$ such that

$$\vec{H}_{\Sigma_1} = \frac{F_1^\perp}{2} \qquad (\dagger)$$

(also an Euler–Lagrange equation), but then there are *no* closed expanders! Heuristically:

"Shrinkers $\rightsquigarrow$ minimal surfaces in spheres / expanders $\rightsquigarrow$ minimal surfaces in hyperbolic space."

In $\mathbb{R}^2$, (†) becomes a second order ODE (which has a 'first integral', so can be reduced to *one* first order ODE). Fixing $\alpha \in [0, \frac{\pi}{2})$, let $C_\alpha = \text{graph}(\tan(\alpha)|x|)$ be the cone whose rays make angle α with the x-axis; there is then an expander, Γ^1_α (its time 1 slice), asymptotic to C_α, and for 'steeper' α we get 'steeper' Γ^1_α (Figure 1.3). Note then that

$$\lim_{t\to 0^+} \sqrt{t}\,\Gamma^1_\alpha = C_\alpha$$

in the sense that for some $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, Lipschitz in space and continuous in time, we have

$$\text{graph}(u(t,\cdot)) = \begin{cases} \sqrt{t}\,\Gamma^1_\alpha & \text{for } t > 0, \\ C_\alpha & \text{for } t = 0; \end{cases}$$

the flow 'achieves its initial data'.

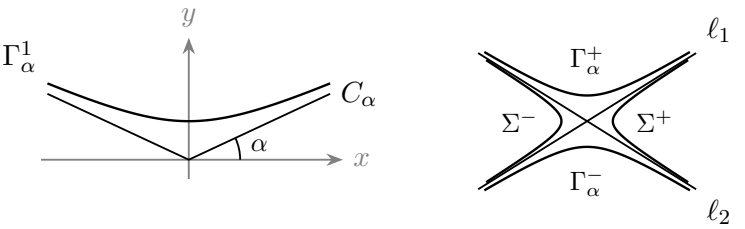


Figure 1.3. Left: an expander Γ^1_α asymptotic to the cone C_α. Right: expanders in the four sectors of $C = \ell_1 \cup \ell_2$, giving different evolutions out of C (schematic).

Moreover, if $C = \ell_1 \cup \ell_2$ is a pair of lines crossing at the origin, each making angle α with the x-axis, let Γ^+_α be the expander asymptotic to the upper half of C, $\Gamma^-_\alpha = -\Gamma^+_\alpha$, and let Σ^+ and Σ^- be the (rotated) expanders asymptotic to the right and left halves of C. Then one possible evolution out of C is

$$\Gamma^1_t = \sqrt{t}\,\Gamma^+_\alpha \cup \sqrt{t}\,\Gamma^-_\alpha \quad \text{for } t > 0,$$

another is

$$\Sigma^1_t = \sqrt{t}\,\Sigma^+ \cup \sqrt{t}\,\Sigma^- \quad \text{for } t > 0,$$

and another is the static flow

$$\Omega^1_t = \ell_1 \cup \ell_2 = C.$$

(Also, any union of these flows out of C, as a set!) Thus, there is *no* 'canonical'/unique evolution out of C!

1.3 Basic facts about mean curvature flow

1.3.1 Existence and uniqueness

Theorem 1. *(Short time existence and uniqueness) Let M^n be closed and $F_0 : M^n \to \mathbb{R}^{n+1}$ an embedding (i.e. a parametrisation of a closed hypersurface). Then there exists a unique maximal time of existence $T \in (0, \infty)$ and a unique solution $F : [0, T) \times M \to \mathbb{R}^{n+1}$ to (MCF) with $F(0, \cdot) = F_0$, such that*

$$\lim_{t \to T^-} \sup_M |A_{F(t,\cdot)}| = \infty.$$

Proof. 'Standard' quasilinear parabolic theory. Later, we may see a proof of this from Evans–Spruck [ES91]. The idea is to reduce it to a scalar problem over a tubular neighbourhood of Σ_0; this should work for higher codimension also. Note that we are working with embedded objects, so we do need to show that embeddedness is preserved along the flow (this fails in high codimension). □

This gives us a plethora of examples! In the non-closed case this is more complicated; this is to be expected, since even for the heat equation (without growth assumptions) we can see non-uniqueness (Tychonoff [Tyc35]). There are also examples of initial data that seem to become singular in finite time.

1.3.2 The avoidance principle and embeddedness

Theorem 2. *(Avoidance principle) (Codimension one only.) Let $\{\Sigma_t^n\}_{t \in [0,T)}$ and $\{\Gamma_t^n\}_{t \in [0,T)}$ be two MCFs in $\mathbb{R}^{n+1}$ with Σ_t closed and Γ_t proper for all $t \in [0, T)$. Then*

$$\Sigma_0 \cap \Gamma_0 = \emptyset \implies \Sigma_t \cap \Gamma_t = \emptyset \text{ for all } t \in [0, T).$$

Theorem 3. *(Preservation of embeddedness) If $F : [0, T] \times M^n \to \mathbb{R}^{n+1}$, with M closed, is smooth, $F(t, \cdot)$ is an immersion for each t, and $F : [0, T) \times M^n \to \mathbb{R}^{n+1}$ is a MCF, then $F(T, \cdot) : M \to \mathbb{R}^{n+1}$ is an embedding.*

"We should study quantities preserved by the flow!"

Remark 9. *Both are codimension one phenomena: e.g. in $\mathbb{R}^3$ take round circles in the xz- and xy-planes, or one grim reaper in the xz-plane and one in the xy-plane.*

As a consequence we can make the dumbbell argument precise (Figure 1.4). Let Σ_0 be a (closed) dumbbell, so that its flow exists. Put spheres $\mathbb{S}^2(R)$ in the lobes; then the flow of Σ_0 outlives these (e.g. for $t < \frac{R^2}{4}$, the maximal existence time of the spheres, the flow is not over). Now surround the neck by an Angenent doughnut which is sufficiently small; the avoidance principle then forces a 'neck pinch'. More precisely, there is some singular time $T < \tau$, where τ is the collapsing time of the doughnut!

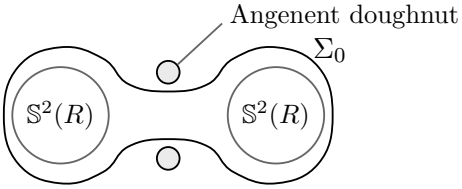


Figure 1.4. A cross-section of the dumbbell argument: spheres in the lobes outlive a small Angenent doughnut around the neck, so the flow must become singular before the doughnut collapses.

By choosing 'unbalanced' lobes (one large and one small) you can make the pinch happen at the edge of a lobe.

To prove both theorems we use the following result:

Lemma 1. *(Strict parabolic maximum principle) Let $u_1, u_2 : [0,T] \times B_R \to \mathbb{R}$ be smooth and satisfy:*

1. *u_1, u_2 solve (GMCF), i.e. $\frac{\partial u_i}{\partial t} = \sqrt{1+|Du_i|^2}\,\mathrm{div}\left(\frac{Du_i}{\sqrt{1+|Du_i|^2}}\right)$ for $i = 1, 2$.*
2. *$|Du_1| + |D^2u_1| + |Du_2| + |D^2u_2| < 1$.*
3. *$u_1 < u_2$ on $[0,T) \times B_R$.*

Then $u_1(T,\cdot) < u_2(T,\cdot)$, and both functions are bounded.

Proof. Let $w = u_2 - u_1$, so that by (3) $w > 0$ on $[0,T) \times B_R$, and by (1) w solves

$$\frac{\partial w}{\partial t} = a^{ij} D^2_{ij} w + b^i D_i w$$

(which is strictly parabolic) for some (bounded) coefficients

$$a^{ij} = a^{ij}(Du_1, Du_2) \quad \text{and} \quad b^i = b^i(Du_1, Du_2, D^2u_1, D^2u_2),$$

with a^{ij} strictly elliptic. To see this, write $W_i = \sqrt{1+|Du_i|^2}$ and note that

$$\frac{\partial w}{\partial t} = (W_2 - W_1)\,\mathrm{div}\left(\frac{Du_1}{W_1}\right) + W_2\,\mathrm{div}\left(\frac{Du_2}{W_2} - \frac{Du_1}{W_1}\right),$$

where

$$W_2 - W_1 = \frac{D(u_1 + u_2) \cdot Dw}{W_1 + W_2},$$

and continue. The strict maximum principle applied to w then gives the result. □

Proof of the avoidance principle. As $\Sigma_0 \cap \Gamma_0 = \emptyset$ and Σ_0 is closed, if $\Sigma_t \cap \Gamma_t \neq \emptyset$ for some $t > 0$, then some (t_0, p_0) exists with both

$$p_0 \in \Sigma_{t_0} \cap \Gamma_{t_0} \quad \text{and} \quad \Sigma_t \cap \Gamma_t = \emptyset \text{ for } t < t_0,$$

and, 'near' p_0, Σ_t lies on one side of Γ_t for $t \leq t_0$ close to t_0. Writing both flows locally as graphs over the common tangent plane at p_0, we can apply the lemma above to reach a contradiction! □

Remark 10. *(On short time existence) If we had a 'family' of embeddings, $F : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ with $F(\cdot, s)$ an embedding for each $s \in (-\varepsilon, \varepsilon)$, then again we get a unique evolution for each member of the family, i.e. $F^s(t, \cdot)$ with $F^s(0, \cdot) = F(\cdot, s)$ for $s \in (-\varepsilon, \varepsilon)$. Moreover, for each $0 < T' < T_0$, where T_0 is the maximal existence time for F^0, there is some $\varepsilon_0 > 0$ such that $(t, p, s) \mapsto F^s(t, p)$ is continuous on $[0, T'] \times M \times [-\varepsilon_0, \varepsilon_0]$ ('families of flows vary continuously'); i.e. the problem is* ***well posed*** *(c.f. $y' = \sqrt{y}$, which is not well posed as it has non-uniqueness). Note however that the extinction time, T_s, does* not *depend continuously on s! (Find an example.)*

Remark 11. *(On the avoidance principle) As a corollary of the avoidance principle we have that there are 'no flows from infinity': suppose that $F : (0, T) \times M \to \mathbb{R}^{n+1}$ is a smooth map such that $F : [T_0, T) \times M \to \mathbb{R}^{n+1}$ is a MCF for each $T_0 \in (0, T)$. Then there is some $R > 0$ with*

$$F(t, M) \cap B_R(0) \neq \emptyset \quad \textit{for all } t \in (0, \varepsilon), \textit{ with } \varepsilon \textit{ sufficiently small}$$

(i.e. there are no flows from infinity/nothing!).

Sketch of proof. Fix $t_0 \in (0, T)$ and note that there is then some $R_0 > 0$ with $\Sigma_{t_0} \cap B_{R_0}(0) \neq \emptyset$. If no $R > 0$ as in the statement existed, we could consider flows of spheres $\mathbb{S}^n(R)$, starting at times $t < t_0$, which equal $\mathbb{S}^n(R_0) = \partial B_{R_0}(0)$ at time t_0 (so $R^2 = R_0^2 + 2n(t_0 - t)$). Applying avoidance, we see that if $B_R(0) \cap \Sigma_t = \emptyset$ for some $t < t_0$, then $B_{R_0}(0) \cap \Sigma_{t_0} = \emptyset$ also, a contradiction! □

Remark 12. *This fails for manifolds. For example, take $M = \mathbb{R}^2$ with $g = dr^2 + f(r)^2 d\theta^2$, with f growing rapidly as $r \to \infty$. The circles $\{r = const\}$*

are preserved by the flow, and for appropriate f their geodesic curvature grows in r; this means that a MCF can start at infinity (from circles) and propagate inwards at 'infinite' speed. This is because spheres evolve at a fixed speed in Euclidean space, but not in general manifolds! (A lower Ricci bound, $\mathrm{Ric} \geq -c$, would be fine; see later!) (A grim reaper can move 'out' with infinite speed, but not in.)

1.3.3 Monotonicity formulae

Theorem 4. *(Area monotonicity) Let $t \in [0,T] \mapsto \Sigma_t \subset \mathbb{R}^{n+1}$ be a MCF of closed hypersurfaces. Then*

$$\mathrm{Vol}_n(\Sigma_t) = \mathrm{Vol}_n(\Sigma_0) - \int_0^t \int_{\Sigma_s} |\vec{H}|^2 \, dV \, ds,$$

and so $\mathrm{Vol}_n(\Sigma_t) \leq \mathrm{Vol}_n(\Sigma_0)$, with strict inequality (since there are no closed minimal hypersurfaces!). More compactly, $\frac{d}{dt}\mathrm{Vol}_n(\Sigma_t) = -\int_{\Sigma_t} |\vec{H}|^2 \, dV < 0$.

Proof. This is a consequence of the first variation of area formula: if $G : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^{n+1}$ is smooth, with $G(s,\cdot)$ a proper embedding and $\frac{\partial G}{\partial s}$ compactly supported, then

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{Vol}_n(G(s,\cdot)) = -\int_{\Sigma_0} \frac{\partial G}{\partial s}(0,\cdot) \cdot \vec{H}(G(0,\cdot)) \, dV$$

(exercise: compute this; the minus sign is a convention on $\vec{H}$). For a MCF we then have $\frac{\partial G}{\partial s} \cdot \vec{H}(G(0,\cdot)) = |\vec{H}|^2$, as $\vec{H}$ is normal! □

This also motivates MCF as the (negative) gradient flow of area; area is fundamentally important!

Remark 13. *We can 'localise' this in the non-closed case by using test functions!*

Theorem 5. *(Huisken's monotonicity formula [Hui90]) Fix $x_0 \in \mathbb{R}^{n+1}$ and $t_0 \in \mathbb{R}$, and set*

$$\Phi^{(t_0,x_0)}(t,x) = (4\pi(t_0-t))^{-\frac{n}{2}} \exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right)$$

on $(-\infty, t_0) \times \mathbb{R}^{n+1}$. On any hyperplane in $\mathbb{R}^{n+1}$ through x_0 (or on any n-plane through x_0 if we are in $\mathbb{R}^{n+k}$) this is the backwards heat kernel, i.e. the fundamental solution of the backwards heat equation on an n-plane.

Then, if $t \in [T_0, T_1) \mapsto \Sigma_t \subset \mathbb{R}^{n+1}$ is a MCF and $T_0 \le t < \min(T_1, t_0)$, we have

$$\int_{\Sigma_{T_0}} \Phi^{(t_0,x_0)}(T_0,\cdot)\,dV = \int_{\Sigma_t} \Phi^{(t_0,x_0)}(t,\cdot)\,dV + \int_{T_0}^{t}\int_{\Sigma_s} \left|\vec{H}_{\Sigma_s} + \frac{(x-x_0)^\perp}{2(t_0-s)}\right|^2 \Phi^{(t_0,x_0)}(s,\cdot)\,dV\,ds.$$

In other words,

$$\frac{d}{dt}\int_{\Sigma_t} \Phi^{(t_0,x_0)}(t,\cdot)\,dV = -\int_{\Sigma_t} \left|\vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 \Phi^{(t_0,x_0)}(t,\cdot)\,dV.$$

Remark 14. *If $t \mapsto \Sigma_t$ is a MCF, then so are the translates $t \mapsto \Sigma_t + v$, the time shifts $t \mapsto \Sigma_{t-t_0}$, and the parabolic dilations $t \mapsto \lambda\Sigma_{\lambda^{-2}t}$ (with appropriate changes in the time intervals). Hence we can always compute the Huisken monotonicity for flows with $t_0 = 0$ and $x_0 = 0$ (without loss of generality).*

By the above remark,

$$\frac{d}{dt}\int_{\Sigma_t} \Phi^{(0,0)}(t,\cdot)\,dV = -\int_{\Sigma_t} \left|\vec{H} + \frac{x^\perp}{-2t}\right|^2 \Phi^{(0,0)}(t,\cdot)\,dV$$

for $T_0 < t < 0$, and the right hand side vanishes if and only if $\vec{H} + \frac{x^\perp}{-2t} = 0$, if and only if $\Sigma_t = \sqrt{-t}\,\Sigma_{-1}$, where Σ_{-1} satisfies $\vec{H}_{\Sigma_{-1}} + \frac{x^\perp}{2} = 0$ (a **self-shrinker**!). Hence the Gaussian area is strictly decreasing unless Σ_t is a self-shrinker!

Proof of Huisken's monotonicity formula. One can compute it via the first variation, but, using the remark, we let $F : [T_0, T_1) \times M \to \mathbb{R}^{n+1}$ be a parametrisation of the MCF and assume without loss of generality that $T_1 \le 0 = t_0$ and $x_0 = 0$. We use **self-similar coordinates**,

$$G(s,\cdot) = e^{\frac{s}{2}} F(-e^{-s},\cdot), \qquad G : [-\log(-T_0), -\log(-T_1)) \times M \to \mathbb{R}^{n+1}$$

(if $T_1 = 0$ then $-\log(-T_1) = +\infty$). Then

$$\left(\frac{\partial G}{\partial s}\right)^\perp = \vec{H}(G) + \frac{G^\perp}{2}$$

(check!), i.e. G is a **rescaled MCF**. We can also check that if $\Gamma_s = e^{\frac{s}{2}}\Sigma_{-e^{-s}}$ then

$$\int_{\Sigma_t} \Phi^{(0,0)}(t,\cdot)\,dV = (4\pi)^{-\frac{n}{2}} \int_{\Gamma_s} e^{-\frac{|x|^2}{4}}\,dV \qquad (t = -e^{-s}),$$

so one can use the weighted form of the first variation to see that

$$\begin{aligned}\frac{d}{ds}\int_{\Gamma_s} e^{-\frac{|x|^2}{4}}\,dV &= -\int_{\Gamma_s}\left(\vec{H}+\frac{x^\perp}{2}\right)\cdot\left(\frac{\partial G}{\partial s}\right)^\perp e^{-\frac{|x|^2}{4}}\,dV \\ &= -\int_{\Gamma_s}\left|\vec{H}+\frac{x^\perp}{2}\right|^2 e^{-\frac{|x|^2}{4}}\,dV;\end{aligned}$$

checking the change of variables gives the result! □

The long time behaviour of G is the small scale behaviour of F near $(0,0)$; i.e. it amounts to 'zooming in' on Σ_t as $t \to 0$.

Remark 15. *Say $G : [0,\infty) \times M \to \mathbb{R}^{n+1}$, parametrising Γ_s, solves the rescaled MCF $\left(\frac{\partial G}{\partial s}\right)^\perp = \vec{H}(G) + \frac{G^\perp}{2}$, and suppose that along a sequence $\sigma_i \to \infty$,*

$$\lim_{i\to\infty} G(s+\sigma_i,\cdot) = G^\infty(s,\cdot) \quad \textit{(in some sense)}, \tag{‡}$$

where G^∞ parametrises Γ_s^∞. Then the Huisken monotonicity and its sign imply that

$$\lim_{s\to\infty}\int_{\Gamma_s} e^{-\frac{|x|^2}{4}}\,dV = C \geq 0$$

exists (as it is monotone and bounded below!), and so the assumption (‡) gives

$$\int_{\Gamma_s^\infty} e^{-\frac{|x|^2}{4}}\,dV \equiv C \text{ is constant} \implies \Gamma_s^\infty = \Gamma_0^\infty \text{ is a self-shrinker!}$$

Thus dilations are self-shrinkers, and so 'model' the flow! We need to justify (‡) in general, for which we need some notion of compactness! One could also ask whether the limits are independent of the sequence σ_i!

1.4 More sophisticated facts about mean curvature flow

1.4.1 Convex hypersurfaces and embedded curves

Theorem 6. *(Huisken $n \geq 2$ [Hui84], Gage–Hamilton $n = 1$ [GH86]) If $\Sigma_0^n \subset \mathbb{R}^{n+1}$ is closed and (strictly) convex (the eigenvalues of A_{Σ_0} are positive), then the MCF $t \in [0,T) \mapsto \Sigma_t$ of Σ_0, with T the maximal time of existence, is such that:*

1. *Σ_t is strictly convex for all $t \in [0,T)$ (as is mean convexity, and k-convexity).*

2. *Σ_t becomes singular at time T in a 'round point' x_0, i.e. $\widetilde{\Sigma}_t = \Sigma_{t+T} - x_0$ becomes singular at $(0,0)$, and the rescaled MCF $s \in [s_0, \infty) \mapsto \Gamma_s$ of $\widetilde{\Sigma}_t$ (with $s_0 = -\log(T)$) is such that $\Gamma_s \to \mathbb{S}^n(\sqrt{2n})$ in a strong sense, with estimates.*

Idea of proof. For (1), we use a maximum principle for the second fundamental form, A_{Σ_t}, after showing that it satisfies a heat-like equation (exercise). For (2), we prove estimates forcing the trace free part of A_{Σ_t}, $\mathring{A}_{\Sigma_t}$, to be small relative to A_{Σ_t}. □

Theorem 7. *(Grayson [Gra87]) If $\sigma_0 \subset \mathbb{R}^2$ is a closed embedded curve, then the MCF $t \in [0,T) \mapsto \sigma_t$ of σ_0 is such that:*

1. *σ_t is embedded for $t \in [0,T)$ (by the avoidance principle).*

2. *σ_t becomes singular at time T in a round point (the main idea is to show that σ_t is convex for some $t < T$).*

Remark 16.

- *This is not true for $n \geq 2$ (see the dumbbell from before!).*
- *(2) gives a totally canonical flow (an alternate proof is due to Huisken using maximum principles, written up by Andrews [Hui98, AB11]; these use two-point maximum principles and a quantitative measure of embeddedness).*
- *It is not clear exactly at* what *time σ_t becomes convex.*
- *If $\sigma_0 = \partial\Omega_0$, where Ω_0 is closed and compact, then $\sigma_t = \partial\Omega_t$ is a flow of boundaries of compact sets Ω_t. Then*

$$\frac{d}{dt}A(\Omega_t) = -\int_{\sigma_t} \kappa_{\sigma_t} = \int_{\sigma_t} \vec{H}_{\sigma_t} \cdot \vec{n}_t,$$

where $\kappa_{\sigma_t} = H_{\sigma_t}$ is the geodesic curvature, $\vec{n}_t$ is the outward normal and $\vec{H}_{\sigma_t} = -H_{\sigma_t}\vec{n}_t$ (more generally, for $n \geq 2$, $\frac{d}{dt}\mathrm{Vol}(\Omega_t) = -\int_{\Sigma_t} H_{\Sigma_t}$ if $\Sigma_t = \partial\Omega_t$). By Gauss–Bonnet (with boundary), or the winding number of the tangent,

$$\int_{\sigma_t} \kappa_{\sigma_t} = 2\pi\,(\#\text{ components of } \sigma_t).$$

Hence, assuming σ_0 is connected, $A(\Omega_t) = A(\Omega_0) - 2\pi t$ and so $T \leq \frac{A(\Omega_0)}{2\pi}$; but Grayson's result gives $T = \frac{A(\Omega_0)}{2\pi}$.

1.4.2 Ill-posedness out of singular initial data

Example 3. *(Figure eight) Consider the figure eight, σ_0, as an immersion of $\mathbb{S}^1$. The winding number of its unit tangent is* 0*, which is invariant under homotopy through immersions (a.k.a. regular homotopy). Thus, under the flow of immersions by MCF the conclusion of Grayson's theorem does* not *hold! We think of σ_0 as a singular curve, singular at the crossing point.*

But one can 'outwardly' perturb σ_0 to an embedded curve, $\sigma_0^{+\varepsilon}$, or 'inwardly' perturb it to a pair of embedded curves, $\sigma_0^{-\varepsilon}$ (Figure 1.5), with $\sigma_0^{\pm\varepsilon} \to \sigma_0$ in some sense as $\varepsilon \to 0$. Then the $\sigma_0^{\pm\varepsilon}$ are embedded curves to which we can apply Grayson's theorem to get flows $t \mapsto \sigma_t^{\pm\varepsilon}$ for $t \in [0, T^{\pm\varepsilon})$. Letting $\Omega^\varepsilon(t)$ be the region between $\sigma_t^{+\varepsilon}$ and $\sigma_t^{-\varepsilon}$, we have (using the outward normal)

$$\frac{d}{dt}A(\Omega^\varepsilon(t)) = -\int_{\sigma_t^{+\varepsilon}} \kappa_g + \int_{\sigma_t^{-\varepsilon}} \kappa_g = -2\pi + 2\cdot(2\pi) = 2\pi,$$

and so $A(\Omega^\varepsilon(t)) = A(\Omega^\varepsilon(0)) + 2\pi t$. As $\varepsilon \to 0$ we expect that $A(\Omega^\varepsilon(0)) \to 0$ but $A(\Omega^\varepsilon(t)) \to 2\pi t > 0$ for $t > 0$. So as $\varepsilon \to 0$, $\sigma_t^{+\varepsilon}$ and $\sigma_t^{-\varepsilon}$ *cannot* coincide, which is an example of **ill-posedness** for flows out of the figure eight! (This is an example of **fattening**/non-uniqueness, and is an indicator of difficulties for weak flows!)

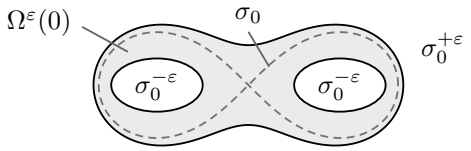


Figure 1.5. The figure eight σ_0 (dashed), its outward perturbation $\sigma_0^{+\varepsilon}$ and its inward perturbation $\sigma_0^{-\varepsilon}$ (two curves); the region $\Omega^\varepsilon(0)$ between them is shaded.

Remark 17. *If we instead flow the figure eight as an immersion, with loops Ω_1 and Ω_2 meeting at the crossing with interior angle $\theta(t)$, then by Gauss–Bonnet*

$$\frac{d}{dt}A(\Omega_1(t)) = -\int_{\partial\Omega_1(t)} \kappa_g = -2\pi + (\pi - \theta(t)) = -\int_{\partial\Omega_2(t)} \kappa_g = \frac{d}{dt}A(\Omega_2(t)).$$

So, if $A(\Omega_1(0)) < A(\Omega_2(0))$, then at the singular time T we have $A(\Omega_1(T)) = 0$ but $A(\Omega_2(T)) > 0$: the flow becomes singular without disappearing! The curve stops being an immersion, and zooming in one should see a multiplicity two line as a shrinker.

Example 4. *(Due to Ilmanen) Fix $\beta > 0$ and consider $h(x) = (1 + |x|)^{-1-\beta}$ on $\mathbb{R}$. Let Ω_0 be the region between the graphs of h and $-h$ (Figure 1.6), so that $\sigma_0^G = \text{graph}(h) \cup \text{graph}(-h) = \partial\Omega_0$ and*

$$A(\Omega_0) = \frac{4}{\beta} < \infty.$$

We can evolve graph$(\pm h)$ *by graphical MCF (which was shown to remain graphical by Ecker–Huisken [EH89, EH91]) to give a flow $t \mapsto \sigma_t^G$, made up of these graphs, which exists for all $t \geq 0$. On the other hand, we can consider the compact sets Ω_0^R, formed by taking $B_R \cap \Omega_0$ and 'smoothing' out these sets, so that the $\sigma_0^R = \partial\Omega_0^R$ are smooth closed curves. Grayson's theorem then gives that $t \in [0, T_R) \mapsto \sigma_t^R$ is a MCF with $T_R \leq \frac{2}{\pi\beta} < \infty$, independently of R, and on compact sets*

$$\sigma_0^R \to \sigma_0^G \quad \text{as } R \to \infty,$$

but σ_t^R is not *related to σ_t^G! Again, this shows the ill-posedness of the flow out of $\partial\Omega_0$!*

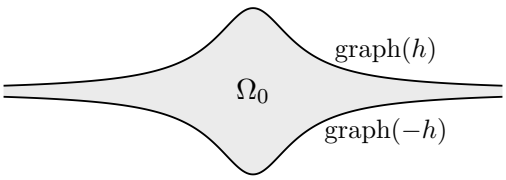


Figure 1.6. Ilmanen's example (schematic): the region Ω_0 between the graphs of $\pm h$ has finite area $\frac{4}{\beta}$.

1.4.3 Mean convexity

Theorem 8. *(Preservation of mean convexity) If $\Sigma_0 = \partial\Omega_0$ is a closed hypersurface, with Ω_0 compact and Σ_0 mean convex (i.e. $\vec{H}_{\Sigma_0}$ points into Ω_0, so $H_{\Sigma_0} \geq 0$), and if $t \mapsto \Sigma_t$ is a MCF of Σ_0 with $\Sigma_t = \partial\Omega_t$ for compact Ω_t, then $H_{\Sigma_t} > 0$ for $t > 0$ (i.e. Σ_t is strictly mean convex).*

Remark 18.

- *The singularities of the flow are given by generalised cylinders, i.e. $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n$; this is ultimately shown by much harder techniques!*

- *This means that if $0 \leq s < t < T$ then $\Omega_t \subset \text{int}(\Omega_s)$, so the flow 'moves inwards' (strictly) as $H_{\Sigma_t} > 0$. The other direction only gives $H_{\Sigma_t} \geq 0$ (i.e. inward flow implies $H_{\Sigma_t} \geq 0$).*

Proof. One can compute the evolution of H_{Σ_t} (this is long; one can use Simons' identity):

$$\frac{\partial}{\partial t} H_{\Sigma_t} = \Delta_{\Sigma_t} H_{\Sigma_t} + |A_{\Sigma_t}|^2 H_{\Sigma_t}, \tag{♯}$$

from which one can apply the strict parabolic maximum principle.

Alternatively (for a weaker result), if $H_{\Sigma_0} > 0$ there is some $\delta > 0$ such that for $0 < t < \delta$ we have

$$\Omega_t \subset \mathrm{int}(\Omega_0) \implies \Sigma_t \cap \Sigma_0 = \emptyset,\ \Sigma_t \subseteq \Omega_0.$$

Then, for $0 < \tau < \delta$, the time-shifted flow $\Sigma_t^\tau = \Sigma_{t+\tau}$ for $t \in [0, T-\tau)$ is such that

$$\Sigma_0^\tau \cap \Sigma_0 = \emptyset \overset{\text{A.P.}}{\implies} \Sigma_t^\tau \cap \Sigma_t = \emptyset \text{ for } t \geq 0.$$

So $\Omega_{t+\tau} \subseteq \Omega_t$ for $0 < \tau < \delta$ and any $t \geq 0$, which implies that $H_{\Sigma_t} \geq 0$ (one can upgrade this to $H_{\Sigma_t} > 0$). □

Remark 19.

- *If $H_{\Sigma_0} \geq \delta_0 > 0$ then $H_{\Sigma_t} \geq \delta_0$ for all $t \geq 0$ (maybe even $H_{\Sigma_t} > \delta_0$ here!); we refer to this statement as $(\star_{\delta_0})$.*
- *In (♯) we have $|A_{\Sigma_t}|^2 \geq c(n) H_{\Sigma_t}^2$ (c.f. Ricci flow, but this is* not *signed!), which gives a rate of blow up for $|A_{\Sigma_t}|$.*
- *The weaker result, i.e. $H_{\Sigma_0} \geq 0 \implies H_{\Sigma_t} \geq 0$, just needed the avoidance principle, and so adapts well to weaker formulations of the MCF.*

To prove $(\star_{\delta_0})$, and to get closer to $H_{\Sigma_0} \geq 0 \implies H_{\Sigma_t} > 0$ (i.e. $\Omega_s \subset \mathrm{int}(\Omega_t)$ for $s > t$), we prove:

Theorem 9. *(Quantitative avoidance principle) If $t \in [0, T) \mapsto \Sigma_t$ is a MCF of closed hypersurfaces and $t \in [0, T) \mapsto \Gamma_t$ is a MCF of proper hypersurfaces, then*

$$t \mapsto \mathrm{dist}(\Gamma_t, \Sigma_t) \text{ is monotone non-decreasing}$$

(which implies the avoidance principle if $\Gamma_0 \cap \Sigma_0 = \emptyset$). In fact, it is strictly increasing. (Strictly increasing needs compactness! This also works in manifolds; see later.)

Proof. This follows from the avoidance principle and the translation invariance of MCF (the latter is specialised to Euclidean space, however!).

Without loss of generality we can suppose that $\operatorname{dist}(\Gamma_0, \Sigma_0) > 0$, and if the result fails then we find $t_0 > 0$ with

$$\delta_0 := \operatorname{dist}(\Gamma_{t_0}, \Sigma_{t_0}) < \operatorname{dist}(\Gamma_0, \Sigma_0).$$

Now as Γ_{t_0} is proper and Σ_{t_0} is compact, we can find $x_0 \in \Sigma_{t_0}$ and $y_0 \in \Gamma_{t_0}$ with $\operatorname{dist}(\Gamma_{t_0}, \Sigma_{t_0}) = |x_0 - y_0|$. Letting $v_0 = \frac{x_0 - y_0}{|x_0 - y_0|}$ and $\Gamma'_t = \Gamma_t + \delta_0 v_0$, we have $\Gamma'_{t_0} \cap \Sigma_{t_0} \neq \emptyset$; but $\operatorname{dist}(\Gamma_0, \Sigma_0) > \delta_0$ guarantees that $\Gamma'_0 \cap \Sigma_0 = \emptyset$, and so by the avoidance principle $\Gamma'_{t_0} \cap \Sigma_{t_0} = \emptyset$; a contradiction!

Now suppose that $\operatorname{dist}(\Gamma_{t_0}, \Sigma_{t_0}) = \operatorname{dist}(\Gamma_0, \Sigma_0) = \delta_0$ for some $t_0 > 0$. With Γ'_t as above we then have $\Gamma'_{t_0} \cap \Sigma_{t_0} \neq \emptyset$, and the strict maximum principle (as in the proof of the avoidance principle) gives $\widehat{\Gamma}'_t = \widehat{\Sigma}_t$ for some connected components of Γ'_t and Σ_t, for all $0 \leq t \leq t_0$. Then $K := \widehat{\Sigma}_0 = \widehat{\Gamma}'_0 = \widehat{\Gamma}_0 + w$ with $w = \delta_0 v_0$, so $\operatorname{dist}(K - w, K) \geq \operatorname{dist}(\Gamma_0, \Sigma_0) = |w|$. This is impossible for a closed connected hypersurface $K = \partial\Omega$. Indeed, if $K \cap (K - w) = \emptyset$, then $\Omega \cap (\Omega - w) = \emptyset$: otherwise a connectedness argument shows that one of Ω, $\Omega - w$ contains the other, which is impossible as they have the same volume and different boundaries. Now for $p \in \Omega - w$ we have $p + w \in \Omega$, and the segment from p to $p + w$ leaves $\overline{\Omega - w}$ at some $a \in K - w$ and only then enters $\overline{\Omega}$ at some $b \in K$, so that $|a - b| < |w|$. This contradiction gives the strictly increasing conclusion. □

We use this to prove $(\star_{\delta_0})$. If $H_{\Sigma_0} \geq \delta_0 > 0$ then $\operatorname{dist}(\Sigma_\tau, \Sigma_0) \geq \delta_0 \tau + o(\tau)$ for small $\tau > 0$, and the quantitative avoidance principle then gives $\operatorname{dist}(\Sigma_{t+\tau}, \Sigma_t) \geq \delta_0 \tau + o(\tau)$, which implies that $H_{\Sigma_t} \geq \delta_0$ (or else we reach a contradiction).

Question 2. *Why does $H_{\Sigma_0} \geq 0$ imply that $\Omega_s \subset \operatorname{int}(\Omega_t)$ for $s > t$?*

If we choose some compactly supported vector field Z on $\mathbb{R}^{n+1}$ pointing into Ω_0 (i.e. $Z = -\vec{n}_{\Sigma_0}$ on Σ_0) and consider flows $t \mapsto \Sigma^\varepsilon_t$ defined by

$$\left(\frac{\partial x}{\partial t}\right)^\perp = \vec{H}_{\Sigma^\varepsilon_t} + \varepsilon Z^\perp \quad \text{for } \varepsilon \geq 0 \tag{MCF$_\varepsilon$}$$

(MCF with transport, where $\varepsilon = 0$ gives MCF), then standard parabolic theory gives a unique solution up to a time T^ε, with $\Sigma^\varepsilon_0 = \Sigma_0$ and $\Sigma^\varepsilon_t = \partial\Omega^\varepsilon_t$, such that $\liminf_{\varepsilon \to 0} T^\varepsilon \geq T^0$, and the avoidance principle holds for (MCF$_\varepsilon$).

Remark 20. *(MCF$_\varepsilon$) does* not *necessarily satisfy translation invariance! Also, it may not be the gradient flow of some functional unless Z comes from ∇f for some function f!*

Then for all $0 \leq t < T^\varepsilon$ we have $\Sigma_t^\varepsilon \to \Sigma_t$ as $\varepsilon \to 0$ in a nice sense; i.e. we recover the original flow! Now there is some $\tau(\varepsilon) > 0$ such that

$$\Omega^\varepsilon_{\tau(\varepsilon)} \subseteq \operatorname{int}(\Omega_0^\varepsilon) = \operatorname{int}(\Omega_0) \implies \Omega^\varepsilon_{t+\tau} \subseteq \operatorname{int}(\Omega_t^\varepsilon)$$

for all $0 \leq t \leq T^\varepsilon$ and $\tau \leq \tau(\varepsilon)$, thus $\Omega_s^\varepsilon \subset \operatorname{int}(\Omega_t^\varepsilon)$ for $0 \leq t < s \leq T^\varepsilon$. As $\varepsilon \to 0$ we see that

$$\Omega_s = \Omega_s^0 \subseteq \Omega_t^0 = \Omega_t \quad \text{for } 0 \leq t < s < T$$

(the **nesting property**!). To conclude, if $\Omega_s \subseteq \operatorname{int}(\Omega_t)$ fails for some $0 < t < s < T$, then $\widehat{\Omega}_s = \widehat{\Omega}_{t'}$ for some components and all $0 \leq t' \leq s$, but then the map $t' \mapsto \partial\widehat{\Omega}_{t'}$ is static, i.e. a closed minimal surface, a contradiction! Thus $\Omega_s \subset \operatorname{int}(\Omega_t)$ for all $0 \leq t < s < T$; to summarise,

$$H_{\Sigma_0} \geq 0 \implies \Omega_s \subset \operatorname{int}(\Omega_t) \quad \text{('strictly monotone for sets').}$$

"This did not require derivatives, so adapts to weak settings!"

CHAPTER II

LEVEL SET FLOW

We now turn to weak formulations of mean curvature flow, which allow us to flow through singularities. Following Evans–Spruck [ES91], we define the level set flow using weak (viscosity) solutions of a degenerate parabolic PDE, and prove their existence and uniqueness.

2.1 Weak formulations of MCF

We want to be able to handle singularities, either to smooth out 'rough' objects or to continue flows past singularities, i.e. to 'flow' through them. There are, at least, two broad perspectives on this:

	Variational point of view	**Viscosity point of view**
	Measures/varifolds associated to Σ, e.g. $\mathcal{H}^n \llcorner \Sigma$ (GMT).	Think just of a set; forget smoothness.
	Gradient flow perspective.	Avoidance principle.
	Brakke flow (the MCF analogue of a stationary varifold).	Level set flow (PDE)/biggest flow (geometric).
Pros	Good compactness properties, helps for singularity analysis, and is very general.	Canonical flow, i.e. a lot of uniqueness; nice properties in some situations.
Cons	Lots of non-uniqueness, and regularity is harder; also technically demanding.	Restricted to codimension one, and has 'fattening', so we have to allow flows that do *not* look like hypersurfaces.

We will focus on the viscosity point of view, starting with Evans–Spruck, then Ilmanen's perspective on it, and then, if time allows, we will compare it to Brakke flows (again following Ilmanen).

2.2 The level set equation

We use the theme of the avoidance principle to define weak notions of MCF, i.e. a viscosity approach as mentioned above.

For $\Sigma_0 = \partial\Omega_0 \subset \mathbb{R}^{n+1}$ a closed hypersurface, choose some smooth function $u_0 : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\Sigma_0 = \{u_0 = 0\}$ and $Du_0 \neq 0$ on Σ_0. We also impose that $u_0 < 0$ on Ω_0, and that u_0 is bounded and constant outside of a compact set (one could instead ask that u_0 be proper); our constraints are *not* unique, and there is lots of flexibility! For $|h| \ll 1$ the level sets $\Sigma_h = \{u_0 = h\}$ are then smooth hypersurfaces near Σ_0 (Figure 2.1).

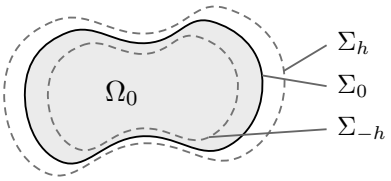


Figure 2.1. Level sets $\Sigma_h = \{u_0 = h\}$ of the initial function near $\Sigma_0 = \partial\Omega_0$, with $u_0 < 0$ on Ω_0.

We can do some geometric computations to see that the outward normal to Σ_0 is $\vec{\nu}_0 = \frac{Du_0}{|Du_0|}$, which we can extend to a neighbourhood of Σ_0. One can also check that, for $|h| \ll 1$, the mean curvature of Σ_h is given by

$$H_{\Sigma_h} = \mathrm{div}_{\mathbb{R}}(\vec{\nu}_0) = \mathrm{div}_{\mathbb{R}}\left(\frac{Du_0}{|Du_0|}\right),$$

so that $\vec{H}_{\Sigma_h} = -\mathrm{div}_{\mathbb{R}}\left(\frac{Du_0}{|Du_0|}\right)\frac{Du_0}{|Du_0|}$.

Supposing we had a MCF $t \mapsto \Sigma_t$ for $t \in [0, \varepsilon)$, we can also find a function $u : [0, \varepsilon) \times \mathbb{R}^{n+1} \to \mathbb{R}$ with $u(0, \cdot) = u_0$, $\Sigma_t = \{u(t, \cdot) = 0\}$, and $Du(t, \cdot) \neq 0$ in a neighbourhood of Σ_t (or in a neighbourhood of Σ_0, when $\varepsilon > 0$ is small). Following a point $x_0 \in \Sigma_0$ along the MCF gives an ODE on a neighbourhood of Σ_0:

$$\begin{cases} \dot{x}(s) = -\mathrm{div}_{\mathbb{R}}(\vec{\nu})\,\vec{\nu} = -\mathrm{div}_{\mathbb{R}}\left(\frac{Du}{|Du|}\right)\frac{Du}{|Du|} & \text{at } (s, x(s)), \\ x(0) = x_0, \end{cases} \tag{$*$}$$

where $\vec{\nu} = \frac{Du}{|Du|}$, so that $\vec{\nu}(t, \cdot)$ is the outward normal on Σ_t. Now $-\mathrm{div}_{\mathbb{R}}(\vec{\nu})\,\vec{\nu}$ at $(s, x(s))$ is the mean curvature vector of the level set $\{u = u(s, x(s))\}$,

and $t \mapsto \Sigma_t$ being a MCF implies that $x(s) \in \Sigma_s = \{u(s,\cdot) = 0\}$. Thus $u(s, x(s)) = 0$ when $x(s)$ solves $(*)$, and then by the chain rule we see that

$$u_t(s, x(s)) + Du(s, x(s)) \cdot \dot{x}(s) = 0, \qquad (**)$$

i.e. we have $u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right)$ holding at $(s, x(s))$ (along a trajectory where $|Du| \neq 0$). Conversely, if $u : [0, \varepsilon) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is smooth and solves this equation, with $|Du| \neq 0$ for $t \in [0, \varepsilon)$ in a neighbourhood of $\Sigma_0 = \{u(0,\cdot) = 0\}$, then for $t \in [0, \varepsilon')$ the flow $t \mapsto \Sigma_t = \{u(t,\cdot) = 0\}$ is a MCF! Let's use this as motivation for a definition.

Thus we turn to studying the PDE problem (summing over repeated indices)

$$\begin{cases} u_t = |Du| \operatorname{div}_{\mathbb{R}}\left(\frac{Du}{|Du|}\right) = \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}\right) D^2_{ij} u, \\ u(0,\cdot) = u_0, \end{cases} \qquad \text{(LSF)}$$

and we study weak solutions of (LSF) to define weak flows.

Remark 21. *(Issues)*

- *When* $|Du| = 0$ *this is* not *classically defined (and in general we* cannot *expect* $|Du| \neq 0$ *everywhere!).*
- *(LSF) is also* very *degenerate: if* $\psi : \mathbb{R} \to \mathbb{R}$ *is a smooth map and* u *solves (LSF), then* $\psi(u)$ *solves (LSF) also! (We did not ask the level sets to 'interact' with one another, so we can only get estimates along level sets, and* not *transverse to them; there are no local a priori estimates!)*

Our steps will be as follows:

1. We need a weak notion of a solution.
2. Basic properties of weak solutions.
3. Uniqueness of weak solutions (from given initial data); this is the *important* part, as it gives a canonical way to evolve things!
4. Existence of weak solutions.
5. Relation to MCF (to what extent do MCF properties hold for solutions of (LSF)?).

2.3 Weak solutions

Let's start with (1).

Definition 2. *(Weak solutions) A function $u \in C^0([0,\infty)\times\mathbb{R}^{n+1}) \cap L^\infty([0,\infty)\times\mathbb{R}^{n+1})$ is a **weak subsolution** of (LSF) if for any $\varphi \in C^\infty((-\infty,\infty)\times\mathbb{R}^{n+1})$ such that $u-\varphi$ has a local maximum at a point $z_0=(t_0,x_0)\in(0,\infty)\times\mathbb{R}^{n+1}$, we have either:*

1. $\varphi_t \le \left(\delta_{ij} - \frac{D_i\varphi D_j\varphi}{|D\varphi|^2}\right) D^2_{ij}\varphi$ *at* z_0, *if* $|D\varphi|(z_0)\neq 0$, *or*
2. $\varphi_t \le (\delta_{ij}-\xi_i\xi_j)D^2_{ij}\varphi$ *at* z_0 *for some* $\xi\in\mathbb{R}^{n+1}$ *with* $|\xi|\le 1$, *if* $|D\varphi|(z_0)=0$ *(having* $|\xi|\le 1$, not $|\xi|=1$, *is important; see later!).*

*We similarly say that u is a **weak supersolution** if whenever $u-\varphi$ has a local* minimum *at z_0, (1) or (2) hold with the inequalities reversed. We say that u is a **weak solution** if it is both a weak subsolution and a weak supersolution.*

Then, if u is smooth at z_0 with $Du(z_0)\neq 0$ and is a classical sub/supersolution there, i.e.

$$u_t \le \left(\delta_{ij} - \frac{D_iuD_ju}{|Du|^2}\right) D^2_{ij}u \quad \text{(subsolution)},$$
$$u_t \ge \left(\delta_{ij} - \frac{D_iuD_ju}{|Du|^2}\right) D^2_{ij}u \quad \text{(supersolution)},$$

and $u-\varphi$ has a local maximum (minimum) at z_0, then $\partial_t u=\partial_t\varphi$, $Du=D\varphi$ and $D^2u\le D^2\varphi$ ($D^2u\ge D^2\varphi$) at z_0; noting that $\left(\delta_{ij}-\frac{D_iuD_ju}{|Du|^2}\right)$ is positive semi-definite, we get (1). Thus classical solutions are weak solutions!

Remark 22. *(Further justification) If u is smooth and a classical subsolution on $\{Du\neq 0\}$, then we claim that the weak definition is consistent even on $\{Du=0\}$. If $Du(z_0)=0$, with $z_0=(t_0,x_0)$, and $z_k=(t_k,x_k)\to z_0$ with $Du(z_k)\neq 0$, then at z_k we have*

$$u_t \le (\delta_{ij}-\xi^k_i\xi^k_j)D^2_{ij}u \quad \text{for } \xi^k=\frac{Du(z_k)}{|Du(z_k)|},\ |\xi^k|=1.$$

Then, up to a subsequence, $\xi^k\to\xi$ with $|\xi|=1$, and hence at z_0 we have $u_t\le(\delta_{ij}-\xi_i\xi_j)D^2_{ij}u$ by continuity (so u is a weak subsolution there also!). Moreover, if z_0 belongs to the interior of $Z=\{Du=0\}$, then u is constant in space and $u=U(t)$ near z_0; this extends to a point z_0' on $\partial(Z^\circ)$ with $D^2u(z_0')=0$, and thus $u_t(z_0')\le 0$, so that $U'(t_0)\le 0$.

Equivalently to the definition of weak subsolutions, we can say that u is a weak subsolution of (LSF) if whenever $z_0 = (t_0, x_0) \in (0,\infty) \times \mathbb{R}^{n+1}$ and

$$u(t,x) \le u(t_0,x_0) + p \cdot (x - x_0) + q(t - t_0) \\ + \tfrac{1}{2}(z - z_0)^T R\,(z - z_0) + o(|z - z_0|^2)$$

as $z = (t,x) \to z_0$, where $p \in \mathbb{R}^{n+1}$, $q \in \mathbb{R}$, and R is a symmetric $(n+2)\times(n+2)$ matrix (with $n+1$ space directions and one time direction, the last), then

$$\begin{cases} q \le \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) R_{ij} & \text{if } |p| \neq 0, \\ q \le (\delta_{ij} - \xi_i \xi_j) R_{ij} \text{ for some } |\xi| \le 1 & \text{if } |p| = 0, \end{cases}$$

where the sums are over the spatial entries R_{ij}, $1 \le i,j \le n+1$. (There is a similar idea for weak supersolutions; see Jensen and Ishii. One uses Taylor polynomials.)

Theorem 10. *(Stability) If (u_k) is a sequence of weak solutions of (LSF) with both*

1. *$\|u_k\|_{L^\infty} \le C$ uniformly, and*
2. *$u_k \to u$ locally uniformly, i.e. in $C^0_{loc}([0,\infty) \times \mathbb{R}^{n+1})$,*

then u is also a weak solution of (LSF). The same is true for weak sub/supersolutions.

We will also show:

Theorem 11. *(Invariance under reparametrisation) If u is a weak solution of (LSF) and $\psi : \mathbb{R} \to \mathbb{R}$ is continuous, then $\psi(u)$ is also a weak solution of (LSF) (ψ does not need to be invertible!).*

Remark 23. *If ψ is strictly increasing, then ψ takes subsolutions to subsolutions and supersolutions to supersolutions.*

Proof of Theorem 10. From the definition, we will show that if the u_k are subsolutions then so is u (the same idea works for supersolutions/solutions). We pick some test function $\varphi \in C^\infty((0,\infty) \times \mathbb{R}^{n+1})$ and for now suppose that $u - \varphi$ has a *strict* local maximum at $z_0 = (t_0, x_0)$. As $u_k \to u$ in C^0_{loc}, we see that $u_k - \varphi$ has a local maximum at some z_k near z_0 (for sufficiently large k), with $z_k \to z_0$. Thus, as u_k is a weak subsolution, either:

1. $D\varphi(z_k) \neq 0$ and $\varphi_t \le \left(\delta_{ij} - \frac{D_i\varphi D_j\varphi}{|D\varphi|^2}\right) D^2_{ij}\varphi$ at z_k, or

2. $D\varphi(z_k) = 0$ and $\varphi_t \leq (\delta_{ij} - \xi_i^k \xi_j^k) D^2_{ij}\varphi$ at z_k, for some $|\xi^k| \leq 1$.

As $z_k \to z_0$, if $D\varphi(z_0) \neq 0$ then $D\varphi(z_k) \neq 0$ for large k, and taking limits in (1) we see that (1) holds for φ at z_0. Now if $D\varphi(z_0) = 0$ we set

$$\theta^k = \begin{cases} \frac{D\varphi}{|D\varphi|}(z_k) & \text{if } D\varphi(z_k) \neq 0, \\ \xi^k & \text{if } D\varphi(z_k) = 0, \end{cases}$$

so that $|\theta^k| \leq 1$, and (up to a subsequence, not relabelled) $\theta^k \to \theta$ with $|\theta| \leq 1$. Thus (1) or (2) holds with θ^k, so we can pass to the limit to get (2) with θ at z_0.

Now if $u - \varphi$ only has a local maximum at z_0, we add a high enough order polynomial to force a strict local maximum:

$$\widetilde{\varphi} = \varphi + (t - t_0)^4 + |x - x_0|^4,$$

so that $u - \widetilde{\varphi}$ has a strict local maximum at z_0. Thus (1) or (2) hold for $\widetilde{\varphi}$, which is equivalent to (1) or (2) holding for φ (as we only affect third order/higher behaviour). □

Proof of Theorem 11. We first assume that ψ is smooth and strictly increasing, i.e. $\psi' > 0$. Let us denote $\Phi = \psi^{-1}$, which is smooth with $\Phi' > 0$, and let $v = \psi(u)$. Pick $\varphi \in C^\infty((0,\infty) \times \mathbb{R}^{n+1})$ such that $v - \varphi$ has a local maximum at $z_0 = (t_0, x_0)$; without loss of generality $v(z_0) = \varphi(z_0)$ and $v(z) \leq \varphi(z)$ near z_0. Let $\Psi = \Phi(\varphi) = \psi^{-1}(\varphi)$, which is such that $\Psi(z_0) = u(z_0)$ and $u(z) \leq \Psi(z)$ near z_0 (since $\psi' > 0$). Thus $u - \Psi$ has a local maximum at z_0, so if u is a weak subsolution then either:

1. $D\Psi(z_0) \neq 0$ and $\Psi_t \leq \left(\delta_{ij} - \frac{D_i\Psi D_j\Psi}{|D\Psi|^2}\right) D^2_{ij}\Psi$ at z_0, or

2. $D\Psi(z_0) = 0$ and $\Psi_t \leq (\delta_{ij} - \xi_i\xi_j) D^2_{ij}\Psi$ at z_0, for some $|\xi| \leq 1$.

Note that $D\Psi(z_0) = \Phi'(\varphi(z_0)) D\varphi(z_0)$, and since $\Phi' > 0$, if $D\varphi(z_0) \neq 0$ then by case (1) above we have, at z_0,

$$\Phi'(\varphi(z_0))\varphi_t \leq \left(\delta_{ij} - \frac{(\Phi')^2 D_i\varphi D_j\varphi}{(\Phi')^2|D\varphi|^2}\right)\left(\Phi' D^2_{ij}\varphi + \Phi'' D_i\varphi D_j\varphi\right),$$

where the Φ'' term gives 0! As $\Phi' > 0$ this gives $\varphi_t \leq \left(\delta_{ij} - \frac{D_i\varphi D_j\varphi}{|D\varphi|^2}\right) D^2_{ij}\varphi$, so (1) holds for v! But if $D\varphi(z_0) = 0$, then by case (2) above (2) similarly holds for v! Thus v is a subsolution (and similarly v is a supersolution)!

Now if ψ is smooth but $\psi' < 0$, all is the same by reversing local maxima to minima and the inequalities (sub $\leftrightarrow$ super). If ψ is smooth but only

$\psi' \geq 0$ (or $\psi' \leq 0$), then we use the above for ψ_k with $\psi_k' \neq 0$ and $\psi_k \to \psi$ uniformly, so that $\psi_k(u) \to \psi(u)$; we can then apply Theorem 10 to see that $\psi(u)$ is a weak sub/supersolution if u is (since $u \in L^\infty$!).

Next, if ψ is smooth and $-\infty = a_0 < a_1 < \cdots < a_m < a_{m+1} = \infty$, with ψ monotone on each (a_j, a_{j+1}) and ψ constant on $(a_j - \sigma, a_j + \sigma)$ for some $\sigma > 0$ (for $j = 1, \ldots, m$), then since we only required local monotonicity in the earlier arguments, we get that $\psi(u)$ is a weak solution. Finally, if ψ is just continuous, we can approximate it by functions as above, so that $\psi(u)$ is a weak solution. □

2.4 Uniqueness

We now turn to uniqueness.

Theorem 12. *(Comparison principle for weak solutions) Assume that u is a weak subsolution and v is a weak supersolution of (LSF), with $u \leq v$ at $t = 0$, and with u and v constant on $\{t + |x| \geq R\}$ (with $u \leq v$ there), for some fixed $R > 0$ and $t \geq 0$. Then $u \leq v$ on $[0, \infty) \times \mathbb{R}^{n+1}$.*

Being constant after a finite time is consistent with MCF (see the remark after Lemma 2).

Remark 24.
- *This implies uniqueness for appropriately restricted weak solutions of (LSF) (e.g. with $u = v$ on $\{t + |x| \geq R\}$).*
- *The difficulty is that we* cannot *use u or v as test functions, so we will need to* regularise *them for the proof!*

Theorem 13. *(Contraction principle) If u and v are weak solutions of (LSF) with $u = v = c$ on $\{|x| + t \geq R\}$, then*

$$\sup_{t \geq 0} \|u(t, \cdot) - v(t, \cdot)\|_{L^\infty} = \|u(0, \cdot) - v(0, \cdot)\|_{L^\infty}.$$

For the regularisation we use **sup/inf-convolution** (or just max/min, since our functions are bounded).

Definition 3. *(Sup/inf-convolution) Let $w : [0, \infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ be continuous and bounded, and $\varepsilon > 0$. We set*

$$w^\varepsilon(t, x) = \sup_{y \in \mathbb{R}^{n+1},\, s \geq 0} \left(w(s, y) - \varepsilon^{-1}(|x - y|^2 + |t - s|^2) \right),$$

and similarly $w_\varepsilon(t, x) = \inf_{y \in \mathbb{R}^{n+1},\, s \geq 0} \left(w(s, y) + \varepsilon^{-1}(|x - y|^2 + |t - s|^2) \right)$.

Example 5. *Let $w(x) = -|x|$ (which is not bounded, but...). Then w^ε smooths out the corner of w but w_ε does not (Figure 2.2): one can check that*

$$w^\varepsilon(x) = \begin{cases} -\varepsilon^{-1}|x|^2 & \text{for } |x| \le \frac{\varepsilon}{2}, \\ -|x| + \frac{\varepsilon}{4} & \text{for } |x| > \frac{\varepsilon}{2}, \end{cases} \qquad \text{and} \qquad w_\varepsilon(x) = -|x| - \frac{\varepsilon}{4}.$$

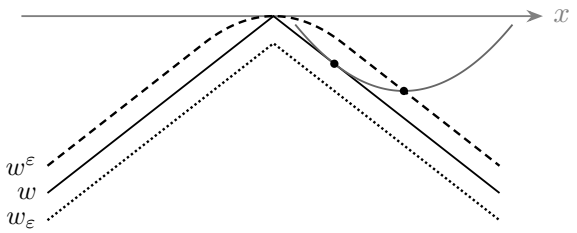


Figure 2.2. Sup/inf-convolution of $w(x) = -|x|$ (solid): w^ε (dashed) and w_ε (dotted). The paraboloid $\varepsilon^{-1}|x-y|^2 + w^\varepsilon(x)$, with vertex on the graph of w^ε, touches the graph of w from above.

What does w^ε mean? For a fixed x (ignoring t), the paraboloid $P(y) = \varepsilon^{-1}|x-y|^2 + \alpha$ lies above the graph of w for $\alpha \ge w^\varepsilon(x)$, touching it when $\alpha = w^\varepsilon(x)$. Then the union, R^ε, of the regions below the downward paraboloids $x \mapsto w(y) - \varepsilon^{-1}|x-y|^2$ (whose vertices lie on the graph of w) is such that $\partial R^\varepsilon = \text{graph}(w^\varepsilon)$ (similarly, using upward paraboloids touching from below, $\partial R_\varepsilon = \text{graph}(w_\varepsilon)$).

Lemma 2. *(Properties of sup/inf-convolution) Given a continuous bounded $w : [0,\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$, there are $A, B, C > 0$, depending on n and $\|w\|_{L^\infty}$, such that:*

1. $w_\varepsilon \le w \le w^\varepsilon$.
2. $\|w^\varepsilon\|_{L^\infty}, \|w_\varepsilon\|_{L^\infty} \le A$.
3. *If $w^\varepsilon(t,x) = w(s,y) - \varepsilon^{-1}(|x-y|^2 + |t-s|^2)$ for some (s,y), with (t,x) fixed, then $|x-y|, |t-s| \le C\varepsilon^{\frac{1}{2}} =: \sigma(\varepsilon)$ (and the same holds for w_ε).*
4. *$w^\varepsilon, w_\varepsilon \to w$ uniformly on compact sets as $\varepsilon \to 0$.*
5. *$\text{Lip}(w^\varepsilon), \text{Lip}(w_\varepsilon) \le \frac{B}{\varepsilon}$ for $\varepsilon > 0$ (even though w is only continuous!).*
6. *$(t,x) \mapsto w^\varepsilon(t,x) + \varepsilon^{-1}(|x|^2 + t^2) = Z^\varepsilon(t,x)$ is convex, i.e. for $\lambda \in [0,1]$,*

$$Z^\varepsilon(\lambda(t,x) + (1-\lambda)(\tilde{t},\tilde{x})) \le \lambda Z^\varepsilon(t,x) + (1-\lambda)Z^\varepsilon(\tilde{t},\tilde{x}).$$

Similarly, $(t,x) \mapsto w_\varepsilon(t,x) - \varepsilon^{-1}(|x|^2+t^2)$ is concave (this gives weak control on second derivatives).

7. *If w is a weak subsolution (supersolution) of (LSF) on $[0,\infty)\times\mathbb{R}^{n+1}$, then w^ε is a weak subsolution (w_ε is a weak supersolution) on $[\sigma(\varepsilon),\infty)\times\mathbb{R}^{n+1}$.*

8. *w^ε is twice differentiable a.e., and if w is a weak subsolution, then wherever $D^2w^\varepsilon(z_0)$ exists and $Dw^\varepsilon(z_0)\neq 0$ we have*

$$w_t^\varepsilon \leq \left(\delta_{ij} - \frac{D_i w^\varepsilon D_j w^\varepsilon}{|Dw^\varepsilon|^2}\right) D_{ij}^2 w^\varepsilon \quad \text{at } z_0$$

(and similarly, with the inequality reversed, for w_ε).

Proof.

- (1) and (2) follow from the definitions.
- For (3), $\|w - w^\varepsilon\|_{L^\infty} \leq A + \|w\|_{L^\infty} \leq 2A$, so that $\varepsilon^{-1}(|x-y|^2+|t-s|^2) = w(s,y) - w^\varepsilon(t,x) \leq 2A$, and thus $|x-y|, |t-s| \leq \sqrt{2A}\,\varepsilon^{\frac{1}{2}} = \sigma(\varepsilon)$.
- (4) follows from (3) and the uniform continuity of w on compact subsets.
- (5) follows from (3) and w^ε being bounded; thus we work at small scales. We know that $w^\varepsilon(t,x) = w(s,y) - \varepsilon^{-1}(|x-y|^2+|t-s|^2)$ for some (s,y) with $|x-y|, |t-s| < \sigma(\varepsilon)$. If $(\tilde{t},\tilde{x})$ is another point, then by definition

$$\begin{aligned}
w^\varepsilon(\tilde{t},\tilde{x}) &\geq w(s,y) - \varepsilon^{-1}(|\tilde{x}-y|^2 + |\tilde{t}-s|^2) \\
&= w^\varepsilon(t,x) + \varepsilon^{-1}(|x-y|^2+|t-s|^2) \\
&\quad - \varepsilon^{-1}(|\tilde{x}-y|^2+|\tilde{t}-s|^2) \\
&= w^\varepsilon(t,x) + \varepsilon^{-1}(x-\tilde{x})\cdot\big((x-y)+(\tilde{x}-y)\big) \\
&\quad + \varepsilon^{-1}(t-\tilde{t})\big((t-s)+(\tilde{t}-s)\big) \\
&\geq w^\varepsilon(t,x) - \varepsilon^{-1}|x-\tilde{x}|\big(2\sigma(\varepsilon)+|x-\tilde{x}|\big) \\
&\quad - \varepsilon^{-1}|t-\tilde{t}|\big(2\sigma(\varepsilon)+|t-\tilde{t}|\big),
\end{aligned}$$

  using $|x-y|, |t-s| \leq \sigma(\varepsilon)$. So for $|x-\tilde{x}|, |t-\tilde{t}| \leq 1$ we get $w^\varepsilon(\tilde{t},\tilde{x}) \geq w^\varepsilon(t,x) - \frac{2\sigma(\varepsilon)+1}{\varepsilon}(|x-\tilde{x}|+|t-\tilde{t}|)$, and reversing the roles of (t,x) and $(\tilde{t},\tilde{x})$ gives the other bound. This is a local Lipschitz bound with constant $\frac{B}{\varepsilon}$, where $B = 2\sqrt{2A}+1$ for $0<\varepsilon\leq 1$, and being Lipschitz on all segments of length at most 1 gives the global bound.

- (6) follows by fixing (s,y): then $(t,x) \mapsto w(s,y) - \varepsilon^{-1}(|x-y|^2 + |t-s|^2) + \varepsilon^{-1}(|x|^2+t^2)$ is an affine function, and the supremum of these over (s,y) is Z^ε; hence Z^ε is convex (its epigraph, the region above its graph, is an intersection of half spaces). The concave case is similar, using the infimum.

- For (7), fix $\varphi \in C^\infty$ with $w^\varepsilon - \varphi$ having a local maximum at $z_0 = (t_0, x_0)$ for $t_0 \in [\sigma(\varepsilon), \infty)$. Pick $w_0 = (s_0, y_0)$ so that

$$w^\varepsilon(z_0) = w(w_0) - \varepsilon^{-1}|z_0 - w_0|^2 \qquad (\text{so } |z_0 - w_0| < \sigma(\varepsilon)),$$

  and set $\psi(t,x) = \varphi(t + t_0 - s_0, x + x_0 - y_0) = \varphi(\cdot + z_0 - w_0)$. Then

$$\begin{aligned} w(s_0,y_0) - \varepsilon^{-1}|z_0 - w_0|^2 - \varphi(t_0,x_0) &= w^\varepsilon(t_0,x_0) - \varphi(t_0,x_0) \\ &\geq w^\varepsilon(t,x) - \varphi(t,x) \qquad (\diamond) \end{aligned}$$

  for (t,x) near (t_0,x_0), as $w^\varepsilon - \varphi$ has a local maximum at z_0! By the definition of w^ε, the right hand side of $(\diamond)$ is at least $w(s,y) - \varepsilon^{-1}(|x-y|^2+(t-s)^2) - \varphi(t,x)$ for all (s,y), and taking $t = s + t_0 - s_0$, $x = y + x_0 - y_0$ (so the penalty terms agree) gives

$$w(s_0,y_0) - \psi(s_0,y_0) \geq w(s,y) - \psi(s,y) \quad \text{for } (s,y) \text{ near } (s_0,y_0).$$

  So $w - \psi$ has a local maximum at (s_0,y_0)! Then, if $D\psi(s_0,y_0) \neq 0$, we get

$$\psi_t \leq \left(\delta_{ij} - \frac{D_i\psi D_j\psi}{|D\psi|^2}\right) D^2_{ij}\psi \quad \text{at } (s_0,y_0),$$

  or if $D\psi(s_0,y_0) = 0$ then $\psi_t \leq (\delta_{ij} - \xi_i\xi_j)D^2_{ij}\psi$ for some $|\xi| \leq 1$ at (s_0,y_0). Now as $D\psi(s_0,y_0) = D\varphi(t_0,x_0)$ (and similarly for $\partial_t\psi$ and $D^2\psi$), we get that w^ε is a subsolution.

- For (8), convex functions are twice differentiable a.e. (Alexandrov), and the final statement was verified before by looking at the 2-jet (the classical case, by (7)).

□

Remark 25. *(On the assumption in the comparison principle) The assumption that u and v are constant on $\{t + |x| \geq R\}$, with $u \leq v$ there, is natural for weak solutions (Figure 2.3). If $u(0,\cdot)$ is constant on $\{|x| \geq R\}$ then $\partial B_R \subset \{|x| \geq R\}$ lies in a level set of $u(0,\cdot)$. The MCF of ∂B_R is $\partial B_{\sqrt{R^2-2nt}}$, so if (LSF) describes MCF then this flow should lie in $\{u = const = c\}$, and so we get finite time extinction/constant functions!*

We now prove the comparison principle.

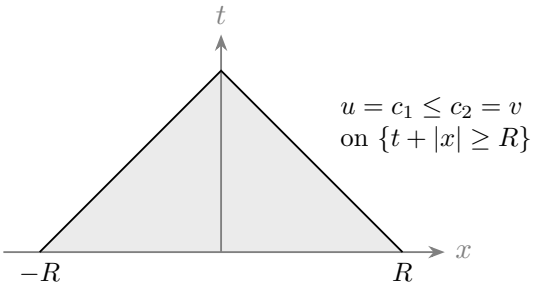


Figure 2.3. Outside the region $\{t + |x| < R\}$ (shaded) both u and v are constant, with $u \leq v$ there (the true boundary of the region where a solution can vary is really a paraboloid).

Proof of Theorem 12. Suppose that $u \leq v$ fails, so that

$$\max_{(t,x)}(u - v) = a > 0$$

(recall that $u - v \leq 0$ on $\{t + |x| \geq R\}$ by assumption). For some small $\alpha > 0$ (penalising the distance from $t = 0$) we then have

$$\max_{(t,x)}(u - v - \alpha t) \geq \frac{a}{2} > 0.$$

We then regularise, considering u^ε and v_ε, which are sub- and supersolutions respectively on $[\sigma(\varepsilon), \infty) \times \mathbb{R}^{n+1}$ for $\varepsilon > 0$ sufficiently small. For small $\varepsilon > 0$, both u^ε and v_ε are constant on $\{t + |x| \geq 2R\}$, say equal to c_1 and c_2. Also, $u^\varepsilon \to u$ and $v_\varepsilon \to v$ uniformly as $\varepsilon \searrow 0$ (since $\{t + |x| \leq 2R\}$ is compact), and thus

$$\max_{(t,x)}(u^\varepsilon - v_\varepsilon - \alpha t) \geq \frac{a}{4} > 0 \quad \text{for small } \varepsilon > 0.$$

Fix $\delta > 0$ (later taken small) and consider the function

$$\Phi(x, y, t, s) = u^\varepsilon(t + s, x + y) - v_\varepsilon(t, x) - \alpha t - \delta^{-1}(|y|^4 + s^4),$$

where y and s are perturbations. As Φ is bounded in terms of $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$, we have

$$\infty > \max\{\Phi(x, y, t, s) \,|\, (t, x),\ (t + s, x + y) \in [0, \infty) \times \mathbb{R}^{n+1}\} \geq \frac{a}{4} > 0$$

also! We can then pick (x_1, y_1, t_1, s_1) where this maximum occurs, so $\Phi(x_1, y_1, t_1, s_1) \geq \frac{a}{4} > 0$.

We observe that $|y_1|$, $|s_1| < C\delta^{\frac{1}{4}}$, for C depending on the sup norms of u and v again. Now, we claim that there are $\varepsilon, \delta > 0$ sufficiently small so that $t_1, t_1 + s_1 > \sigma(\varepsilon)$ (i.e. we are in the region where u^ε and v_ε are sub/supersolutions). If not, then $t_1 \leq \sigma(\varepsilon) \sim \sqrt{\varepsilon}$, and so

$$\begin{aligned}\frac{a}{4} \leq \Phi(x_1, y_1, t_1, s_1) &\leq u^\varepsilon(t_1 + s_1, x_1 + y_1) - v_\varepsilon(t_1, x_1) \\ &\leq u(t_1 + s_1, x_1 + y_1) - v(t_1, x_1) + o(1) \quad (\text{as } \varepsilon \to 0) \\ &\leq u(s_1, x_1 + y_1) - v(0, x_1) + o(1) \\ &\leq u(0, x_1) - v(0, x_1) + o(1) \quad (\text{as } s_1, \delta \to 0) \\ &\leq o(1) \quad (\text{as } u - v \leq 0 \text{ at } t = 0),\end{aligned}$$

which is a contradiction since $\frac{a}{4} > 0$; hence $t_1 > \sigma(\varepsilon)$! Also, for $\delta > 0$ small we ensure that $t_1 + s_1 > \sigma(\varepsilon)$, as $|s_1| < C\delta^{\frac{1}{4}}$. We then fix such $\varepsilon, \delta > 0$ small (and $\alpha > 0$ as before).

Then u^ε and v_ε are weak sub- and supersolutions near $(t_1 + s_1, x_1 + y_1)$ and (t_1, x_1) respectively. We claim that $y_1 \neq 0$; i.e. that $y_1 = 0$ leads to a contradiction. If $y_1 = 0$, then (as we are at a maximum)

$$\begin{aligned}u^\varepsilon(t_1 + s_1, x_1) - v_\varepsilon(t_1, x_1) - \alpha t_1 - \delta^{-1} s_1^4 \geq u^\varepsilon(t + s, x + y) - v_\varepsilon(t, x) - \alpha t \\ - \delta^{-1}(|y|^4 + s^4) \qquad (\diamond\diamond)\end{aligned}$$

for all (t, x, s, y) near $(t_1, x_1, s_1, 0)$. Plugging in $t = t_1$ and $x = x_1$ gives

$$u^\varepsilon(t_1 + s_1, x_1) \geq u^\varepsilon(t_1 + s, x_1 + y) - \delta^{-1}(|y|^4 + s^4) + \delta^{-1} s_1^4,$$

and setting $r = s - s_1$ we get

$$\underbrace{u^\varepsilon(t_1 + s_1, x_1) + \frac{4s_1^3 r}{\delta} + \frac{6s_1^2 r^2}{\delta}}_{\text{quadratic polynomial}} + O(|y|^4 + |r|^3) \geq u^\varepsilon(t_1 + s_1 + r, x_1 + y).$$

As u^ε is a weak subsolution, setting $p = 0$, $q = \frac{4s_1^3}{\delta}$, $r_{n+2,n+2} = \frac{12s_1^2}{\delta}$ and $r_{ij} = 0$ otherwise, the alternative definition of subsolutions implies that

$$\frac{4s_1^3}{\delta} \leq 0 \qquad (\flat)$$

(as the spatial part of the Hessian vanishes, this says $\partial_t u^\varepsilon \leq 0$). Setting $y = x_1 + y_1 - x$ and $s = t_1 + s_1 - t$ in $(\diamond\diamond)$, the u^ε terms cancel and give

$$v_\varepsilon(t, x) \geq v_\varepsilon(t_1, x_1) + \left(\frac{4s_1^3}{\delta} - \alpha\right)(t - t_1)$$

$$- \frac{6s_1^2(t-t_1)^2}{\delta} + O(|x-x_1|^4 + |t-t_1|^3),$$

and as v_ε is a weak supersolution at (t_1, x_1), we set $x_0 = x_1$, $t_0 = t_1$, $p = 0$, $q = \frac{4s_1^3}{\delta} - \alpha$, $r_{n+2,n+2} = -\frac{12s_1^2}{\delta}$ and $r_{ij} = 0$ otherwise, to get

$$\frac{4s_1^3}{\delta} - \alpha \geq 0. \tag{♭♭}$$

But (♭) and (♭♭) give a contradiction, since $\alpha > 0$! Thus $y_1 \neq 0$!

We now use a general fact: if $x \mapsto f(x)$ is convex on $\mathbb{R}^n$, then $(x, y) \mapsto f(x+y)$ is also convex. Hence $(t, s, x, y) \mapsto u^\varepsilon(t+s, x+y) + \varepsilon^{-1}(|x+y|^2 + (t+s)^2)$ is convex, and similarly $(t, s, x, y) \mapsto -(v_\varepsilon(t, x) - \varepsilon^{-1}(|x|^2 + t^2))$ is convex. So $\Phi(x, y, t, s) + C(|x|^2 + |y|^2 + t^2 + s^2)$ is convex near (t_1, s_1, x_1, y_1) for some large $C = C(\varepsilon, \delta) > 0$. As Φ has a local maximum there, we can then appeal to a result of Jensen: there are points $\theta^k = (t^k, s^k, x^k, y^k) \to (t_1, s_1, x_1, y_1)$ at which Φ, u^ε and v_ε are twice differentiable, with $D\Phi(\theta^k) \to 0$ and $D^2\Phi(\theta^k) \leq o(1)\, I_{2n+4}$ as $k \to \infty$. Then

$$D_x\Phi(\theta^k) = Du^\varepsilon(t^k + s^k, x^k + y^k) - Dv_\varepsilon(t^k, x^k) = p^k - \bar{p}^k,$$
$$D_y\Phi(\theta^k) = Du^\varepsilon(t^k + s^k, x^k + y^k) - 4\delta^{-1}|y^k|^2 y^k = p^k - 4\delta^{-1}|y^k|^2 y^k.$$

Now as $y^k \to y_1 \neq 0$ as $k \to \infty$, we get $p^k, \bar{p}^k \to \frac{4}{\delta}|y_1|^2 y_1 = p \neq 0$. Then

$$\Phi_t(\theta^k) = u_t^\varepsilon(t^k + s^k, x^k + y^k) - (v_\varepsilon)_t(t^k, x^k) - \alpha = q^k - \bar{q}^k - \alpha \to 0,$$

and since u^ε and v_ε are Lipschitz, up to a subsequence $q^k \to q$ and $\bar{q}^k \to \bar{q}$ with $q - \bar{q} = \alpha > 0$. Now,

$$D_x^2\Phi(\theta^k) = D^2u^\varepsilon(t^k + s^k, x^k + y^k) - D^2v_\varepsilon(t^k, x^k) = R^k - \bar{R}^k \leq \varepsilon_k I,$$

where $\varepsilon_k \to 0$, and $R^k \geq -CI$, $\bar{R}^k \leq CI$ for a fixed $C > 0$ (by semiconvexity). Thus, up to a subsequence, $R^k \to R$ and $\bar{R}^k \to \bar{R}$, symmetric matrices with $R \leq \bar{R}$. As u^ε is a weak subsolution and v_ε a weak supersolution, with $p^k, \bar{p}^k \neq 0$, we can use

$$q^k \leq \left(\delta_{ij} - \frac{p_i^k p_j^k}{|p^k|^2}\right) R_{ij}^k \quad \text{and} \quad \bar{q}^k \geq \left(\delta_{ij} - \frac{\bar{p}_i^k \bar{p}_j^k}{|\bar{p}^k|^2}\right) \bar{R}_{ij}^k,$$

and pass to the limit (with $p = \bar{p}$) to get

$$q - \bar{q} \leq \underbrace{\left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right)}_{\geq 0} \underbrace{(R_{ij} - \bar{R}_{ij})}_{\leq 0} \leq 0;$$

but $q - \bar{q} > 0$, a contradiction! □

(The same argument also gives the contraction principle!)

2.5 Existence

Theorem 14. *(Existence of weak solutions) Suppose that $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous, with $g = const = c$ on $\{|x| \geq R_0\}$. Then there exists a weak solution of (LSF) with initial data g, i.e. a weak solution of*

$$\begin{cases} \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}\right) D^2_{ij} u = \partial_t u & \text{on } (0,\infty) \times \mathbb{R}^{n+1}, \\ u = g & \text{on } \{0\} \times \mathbb{R}^{n+1}. \end{cases}$$

Moreover, $u \equiv c$ on $\{|x| + t \geq R\}$ for some $R = R(R_0) > 0$; hence, by the uniqueness, u is unique.

We will regularise the LSF equation and apply standard techniques, then approximate to get a weak solution. So we will look at the case $g \in C^\infty(\mathbb{R}^{n+1})$ and consider, for $\varepsilon > 0$,

$$\begin{cases} \left(\delta_{ij} - \frac{D_i u^\varepsilon D_j u^\varepsilon}{\varepsilon^2 + |Du^\varepsilon|^2}\right) D^2_{ij} u^\varepsilon = \partial_t u^\varepsilon & \text{on } (0,\infty) \times \mathbb{R}^{n+1}, \\ u^\varepsilon = g & \text{on } \{0\} \times \mathbb{R}^{n+1} \end{cases} \tag{LSF$_\varepsilon$}$$

(here u^ε is *not* a sup-convolution). The presence of ε^2 ensures that the equation is not degenerate (and the ε^2 will also justify why we allow $|\xi| \leq 1$ when $D\varphi = 0$ in the weak definition).

Theorem 15. *For $g \in C^\infty(\mathbb{R}^{n+1})$ with $g = c$ on $\{|x| \geq R_0\}$, there is a unique bounded smooth solution of (LSF$_\varepsilon$), which we call u^ε, and we have uniform estimates, independent of ε, of the form*

$$\sup_{\varepsilon > 0} \left(\|u^\varepsilon\|_{L^\infty} + \|Du^\varepsilon\|_{L^\infty} + \|\partial_t u^\varepsilon\|_{L^\infty}\right) \leq C \|g\|_{C^{1,1}},$$

with all norms over $(0,\infty) \times \mathbb{R}^{n+1}$.

Proof. Fix $\sigma \in (0,1)$ and consider

$$\begin{cases} \partial_t u^{\varepsilon,\sigma} = a^{\varepsilon,\sigma}_{ij}(Du^{\varepsilon,\sigma}) D^2_{ij} u^{\varepsilon,\sigma}, \\ u^{\varepsilon,\sigma} = g \text{ on } \{0\} \times \mathbb{R}^{n+1}, \end{cases} \qquad \text{for } a^{\varepsilon,\sigma}_{ij}(p) = (1+\sigma)\delta_{ij} - \frac{p_i p_j}{\varepsilon^2 + |p|^2}, \tag{LSF$_{\varepsilon,\sigma}$}$$

which is uniformly elliptic: $\sigma|\xi|^2 \leq a^{\varepsilon,\sigma}_{ij}(p)\xi_i\xi_j$. We use 'standard' PDE theory for parabolic equations to get a bounded, unique, smooth solution $u^{\varepsilon,\sigma}$ with $\|u^{\varepsilon,\sigma}\|_{L^\infty} = \|g\|_{L^\infty}$ (the coefficients are time independent, so the solution exists for all time). By differentiating the equation we get a PDE of the form (for $v_l = D_l u^{\varepsilon,\sigma}$, writing $u = u^{\varepsilon,\sigma}$)

$$(v_l)_t = a^{\varepsilon,\sigma}_{ij}(Du) D^2_{ij} v_l + a^{\varepsilon,\sigma}_{ij,p_k}(Du) D^2_{ij} u \, D_k v_l,$$

with $v_l = D_l g$ on $\{0\} \times \mathbb{R}^{n+1}$, where $a^{\varepsilon,\sigma}_{ij,p_k} = \partial_{p_k} a^{\varepsilon,\sigma}_{ij}$. The maximum principle then tells us that $\|v_l\|_{L^\infty} \leq \|D_l g\|_{L^\infty}$, so $\|Du^{\varepsilon,\sigma}\|_{L^\infty} \leq \|g\|_{C^1}$. The same idea with $\partial_t u^{\varepsilon,\sigma}$ gives $\|\partial_t u^{\varepsilon,\sigma}\|_{L^\infty} \leq C\|g\|_{C^{1,1}}$ (using a Lipschitz bound on Dg). Observing that if $|p| \leq L$ then

$$\Big(\underbrace{1 - \frac{L^2}{\varepsilon^2 + L^2}}_{>0, \text{ independent of } \sigma} \Big) |\xi|^2 \leq a^{\varepsilon,\sigma}_{ij}(p)\xi_i\xi_j,$$

on compact regions of space-time in $(0,\infty) \times \mathbb{R}^{n+1}$ we get $\|D^k u^{\varepsilon,\sigma}\|_{L^\infty} \leq C_k(\varepsilon)\|g\|_{L^\infty}$ for any $k \geq 1$, independently of σ. So as $\sigma \to 0$, $u^{\varepsilon,\sigma} \to u^\varepsilon$ in C^k for all $k \geq 1$, uniformly on compact sets. Thus $\|u^\varepsilon\|_{L^\infty} \leq \|g\|_{L^\infty}$, $\|Du^\varepsilon\|_{L^\infty} \leq \|g\|_{C^1}$, and $\|\partial_t u^\varepsilon\|_{L^\infty} \leq C\|g\|_{C^{1,1}}$. □

Proof of Theorem 14. First assume that $g \in C^\infty(\mathbb{R}^{n+1})$ and apply the previous theorem: for each $\varepsilon > 0$ we have smooth bounded solutions u^ε of (LSF_ε), with uniform estimates independent of ε. Thus, by Arzelà–Ascoli, up to a subsequence we have $u^{\varepsilon_k} \to u$ uniformly on compact subsets of $[0,\infty) \times \mathbb{R}^{n+1}$, with u bounded and Lipschitz. We now show that u is a weak solution. Pick $\varphi \in C^\infty((0,\infty) \times \mathbb{R}^{n+1})$ with $u - \varphi$ having a strict local maximum at $z_0 = (t_0, x_0) \in (0,\infty) \times \mathbb{R}^{n+1}$. As $u^{\varepsilon_k} \to u$ uniformly, there are points $z_k \to z_0$ at which $u^{\varepsilon_k} - \varphi$ has a local maximum. The second derivative test gives

$$Du^{\varepsilon_k}(z_k) = D\varphi(z_k), \quad \partial_t u^{\varepsilon_k}(z_k) = \partial_t \varphi(z_k), \quad D^2 u^{\varepsilon_k}(z_k) \leq D^2\varphi(z_k),$$

and ($\mathrm{LSF}_{\varepsilon_k}$) then gives

$$\varphi_t \leq \left(\delta_{ij} - \frac{D_i\varphi D_j\varphi}{\varepsilon_k^2 + |D\varphi|^2} \right) D^2_{ij}\varphi \quad \text{at } z_k.$$

So if $D\varphi(z_0) \neq 0$, then $D\varphi(z_k) \neq 0$ for k large and thus, as $k \to \infty$,

$$\varphi_t \leq \left(\delta_{ij} - \frac{D_i\varphi D_j\varphi}{|D\varphi|^2} \right) D^2_{ij}\varphi \quad \text{at } z_0.$$

But if $D\varphi(z_0) = 0$, then we set $\xi^k = \frac{D\varphi(z_k)}{\sqrt{\varepsilon_k^2 + |D\varphi(z_k)|^2}}$, so that $|\xi^k| \leq 1$ and $\varphi_t \leq (\delta_{ij} - \xi^k_i\xi^k_j)D^2_{ij}\varphi$ at z_k. So again, up to a further subsequence, $\xi^k \to \xi$ with $|\xi| \leq 1$, and as $k \to \infty$ we get $\varphi_t \leq (\delta_{ij} - \xi_i\xi_j)D^2_{ij}\varphi$ at z_0. If $u - \varphi$ only has a local (not strict) maximum, let $\widetilde{\varphi} = \varphi + |x - x_0|^4 + |t - t_0|^4$, so that $u - \widetilde{\varphi}$ has a strict local maximum; the above then shows that u is a weak subsolution of (LSF) (and similarly a weak supersolution).

We now show that u is constant on $\{|x| + t \geq R\}$ for some $R > 0$. Note that if $D_\lambda(t, x) = (\lambda^2 t, \lambda x)$ for $\lambda > 0$, then for $a > 0$ and $b \in \mathbb{R}$, u is a weak solution of (LSF) if and only if $a(u \circ D_\lambda) + b$ is also. So without loss of generality we assume that $|g| \leq 1$ on B_1 and $g = 0$ on B_1^c. Let $v(t, x) = \psi\left(\frac{|x|^2}{2} + nt\right)$ for $x \in \mathbb{R}^{n+1}$ and $t > 0$, for the cutoff function

$$\psi(s) = \begin{cases} 0 & s \geq 2, \\ (s-2)^3 & 0 \leq s \leq 2, \end{cases}$$

so that the level sets of $\frac{|x|^2}{2} + nt$ are shrinking spheres (moving by MCF), and $\{v = 0\}$ contains all of the large spheres. Then

$$\psi'(s) = \begin{cases} 0 & s \geq 2, \\ 3(s-2)^2 & 0 \leq s \leq 2, \end{cases} \qquad \psi''(s) = \begin{cases} 0 & s \geq 2, \\ 6(s-2) & 0 \leq s \leq 2, \end{cases}$$

with $|\psi''| \leq C|\psi'|^{\frac{1}{2}}$. We now show that v is a 'barrier' for (LSF_ε): one computes that

$$v_t - \left(\delta_{ij} - \frac{D_i v D_j v}{\varepsilon^2 + |Dv|^2}\right) D^2_{ij} v = A + B,$$

where

$$A = -\psi' \frac{\varepsilon^2}{(\psi')^2 |x|^2 + \varepsilon^2} \leq 0 \quad \text{and} \quad |B| = |\psi''| \frac{\varepsilon^2 |x|^2}{(\psi')^2 |x|^2 + \varepsilon^2},$$

and we can check that $B \leq C\varepsilon^{\frac{1}{2}}$ (treating the cases $\psi' \leq \varepsilon$ and $\psi' \geq \varepsilon$ separately; exercise). Thus

$$v_t - \left(\delta_{ij} - \frac{D_i v D_j v}{\varepsilon^2 + |Dv|^2}\right) D^2_{ij} v \leq C\varepsilon^{\frac{1}{2}}.$$

So, setting $w^\varepsilon(t, x) = v(t, x) - Ct\varepsilon^{\frac{1}{2}}$ (*not* a sup-convolution), this is a classical subsolution of (LSF_ε) with $w^\varepsilon \leq g$ on $\{0\} \times \mathbb{R}^{n+1}$ (check; see Figure 2.4). The maximum principle for (LSF_ε) then gives $w^\varepsilon \leq u^\varepsilon$ on $(0, \infty) \times \mathbb{R}^{n+1}$. Similarly, we can find upper barriers $(-w^\varepsilon)$. Sending $\varepsilon_k \to 0$ we get $v \leq u \leq -v$, and so $u = 0$ on $\{|x| + t \geq R\}$ for some $R > 0$. Using the comparison/contraction principles then gives the case of continuous g! □

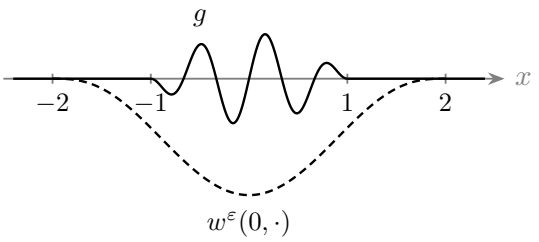


Figure 2.4. The lower barrier w^ε lies below the initial data g (schematic).

2.6 Geometric motivation for LSF_ε

Suppose that $u^\varepsilon(t,x)$ is a smooth solution of (LSF_ε), and write $y = (x_0, x) \in \mathbb{R}^{n+2} = \mathbb{R} \times \mathbb{R}^{n+1}$ to define

$$v^\varepsilon(t,y) = u^\varepsilon(t,x) - \varepsilon x_0$$

(for $\varepsilon = 0$ this is translation invariant in the x_0 direction). One can check that, for $\varepsilon > 0$, u^ε solves (LSF_ε) if and only if v^ε solves (LSF), since $|Dv^\varepsilon|^2 = |Du^\varepsilon|^2 + \varepsilon^2$. Also,

$$\begin{aligned}\Gamma_t^\varepsilon = \{y \in \mathbb{R}^{n+2} \,|\, v^\varepsilon(t,y) = 0\} &= \{y = (\varepsilon^{-1}u^\varepsilon(t,x), x) \,|\, x \in \mathbb{R}^{n+1}\} \\ &= \mathrm{graph}(\varepsilon^{-1}u^\varepsilon(t,\cdot)),\end{aligned}$$

so these are graphs, which get 'steeper' as $\varepsilon \to 0$, and the other level sets of v^ε are translates of Γ_t^ε.

Remark 26. *Ecker–Huisken [EH89, EH91] showed that graphical solutions of MCF are 'well' behaved (i.e. they exist in this class, with estimates).*

If $\Gamma_0 = \{g = 0\}$ is a compact hypersurface, $\Gamma_0 = \partial U_0$ with U_0 a bounded open set, $g < 0$ in U_0 and $g > 0$ in $\overline{U_0}^c$, then $\Gamma_0^\varepsilon = \mathrm{graph}(\varepsilon^{-1}g)$. As $\varepsilon \to 0$ we have $\Gamma_0^\varepsilon \to \mathbb{R} \times \Gamma_0$ (the cylinder), and the mean curvature of Γ_0^ε at $(0,x)$, for $x \in \Gamma_0$, tends to H_{Γ_0}. So the flows Γ_t^ε 'approximate the cylinder', i.e. the MCF of $\mathbb{R} \times \Gamma_0$ (which has no movement in the $\mathbb{R}$ factor).

2.7 Generalised mean curvature flow

Let $\mathcal{K}$ denote the collection of compact subsets of $\mathbb{R}^{n+1}$. For $\Gamma_0 \in \mathcal{K}$, we want a flow by mean curvature, $t \mapsto \mathcal{M}(t)\Gamma_0 \in \mathcal{K}$, such that:

1. (Semigroup property) $\mathcal{M}(t+s)\Gamma_0 = \mathcal{M}(t)(\mathcal{M}(s)\Gamma_0)$ for $t, s \geq 0$, with $\mathcal{M}(0)\Gamma_0 = \Gamma_0$.

2. If $t \mapsto \Sigma_t$ is a smooth MCF with $\Sigma_0 \in \mathcal{K}$, then $\Sigma_t = \mathcal{M}(t)\Sigma_0$.

Definition 4. *(Generalised MCF) Pick g with*

$$g \text{ continuous}, \quad \{g = 0\} = \Gamma_0, \quad g = \text{const on } \{|x| \geq R\}, \tag{$\dagger$}$$

let u be the unique weak solution of (LSF) with initial condition g, and set

$$\mathcal{M}(t)\Gamma_0 = \{u(t, \cdot) = 0\} = \Gamma_t.$$

We call $t \mapsto \Gamma_t$ a ***generalised MCF****.*

We claim that this is what we want! First we need to show that it is well defined.

Remark 27. *We don't expect $(t, \Gamma) \mapsto \mathcal{M}(t)\Gamma$ to be continuous in any reasonable sense! For example, two circles at distance ε apart flow to two shrinking circles, which then disappear, while as $\varepsilon \searrow 0$ their limit, the figure eight, fattens (Figure 2.5).*

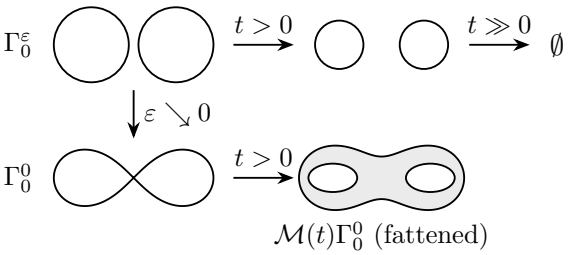


Figure 2.5. No continuity in the initial data: two circles at distance ε apart shrink and disappear, while the limiting figure eight fattens (schematic).

Theorem 16. *(Generalised MCF is well defined) If $\hat{g}$ also satisfies* (†)*, with $\hat{u}$ the corresponding weak solution of (LSF), then $\{\hat{u}(t, \cdot) = 0\} = \hat{\Gamma}_t = \Gamma_t = \{u(t, \cdot) = 0\}$. Hence, the generalised MCF is well defined!*

Proof. Without loss of generality $u, \hat{u}, g, \hat{g} \geq 0$: if we replace g by $|g|$ then (†) still holds, and since uniqueness holds, $|u|$ solves (LSF) with initial data $|g|$ (by Theorem 11) and $\Gamma_t = \{|u(t, \cdot)| = 0\}$. For each $k \geq 1$ let $E_k = \{x \in \mathbb{R}^{n+1} \,|\, g(x) \geq \frac{1}{k}\}$, and let $E_0 = \emptyset$; thus $E_1 \subset E_2 \subset \cdots$ and $\Gamma_0^c = \bigcup_k E_k$. Let $a_k = \sup_{E_{k-1}^c} \hat{g} \geq 0$; then a_k is non-increasing in k, and $a_k \to 0$: as Γ_0 is compact, $g \equiv c > 0$ outside a ball, so the sets $E_{k-1}^c = \{g < \frac{1}{k-1}\}$ lie in a fixed compact set for k large and decrease to $\{g = 0\} = \Gamma_0$, where $\hat{g} = 0$; so $a_k \to 0$ by the continuity of $\hat{g}$. Pick

$\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(\frac{1}{k}) = a_k$, φ linear on each $[\frac{1}{k+1}, \frac{1}{k}]$, and φ constant on $[1,\infty)$. Set $\check{g} = \varphi(g)$ and $\check{u} = \varphi(u)$, so that $\check{u}$ is the unique weak solution of (LSF) with initial data $\check{g}$. By construction $\check{g} = \hat{g}$ $(= 0)$ on Γ_0, and if $x \in E_k \setminus E_{k-1}$ then

$$\check{g}(x) = \varphi(g(x)) \geq \varphi(\tfrac{1}{k}) = a_k \geq \hat{g}(x),$$

so $\check{g} \geq \hat{g}$ on $\mathbb{R}^{n+1}$, and the comparison principle gives $\check{u} = \varphi(u) \geq \hat{u} \geq 0$. So if $x \in \Gamma_t$ then $x \in \hat{\Gamma}_t$! Thus $\Gamma_t \subset \hat{\Gamma}_t$, and reversing the roles gives $\hat{\Gamma}_t \subset \Gamma_t$! □

For (1) we have:

Theorem 17. *(Semigroup property)* $\mathcal{M}(t+s)\Gamma_0 = \mathcal{M}(t)(\mathcal{M}(s)\Gamma_0)$.

Proof. The cases $s = 0$ or $t = 0$ are done, so assume without loss of generality that $s, t > 0$. Pick g as in (†), with weak solution u, so that

$$\Gamma_{t+s} = \{u(t+s,\cdot) = 0\} = \mathcal{M}(t+s)\Gamma_0 \text{ and } \Gamma_s = \{u(s,\cdot) = 0\} = \mathcal{M}(s)\Gamma_0.$$

Then, for $\hat{g}$ satisfying (†) for $\Gamma_s = \mathcal{M}(s)\Gamma_0 \in \mathcal{K}$, we get $\mathcal{M}(t)(\mathcal{M}(s)\Gamma_0) = \{\hat{u}(t,\cdot) = 0\}$ for the weak solution $\hat{u}$. But this is all independent of $\hat{g}$, so we can take $\hat{g} = u(s,\cdot)$; then (by uniqueness) $\hat{u}(t,\cdot) = u(t+s,\cdot)$ for $t \geq 0$, and so $\mathcal{M}(t)(\mathcal{M}(s)\Gamma_0) = \mathcal{M}(t+s)\Gamma_0$. □

Now for (2):

Theorem 18. *(Agreement with smooth flows) Let* Γ_0 *be a smooth closed hypersurface and* $t \in [0,T) \mapsto \Sigma_t$ *a smooth MCF with* $\Sigma_0 = \Gamma_0$. *Then* $\mathcal{M}(t)\Gamma_0 = \Sigma_t$ *for all* $t \in [0,T)$.

Proof. Write $\Gamma_0 = \partial U_0$ for a bounded open set U_0 (Alexander's theorem), and let $\Sigma_t = \partial U_t$ for $t \in [0,T)$. Let $d(t,x)$ be the signed distance function,

$$d(t,x) = \begin{cases} -\operatorname{dist}(x,\Sigma_t) & \text{if } x \in U_t, \\ \operatorname{dist}(x,\Sigma_t) & \text{if } x \notin U_t, \end{cases}$$

and let $S_{t_0} = \bigcup_{0 \leq t \leq t_0} \{t\} \times \Sigma_t \subset [0,\infty) \times \mathbb{R}^{n+1}$ (in space-time), for $t_0 > 0$ small; then S_{t_0} is a smooth hypersurface in space-time. For $\delta_0 > 0$ small let

$$Q^+ = Q^+(t_0,\delta_0) = \{(t,x) \,|\, 0 \leq d(t,x) \leq \delta_0,\ 0 \leq t \leq t_0\},$$
$$Q^- = Q^-(t_0,\delta_0) = \{(t,x) \,|\, -\delta_0 \leq d(t,x) \leq 0,\ 0 \leq t \leq t_0\},$$

which are regions in which d is smooth. Note that (up to shrinking δ_0), for all $(t,x) \in Q^{\pm}$ there is a unique point $y \in \Sigma_t$ with $d(t,x) = \pm|x-y|$.

Extend the outward unit normals $\vec{\nu}$ to Σ_t to $Q^{\pm}$ by setting $\vec{\nu} = Dd$ (the spatial gradient). As the Σ_t move by MCF we compute, on Q^+, that

$$\frac{\partial}{\partial t} d(t,x) = H_{\Sigma_t}(y) = \mathrm{div}_{\mathbb{R}^{n+1}}(\vec{\nu}(t,y)) \qquad (y = \pi_{\Sigma_t}(x)),$$

and, in an orthonormal basis $E_1, \ldots, E_n, \vec{\nu}$ at x (with $E_1, \ldots, E_n \perp \vec{\nu}$),

$$D^2 d(t,x) = \begin{pmatrix} \frac{-\kappa_1}{1-\kappa_1 d} & & & \\ & \ddots & & \\ & & \frac{-\kappa_n}{1-\kappa_n d} & \\ & & & 0 \end{pmatrix},$$

where the κ_i are the principal curvatures of Σ_t at y (at $d = 0$, i.e. on Σ_t, this is the second fundamental form II_{Σ_t}, up to sign). Thus

$$\Delta_{\mathbb{R}^{n+1}} d(t,x) = -\sum_{i=1}^{n} \frac{\kappa_i}{1-\kappa_i d} \quad \text{and} \quad \mathrm{div}_{\Sigma_t}(\vec{\nu}(t,y)) = \sum_{i=1}^{n} (-\kappa_i),$$

which gives

$$\partial_t d - \Delta_{\mathbb{R}^{n+1}} d = \sum_{i=1}^{n} \frac{\kappa_i^2 d}{1-\kappa_i d} \geq 0 \quad \text{in } Q^+$$

if δ_0 is small (depending on Γ_0). Now we set $\underline{d} = \alpha e^{-\lambda t} d$, where $\alpha \in (0,1)$ and $\lambda \gg 1$, so that $(\partial_t - \Delta)\underline{d} \leq 0$ (as $\partial_t d - \Delta d \leq Cd$ for some constant C). As $|Dd|^2 = |\vec{\nu}|^2 = 1$ implies that $D_i d D^2_{ij} d = 0$ in Q^+, we then have that

$$\partial_t \underline{d} - \left(\delta_{ij} - \frac{D_i \underline{d} D_j \underline{d}}{\varepsilon^2 + |D\underline{d}|^2} \right) D^2_{ij} \underline{d} \leq 0,$$

so $\underline{d}$ is a subsolution of $(\mathrm{LSF}_\varepsilon)$ in Q^+.

Pick a Lipschitz $g : \mathbb{R}^{n+1} \to \mathbb{R}$ with $g \geq 0$, $\{g = 0\} = \Gamma_0 = \Sigma_0$, $g = c > 0$ on $\{|x| \geq R_0\}$, and $g = \mathrm{dist}(x, \Sigma_0)$ near Σ_0 (in a δ_0-neighbourhood of Σ_0). By previous work there is a unique solution, u^ε, of $(\mathrm{LSF}_\varepsilon)$ with initial data g, and $u^\varepsilon \to u$ in C^0_{loc}, where u solves (LSF) with data g. Then $\Gamma_t = \{u(t,\cdot) = 0\}$, and $u = g = \delta_0 > 0$ on the set $\{x \,|\, \mathrm{dist}(x, \Sigma_0) = \delta_0\}$ at $t = 0$. So, as u is continuous, $u \geq \frac{\delta_0}{2} > 0$ on $\{(t,x) \,|\, d(t,x) = \delta_0\}$ for all $0 \leq t \leq t_0$, for t_0 small, and so $u^\varepsilon \geq \frac{\delta_0}{4} > 0$ there; picking $\alpha > 0$ small we get $u^\varepsilon \geq \underline{d}$ on this set. We then have $u^\varepsilon \geq \underline{d}$ on the parabolic boundary of Q^+ (since $u^\varepsilon \geq 0$ and $\underline{d} = 0$ on Σ_t, and $\underline{d} = \alpha d \leq g = u^\varepsilon$ at $t = 0$), so the maximum principle gives $u^\varepsilon \geq \underline{d}$ on Q^+! Thus, as $\varepsilon \to 0$, $u \geq \underline{d}$ on Q^+; but $\underline{d} > 0$ on the interior $(Q^+)^\circ$, so $u > 0$ on $(Q^+)^\circ$ also, and $\{u = 0\} \cap Q^+ \subseteq \Sigma_t$. The same idea gives $\{u = 0\} \cap Q^- \subseteq \Sigma_t$, by choosing

$\overline{d} = -\alpha e^{-\lambda t} d$. Also $u > 0$ on $(\mathbb{R}^{n+1} \setminus \{x \,|\, \mathrm{dist}(x, \Sigma_0) \leq \delta_0\}) \times [0, t_0]$ by construction. So $u = 0$ only on Σ_t, i.e. $\Gamma_t \subseteq \Sigma_t$ for $0 \leq t \leq t_0$.

For the reverse inclusion, let $\hat{g} : \mathbb{R}^{n+1} \to \mathbb{R}$ be Lipschitz with $\hat{g} = d(0, \cdot)$ (signed) near Σ_0, $\{\hat{g} = 0\} = \Sigma_0$, and $\hat{g} = c > 0$ on $\{|x| \geq R_0\}$. Let $\hat{u}$ be the weak solution of (LSF) with data $\hat{g}$ and set $\hat{\Gamma}_t = \{\hat{u}(t, \cdot) = 0\}$. As $\hat{g} < 0$ in U_0 we have $\hat{u}(t, x) < 0$ for some $x \in U_t$, for $t \approx 0$ (by continuity), and the same reasoning gives some $x \in U_t^c$ with $\hat{u}(t, x) > 0$. Then fix $x_0 \in \Sigma_t$, for small t, and a curve C such that one endpoint, x_1, has $\hat{u}(t, x_1) > 0$, the other, x_2, has $\hat{u}(t, x_2) < 0$, and $C \cap \Sigma_t = \{x_0\}$ only! By the continuity of $\hat{u}$ on C, $\hat{u}(t, \cdot) = 0$ at some point of C, call it x', with $x' \in \hat{\Gamma}_t \subseteq \Sigma_t$ by the first step, so that $x' = x_0$. But x_0 was arbitrary, so $\Sigma_t \subseteq \hat{\Gamma}_t$, and thus $\Sigma_t = \hat{\Gamma}_t$ for small t ($\leq t_0$); i.e. $\mathcal{M}(t)\Gamma_0 = \Sigma_t$.

Finally, let $S = \sup\{\tau \in (0, T) \,|\, \Sigma_t = \Gamma_t \text{ for } 0 \leq t \leq \tau\} \geq t_0$. If $S < T$ we have $\Sigma_t = \Gamma_t$ for $0 \leq t < S$, and the continuity of solutions of (LSF) tells us that $\Sigma_S \subseteq \Gamma_S$ (i.e. $u = 0$ on Σ_S, by taking limits). However, if we had $x \in \mathbb{R}^{n+1} \setminus \Sigma_S$, then $B_r(x) \subset \mathbb{R}^{n+1} \setminus \Sigma_t$ for $S - \varepsilon \leq t \leq S$, and so $B_r(x) \cap \Gamma_t = \emptyset$ for such $t < S$; one then needs a barrier argument to conclude that $x \notin \Gamma_S$. Thus $\Gamma_S = \Sigma_S$, and since $S < T$ we can start again, a contradiction! □

2.8 Geometry of the generalised MCF

We want to show that our geometric intuition for MCF carries over to the generalised MCF.

Example 6. *(Spheres) Consider $\Gamma_0 = \partial B_{R_0}$ and $\mathcal{M}(t)\Gamma_0 = \Gamma_t$. We claim that*

$$\Gamma_t = \begin{cases} \partial B_{R(t)}, \text{ where } R(t) = \sqrt{R_0^2 - 2nt}, & \text{for } 0 \leq t < T = \frac{R_0^2}{2n}, \\ \{0\} & \text{for } t = T, \\ \emptyset & \text{for } t > T. \end{cases}$$

The first part is clear by Theorem 18, as $\Sigma_0 = \Gamma_0$ evolves smoothly for $0 < t < T$. For the other cases, let $u(t, x) = \psi(|x|^2 + 2nt)$, where $\psi : \mathbb{R} \to \mathbb{R}$ is smooth with $\psi' \geq 0$, $\psi < 0$ on $[0, R_0^2)$, $\psi > 0$ on $(R_0^2, 3R_0^2)$, $\psi(R_0^2) = 0$, and $\psi = 1$ on $[3R_0^2, \infty)$. Then u is a weak solution of (LSF) with $\{u(0, \cdot) = 0\} = \Gamma_0$, and $\Gamma_t = \{u(t, \cdot) = 0\}$ has the desired form above.

Theorem 19. *(Finite extinction and convex hulls) Let $t \mapsto \Gamma_t$ be a generalised MCF. Then:*

1. *If* $\Gamma_0 \subset B_{R_0}$ *then* $\Gamma_t = \emptyset$ *for* $t > \frac{R_0^2}{2n}$*; i.e. the flow disappears in finite time!*

2. $\Gamma_t \subset \mathrm{Conv}(\Gamma_0)$*, the convex hull of* Γ_0*.*

Remark 28. *(1) is a non-trivial remark, since before we only showed that the smooth flow could* not *continue past the collapsing time of the sphere!*

Proof. For (1), suppose first that $\Gamma_0 \subset B_{R_0-\varepsilon}$ for some small $\varepsilon > 0$, and let $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be continuous with $g \geq 0$, $\Gamma_0 = \{g = 0\}$, and $g = 1$ on $\{|x| \geq R_0\}$. Then let $\hat{g} = \psi(|x|^2)$ for ψ as above, chosen so that $\hat{g} \leq g$ on $\mathbb{R}^{n+1}$. By the comparison principle for (LSF) the corresponding weak solutions satisfy $\hat{u} \leq u$, where by uniqueness $\hat{u} = \psi(|x|^2 + 2nt)$, which is positive if $t > T = \frac{R_0^2}{2n}$. So for such t we have $u > 0$, and hence $\Gamma_t = \emptyset$! In general, if $\Gamma_0 \subset B_{R_0}$, replace R_0 by $R_0 + \varepsilon$ and send $\varepsilon \searrow 0$: we get $\Gamma_t = \emptyset$ for $t > \frac{(R_0+\varepsilon)^2}{2n}$ for each $\varepsilon > 0$, so if $t > \frac{R_0^2}{2n}$ then $\Gamma_t = \emptyset$, giving (1).

For (2), if $\Gamma_0 \subset \mathbb{R}^{n+1}_+ = \{x_1 > 0\}$, pick $R_0 \gg 1$ so that $\Gamma_0 \subset B_{R_0}(R_0 e_1) \subseteq \mathbb{R}^{n+1}_+$. Then by (the proof of) (1) we get $\Gamma_t \subseteq B_{\sqrt{R_0^2-2nt}}(R_0 e_1) \subseteq \mathbb{R}^{n+1}_+$. So $\Gamma_t \subset \mathbb{R}^{n+1}_+$, and applying this for any half space containing Γ_0 gives

$$\Gamma_t \subset \mathrm{Conv}(\Gamma_0) = \bigcap \{H \,|\, H \text{ a half space containing } \Gamma_0\},$$

which is (2). □

We now turn to the comparison of sets.

Theorem 20. *(Comparison of sets) Let* Γ_0 *and* $\hat{\Gamma}_0$ *be compact in* $\mathbb{R}^{n+1}$*, with generalised MCFs* $t \mapsto \Gamma_t$ *and* $t \mapsto \hat{\Gamma}_t$*. If* $\Gamma_0 \subset \hat{\Gamma}_0$ *then* $\Gamma_t \subset \hat{\Gamma}_t$ *for all* $t \geq 0$*.*

Remark 29. *Such a result does* not *hold for all notions of weak MCF; e.g. for the Brakke flow given by the blowdown of the grim reaper (a half-line of multiplicity two).*

Proof. Choose $g, \hat{g} \geq 0$ with $\Gamma_0 = \{g = 0\}$ and $\hat{\Gamma}_0 = \{\hat{g} = 0\}$, both constant on $\{|x| \gg 1\}$. Up to replacing g by $g + \hat{g}$ we also get $\hat{g} \leq g$ (since $\Gamma_0 \subset \hat{\Gamma}_0$ gives $\{g + \hat{g} = 0\} = \{g = 0\} \cap \{\hat{g} = 0\} = \Gamma_0$). Then the weak solutions u and $\hat{u}$ of (LSF) are such that $0 \leq \hat{u} \leq u$, so $\{\hat{u} = 0\} \supset \{u = 0\}$, i.e. $\Gamma_t \subset \hat{\Gamma}_t$. □

Theorem 21. *(Distance between generalised flows) Assume that* $\Gamma_0, \hat{\Gamma}_0 \neq \emptyset$ *are compact in* $\mathbb{R}^{n+1}$*, with generalised MCFs* $t \mapsto \Gamma_t, \hat{\Gamma}_t$*. Then* $\mathrm{dist}(\Gamma_0, \hat{\Gamma}_0) \leq \mathrm{dist}(\Gamma_t, \hat{\Gamma}_t)$ *for all* $t \geq 0$*, where we set* $\mathrm{dist}(\Gamma_t, \hat{\Gamma}_t) = \infty$ *if* Γ_t *or* $\hat{\Gamma}_t$ *is empty.*

Proof. Without loss of generality $d = \operatorname{dist}(\Gamma_0, \hat{\Gamma}_0) > 0$, so choose $g : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\begin{cases} \Gamma_0 = \{g = 0\},\ \hat{\Gamma}_0 \subset \{g = 1\}, \\ g \equiv 2 \text{ on } \{|x| \gg 1\}, \end{cases} \qquad \text{and} \quad \operatorname{Lip}(g) \leq d^{-1} \tag{‡}$$

(e.g. if $\Gamma_0 \cup \hat{\Gamma}_0 \subset B_{R-d}$, take $g(x) = \min\{\frac{\operatorname{dist}(x,\Gamma_0)}{d}, 1\} + \min\{\frac{\max\{|x|-R,0\}}{d}, 1\}$; the two terms are never both non-constant). Then let u be the unique weak solution of (LSF) from g, so that $\Gamma_t = \{u = 0\}$ and $\hat{\Gamma}_t \subset \{u = 1\}$: $u - 1$ is the solution for $g - 1$, so $\{u(t, \cdot) = 1\}$ is the generalised MCF of the compact set $\{g = 1\} \supset \hat{\Gamma}_0$, and we can use the comparison of sets (Theorem 20). By the contraction principle for (LSF) (and translation invariance) we have $\operatorname{Lip}(u(t, \cdot)) \leq \operatorname{Lip}(g)$: since $\check{u}(t, x) = u(t, x - x_0)$ also solves (LSF), with data $\check{g} = g(\cdot - x_0)$, comparing u and $\check{u}$ we get

$$\|\check{u}(t, \cdot) - u(t, \cdot)\|_{L^\infty} \leq \|\check{g} - g\|_{L^\infty} \leq \operatorname{Lip}(g)\, |x_0|!$$

Now if $\Gamma_t, \hat{\Gamma}_t \neq \emptyset$, pick $x \in \Gamma_t$ and $\hat{x} \in \hat{\Gamma}_t$ with $|x - \hat{x}| = \operatorname{dist}(\Gamma_t, \hat{\Gamma}_t)$, so that, by (‡),

$$1 = |1 - 0| = |u(t, \hat{x}) - u(t, x)| \leq \operatorname{Lip}(g)\, |x - \hat{x}| \leq \frac{\operatorname{dist}(\Gamma_t, \hat{\Gamma}_t)}{\operatorname{dist}(\Gamma_0, \hat{\Gamma}_0)}.$$

□

Remark 30. *This proof (applied from any starting time, using the semi-group property) shows that* $\operatorname{dist}(\Gamma_t, \hat{\Gamma}_t)$ *is monotone non-decreasing! (We will come back to address strictly increasing!)*

2.9 Mean convex flows

Let $\Gamma_0 = \partial U_0$, for U_0 a bounded open set in $\mathbb{R}^{n+1}$, with Γ_0 smooth and $H_{\Gamma_0} > 0$ with respect to the inward normal; i.e. Γ_0 is **mean convex**. For the smooth MCF $t \mapsto \Sigma_t$ with $\Sigma_0 = \Gamma_0$ we saw that in fact $\Sigma_t \subset U_0$ for all $t \in (0, T)$ (for short times this followed from $H_{\Gamma_0} > 0$, and for other times we used avoidance).

Question 3. *Is this true for* $\mathcal{M}(t)\Gamma_0 = \Gamma_t$*?*

Question 4. *Can we make sense of mean convexity for non-smooth objects (i.e.* $\Gamma_t \subset \Gamma_{\tilde{t}}$ *for* $t > \tilde{t}$*)? (This is addressed by other authors, and is implicit in Evans–Spruck.)*

The answer to the first question is yes, in a very strong sense (with much better behaviour than for general flows). The idea is that in this case there is an elliptic problem that determines everything: solving

$$\begin{cases} -\left(\delta_{ij} - \frac{D_i v D_j v}{|Dv|^2}\right) D^2_{ij} v = 1 & \text{in } U_0, \\ v = 0 & \text{on } \Gamma_0 = \partial U_0, \end{cases} \tag{AT}$$

gives $\Gamma_t = \mathcal{M}(t)\Gamma_0 = \{v = t\} \subset U_0$; i.e. the solution of (AT) gives the 'time of arrival' function for the flow Γ_t. Note that (AT) is equivalent to

$$-\operatorname{div}\left(\frac{Dv}{|Dv|}\right) = \frac{1}{|Dv|} > 0,$$

where the left hand side is the mean curvature of the level set $\{v = \text{const}\}$.

To recap: if $\Gamma_0 = \partial U_0$ (with U_0 open and bounded) is smooth and mean convex, then $t \mapsto \mathcal{M}(t)\Gamma_0 = \Gamma_t$ should retain aspects of smooth MCF, i.e.:

1. $\Gamma_t \subset U_0$ for $t > 0$.
2. $\Gamma_t = \text{'}\partial U_t\text{'}$ with $U_t \subset U_s$ for $t > s \geq 0$ (the flow doesn't fatten); we will need to interpret this.

Claim 1. *There is a unique solution of (AT), with estimates, and $\Gamma_t = \{v = t\}$ (so the graph of v over U_0 records the whole flow).*

To study (AT) we first need a notion of weak solution:

Definition 5. *We say that $v \in C^0(\overline{U_0})$ is a **weak subsolution** (**supersolution**) of (AT) if whenever $v - \varphi$ has a local maximum (minimum) at $x_0 \in U_0$, for some $\varphi \in C^\infty$, then at x_0 either*

$$-\left(\delta_{ij} - \frac{D_i\varphi D_j\varphi}{|D\varphi|^2}\right) D^2_{ij}\varphi \leq 1 \ (\geq 1) \quad \text{when } D\varphi \neq 0,$$

or

$$-(\delta_{ij} - \xi_i\xi_j) D^2_{ij}\varphi \leq 1 \ (\geq 1) \quad \text{for some } |\xi| \leq 1, \text{ when } D\varphi = 0.$$

*A **weak solution** is both a weak subsolution and a weak supersolution.*

Theorem 22. *(The arrival time) There is a unique weak solution, v, of (AT), and moreover there are $0 < a < A$ with*

$$a \operatorname{dist}(x, \Gamma_0) \leq v(x) \leq A \operatorname{dist}(x, \Gamma_0) \quad \text{and} \quad |Dv| \leq A$$

(so v is $C^{0,1}$, and $\{v = 0\} = \Gamma_0$).

Proof. We approximate by a more regular PDE: for $\varepsilon \in (0,1)$ consider

$$\begin{cases} -\left(\delta_{ij} - \frac{D_i v^\varepsilon D_j v^\varepsilon}{\varepsilon^2 + |Dv^\varepsilon|^2}\right) D^2_{ij} v^\varepsilon = 1 & \text{on } U_0, \\ v^\varepsilon = 0 & \text{on } \Gamma_0 = \partial U_0. \end{cases} \tag{AT_ε}$$

Existence and uniqueness for (AT_ε) follow by 'standard' theory; we now want uniform estimates in order to pass to the limit $\varepsilon \to 0$. Note that v^ε has no local minima, by the equation and the maximum principle! We construct barriers of the form $w(x) = \lambda g(d(x))$, where $d(x) = \operatorname{dist}(x, \Gamma_0)$ (which is smooth near Γ_0 and Lipschitz globally) and g, λ are to be determined. We restrict to $V = \{0 < d(x) < 2\delta_0\}$, a small one-sided tubular neighbourhood of ∂U_0 on which d is smooth (for δ_0 small). As Γ_0 is mean convex we have $0 < b < -\Delta d \le B$, where $b > 0$ is a lower bound for H_{Γ_0}; note also that $D_i d D^2_{ij} d = 0$. Then, for w,

$$\begin{aligned} \left(\delta_{ij} - \frac{D_i w D_j w}{\varepsilon^2 + |Dw|^2}\right) D^2_{ij} w &= \lambda(g' \Delta d + g'') - \frac{\lambda^3 (g')^2 g''}{\lambda^2 (g')^2 + \varepsilon^2} \\ &= \lambda g' \Delta d + \frac{\varepsilon^2 \lambda g''}{\lambda^2 (g')^2 + \varepsilon^2}. \end{aligned} \tag{$+$}$$

Setting $\underline{g}(s) = \delta_0^2 - (s - \delta_0)^2$ and $\underline{w} = \lambda \underline{g} \circ d$, we get, for $\underline{w}$,

$$(+) \ge -C\lambda\delta_0 - 2\lambda > -1 \quad \text{for small } \lambda > 0 \text{ (independent of } \varepsilon),$$

so $\underline{w}$ is a subsolution of (AT_ε) on V. The (elliptic) maximum principle (using that v^ε has no interior minima) then gives $v^\varepsilon \ge \underline{w}$ for all $0 < \lambda < \lambda_0$, with λ_0 sufficiently small. Thus $v^\varepsilon \ge a\, d(x)$ in U_0 for a uniform $a > 0$ (using $v^\varepsilon \ge \underline{w}$ on $\{d = \delta_0\}$).

For an upper bound, let $\overline{g}(s) = \log\left(\frac{2\delta_0}{2\delta_0 - s}\right)$ and $\overline{w} = \Lambda \overline{g}(d)$. Then $\overline{g}(0) = 0$, $\overline{g}' \ge \frac{1}{2\delta_0}$, $\overline{g}'' = (\overline{g}')^2$, and $\overline{g}, \overline{g}' \to \infty$ as $s \to 2\delta_0^-$, and $(+)$ for $\overline{g}$ gives

$$(+) \le -c\Lambda + \frac{\varepsilon^2}{\Lambda} < -1 \quad (\text{as } \varepsilon \le 1),$$

so $\overline{w}$ is a supersolution for Λ sufficiently large, independently of ε. Using the maximum principle again (with some care), $v^\varepsilon < \overline{w}$ for all $\varepsilon > 0$ when $\Lambda > \Lambda_0$, independently of ε. Hence $v^\varepsilon \le A\, d(x)$ on V, and

$$a \le \frac{v^\varepsilon(x)}{d(x)} \le A \quad \text{on } V,$$

so $|Dv^\varepsilon| \le A$ on Γ_0 (as v^ε is constant on Γ_0 and $Dd \perp \Gamma_0$; similarly $|Dv^\varepsilon| \ge a$ there). Differentiating (AT_ε), each $D_i v^\varepsilon$ satisfies an equation,

and applying the maximum principle gives $|D_i v^\varepsilon| \leq A$ on U_0 for $i \in \{1, \dots, n+1\}$, so $|Dv^\varepsilon| \leq A$ on U_0 (potentially increasing A). Integrating (and increasing A) we get $v^\varepsilon \leq A\,d$ on U_0 (as U_0 is not necessarily convex). Thus $\|v^\varepsilon\|_{C^{0,1}} \leq A < \infty$ independently of ε, with $|v^\varepsilon| \leq A\,d$, so by Arzelà–Ascoli $v^{\varepsilon_i} \to v$ locally uniformly, with v solving (AT) (and satisfying the same estimates).

For uniqueness, let $t \mapsto \Gamma_t$ be the generalised MCF of Γ_0; we show that $\Gamma_t = \{v = t\}$ (which gives the uniqueness of v, since the generalised MCF is unique!). Set $u(t,x) = v(x) - t$ for $x \in U_0$ and $t > 0$; one can check that u is a solution of (LSF) in $(0,\infty) \times U_0$, i.e. $u_t = \left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2}\right) D^2_{ij} u$. Set $\hat{\Gamma}_t = \{x \in U_0 \,|\, v(x) = t\} = \{u(t,\cdot) = 0\}$ and $\hat{u}(t,x) = |u(t,x)| = |v(x) - t|$, and check that $\hat{u}$ is a weak solution of

$$\begin{cases} \hat{u}_t = \left(\delta_{ij} - \frac{D_i \hat{u} D_j \hat{u}}{|D\hat{u}|^2}\right) D^2_{ij} \hat{u} & \text{in } (0,\infty) \times U_0, \\ \hat{u} = t & \text{on } [0,\infty) \times \partial U_0, \\ \hat{u} = v & \text{on } \{0\} \times \overline{U_0}. \end{cases}$$

Choose $g : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\Gamma_0 = \{g = 0\}$, $g \geq 0$, g growing linearly away from Γ_0, and g constant on $\{|x| \gg 1\}$, and let $w \geq 0$ be the corresponding solution of (LSF). So $\Gamma_t = \{w(t,\cdot) = 0\}$, and w is Lipschitz in t by construction, so that $w(t,x) \leq Ct$ for $x \in \Gamma_0$ and $t > 0$, for some constant $C > 0$. Let $\underline{w} = \alpha w$; then for $\alpha > 0$ sufficiently small we get

$$\underline{w} \leq \begin{cases} v & \text{on } \{0\} \times \overline{U_0} \quad (\text{from } |Dv| \geq a > 0 \text{ on } \Gamma_0), \\ t & \text{on } (0,\infty) \times \partial U_0, \end{cases}$$

and (the proof of) the comparison principle, in a boundary version, gives $\underline{w} \leq \hat{u}$ on $[0,\infty) \times \overline{U_0}$. So if $x \in \hat{\Gamma}_t$ then $x \in \Gamma_t$, i.e. $\hat{\Gamma}_t \subset \Gamma_t$ for $t > 0$.

Similarly, by the same reasoning, we define $\overline{w} = \beta w$ for large $\beta > 0$. From $H_{\Gamma_0} > 0$ we get that $\Gamma_t \subset U_0$ for $t \in (0, t_0)$, for some small $t_0 > 0$ (e.g. as Γ_t agrees with the smooth flow), and then the semigroup property for the generalised MCF gives $\Gamma_t \subset U_0$ for all $t > 0$! Hence $w > 0$ on $(0,\infty) \times \partial U_0$, and so for any fixed $T > 0$ we can choose $\beta \gg 0$, depending on T, so that $\overline{w} \geq \hat{u} \geq 0$ on $[0,T] \times U_0$ (check; near $t = 0$ we use smoothness). This gives $\Gamma_t \subset \hat{\Gamma}_t$ for $0 \leq t \leq T$, but T was arbitrary, so this holds for all t. Thus $\Gamma_t = \hat{\Gamma}_t$ for all t; so $\Gamma_t = \{v = t\}$, and v is unique! □

2.10 Regularity of the arrival time

(This lecture is typeset from Hunter's notes.)

Let $\Gamma_0 = \partial U_0$, with $U_0 \subseteq \mathbb{R}^{n+1}$ bounded and open, and Γ_0 strictly mean convex. Last time (Theorem 22) we saw that there is a unique weak solution $v \in C^{0,1}$ of (AT), with uniform estimates, and v can be interpreted as the time of arrival function of $\Gamma_t = \mathcal{M}(t)\Gamma_0$, i.e. $\Gamma_t = \{v = t\}$. Today: how about the regularity of v?

Remark 31. *(Observation) If $Dv(x_0) \neq 0$ and Dv is C^α near x_0, then $v \in C^\infty$ near x_0 (Schauder theory). In general we need to deal with the* ***singular set****, $S(v)$, the complement of the set of such points x_0.*

Example 7. *(Balls) If $U_0 = B_{R_0}(x_0)$ then*

$$v(x) = \frac{R_0^2 - |x - x_0|^2}{2n},$$

a C^∞ function, with $Dv(x) = 0$ if and only if $x = x_0$.

Remark 32. *As long as we have a smooth MCF $t \mapsto \Sigma_t$ for $0 \leq t < T$, we get regularity of v in $\{0 \leq v < T\}$.*

Example 8. *(Convex domains) Let U_0 be a bounded, convex, smooth subset. By Huisken's theorem there is a smooth MCF $t \mapsto \Sigma_t$, $0 \leq t < T^*$, with $\Sigma_0 = \Gamma_0$, where each Σ_t is convex and the flow disappears in a round point at time T^* at x_0. Then $v \in C^\infty(U_0 \setminus \{x_0\}) \cap C^{0,1}(U_0)$; in fact, $v \in C^{2,\alpha_n}(U_0)$ for $n \geq 2$. But for generic U_0 we cannot get C^3.*

Why? Flows near the round sphere have a certain behaviour under rescaled MCF. What can we say about v near x_0?

Theorem 23. *(Strehlke; see also Sesum) Near x_0 (where the flow disappears, at time T^*) the arrival time has an expansion of the form*

$$v(x) = T^* - \frac{|x - x_0|^2}{2n} + |x - x_0|^{\frac{k(k-1)}{n}} P(x - x_0) + O\left(|x - x_0|^{\sigma + k + \frac{k(k-1)}{n}}\right),$$

for some $\sigma > 0$, where n is the dimension of the hypersurface, $k \geq 2$, and P is a homogeneous harmonic polynomial of degree k. Moreover, given any such k and P, one can find a convex domain U_0 for which this expansion occurs (and Sun–Xue showed genericity).

Remark 33. *(Observation) When $n \geq 2$, if $k = 2$ and $P \neq 0$ we get the lack of C^3. When $n = 1$ we get C^3 (but not more in general).*

Idea of proof. Consider the rescaled MCF around x_0 at time T^*,

$$\Psi_s = (T^*-t)^{-\frac{1}{2}}(\Gamma_t - x_0), \quad s = -\log(T^*-t), \quad \text{where} \quad \frac{\partial \vec{x}}{\partial s} = \vec{H}_{\Psi_s} + \frac{\vec{x}^{\perp}}{2}.$$

Then $\Psi_s \to \mathbb{S}^n(\sqrt{2n})$ as $s \to \infty$, so we can write Ψ_s as a graph over $\mathbb{S}^n(\sqrt{2n})$:

$$\Psi_s = \left\{ \vec{x}(p) + \frac{\vec{x}}{|\vec{x}|}(p)\, w(s,p) \,\middle|\, p \in \mathbb{S}^n(\sqrt{2n}) \right\}.$$

For $s \gg 1$,

$$\frac{\partial w}{\partial s} = \underbrace{\Delta_{\mathbb{S}^n(\sqrt{2n})} w + \tfrac{1}{2} w}_{\text{linearisation of } H} + \underbrace{\tfrac{1}{2} w}_{\text{from } \frac{\vec{x}^{\perp}}{2}} + \underbrace{Q(w, Dw)}_{\text{quadratic}},$$

where the first $\frac{1}{2}$ is $|A_{\mathbb{S}^n(\sqrt{2n})}|^2$, and we can 'ignore' the quadratic term Q. So we want to know the spectrum of

$$L = \Delta_{\mathbb{S}^n(\sqrt{2n})} + 1.$$

The eigenvalues of $-L$ are

$$-1 = \mu_0 < -\tfrac{1}{2} = \mu_1 = \cdots = \mu_{n+1} < \mu_{n+2} = \tfrac{n+1}{n} - 1 = \tfrac{1}{n} \leq \cdots,$$

with eigenfunction 1 for μ_0 and the coordinate functions x_i for $\mu_1, \ldots, \mu_{n+1}$. So we get solutions w that are formally linear combinations $\sum_j c_j \varphi_j e^{-\mu_j s}$, whose modes are:

- $e^{-\mu_0 s} = e^s$, which is killed off by time invariance;
- $x_i e^{\frac{1}{2}s}$ for $1 \leq i \leq n+1$, which are killed off by translation invariance;
- $\varphi_{n+2} e^{-\frac{1}{n}s}$, the good modes.

The bad modes get killed by the nature of the convergence to the round sphere, so that $|w| \leq C e^{-\frac{1}{n}s}$ as $s \to \infty$. □

One can formulate Huisken's result in the level set flow setting (but this does not give everything). Evans–Spruck obtained partial results in this direction: if U_0 is convex with solution v, they show directly that:

1. $w = \sqrt{v}$ is concave;
2. $\{v \geq t\}$ is $C^{1,1}$ and convex.

A priori, $\{v = T^*\}$ could be a line segment. Korevaar and Kennington prove power concavity using a two-point maximum principle.

What about the mean convex world?

Example 9. *(Marriage ring) Let $U_0 = T_\varepsilon(\mathbb{S}^1(1)) = \bigcup_{x_0 \in \mathbb{S}^1(1)} B_\varepsilon(x_0)$, an $\mathbb{S}^1$-invariant solid torus. For $\varepsilon > 0$ sufficiently small, $\Gamma_0 = \partial U_0$ is smooth and mean convex, and we get a unique solution of (AT) for this set. Uniqueness implies that the symmetry is preserved, so the singular set is also $\mathbb{S}^1$-invariant, and the flow becomes singular on a circle of radius ≈ 1. This gives an example where v has a singular set of positive dimension.*

Example 10. *(Dumbbells and the critical peanut) For a dumbbell in $\mathbb{R}^3$, the arrival time v has two maxima at the lobes, where the flow has round ($\mathbb{S}^2$) singularities, and a saddle at the neck, where the neck pinches with a cylindrical ($\mathbb{S}^1 \times \mathbb{R}$) singularity. A less pinched 'peanut' instead shrinks to a single round point, so its arrival time has a single maximum (an $\mathbb{S}^2$ singularity). Interpolating between the two, somewhere in between there is the **critical peanut**, whose arrival time has a single maximum at which the singularity is cylindrical ($\mathbb{S}^1 \times \mathbb{R}$); see Figure 2.6.*

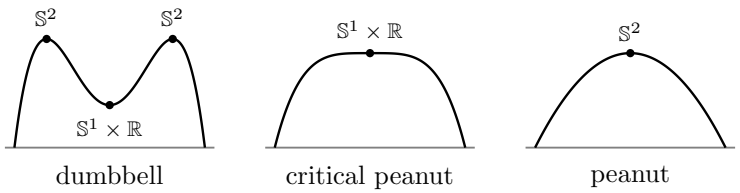


Figure 2.6. The arrival time v along the axis of symmetry (schematic), with the type of singularity at its critical points: a dumbbell, the critical peanut, and a peanut.

We now know the following (from work of Evans–Spruck, Colding–Minicozzi, and others):

Theorem 24. *Let v be the arrival time of a mean convex flow, as above. Then:*

1. $S(v) = \{|Dv| = 0\}$.
2. *v is twice differentiable.*
3. $\dim_{\mathcal{H}} S(v) \leq n - 1$, *and* $\mathcal{H}^{n-1}(S(v)) < \infty$.
4. $v \in C^2$ *if and only if either*

(a) $S(v) = \{v = T^*\}$ *is a unique spherical point, or*

(b) $S(v)$ *is a closed, connected,* k*-dimensional* C^1 *submanifold, for some* $0 \leq k \leq n-1$*, consisting of cylindrical singularities.*

CHAPTER III

ILMANEN'S BIGGEST FLOW

We now restate the Evans–Spruck level set flow approach in manifolds, following Ilmanen [Ilm93]; this does not follow immediately, since the arguments relied on $\mathbb{R}^n$. Throughout this chapter (M, g) is a Riemannian manifold of dimension n.

3.1 Level set flow on manifolds

Fix a Riemannian manifold (M, g) (not necessarily complete), and let $f : M \to \mathbb{R}$ be continuous on M. We consider

$$\begin{cases} \frac{\partial u}{\partial t} - \left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2} \right) \nabla^2_{ij} u = 0 & \text{on } [0,T) \times M, \\ u = f & \text{on } \{t = 0\}. \end{cases} \tag{LSF$_g$}$$

The idea is that $\{u(t, \cdot) = h\}$ evolves by MCF in (M, g) if everything is nice ($|\nabla u| \neq 0$, compact level sets, smooth). We need a weak solution notion for this (the same as for Evans–Spruck):

Definition 6. *(Weak solutions of LSF$_g$) A continuous $u : [0,T) \times M \to \mathbb{R}$ is a **weak subsolution** of (LSF$_g$) if $u(0, \cdot) = f$ and for all $(t_0, x_0) \in (0,T) \times M$, if φ is smooth with $\varphi(t_0, x_0) = u(t_0, x_0)$ and $\varphi \geq u$ near (t_0, x_0), then either*

$$\frac{\partial \varphi}{\partial t} - \left(g^{ij} - \frac{\nabla^i \varphi \nabla^j \varphi}{|\nabla \varphi|^2} \right) \nabla^2_{ij} \varphi \leq 0 \quad \text{at } (t_0, x_0),$$

if $\nabla \varphi(t_0, x_0) \neq 0$, or

$$\frac{\partial \varphi}{\partial t} - (g^{ij} - \xi^i \xi^j) \nabla^2_{ij} \varphi \leq 0 \quad \text{at } (t_0, x_0),$$

for some 1-form ξ with $|\xi| \leq 1$, if $\nabla \varphi(t_0, x_0) = 0$. Weak supersolutions and weak solutions are defined analogously. (Note: we do not *ask u to be bounded!)*

Definition 7. *(Level set flows) We say that a family of closed subsets $t \mapsto \Gamma_t \subset M$, for $t \in [0,T)$, is a **level set flow** (LSF) if there is a weak solution u of (LSF$_g$) with $\{u(t,\cdot) = 0\} = \Gamma_t$.*

This differs from Evans–Spruck: we are *not* claiming that Γ_t is determined by Γ_0!

Ilmanen shows that Evans–Spruck largely carries over to the manifold setting, but he has to deal with the lack of the affine structure of $\mathbb{R}^n$ (which needs other proof ideas). In particular we get the following, where the proofs of the first two follow Evans–Spruck verbatim:

Theorem 25. *(Uniform convergence) If (u_k) is a sequence of weak solutions of (LSF$_g$) with $u_k \to u_\infty$ in $C^0_{loc}([0,T) \times M)$, then u_∞ is also a weak solution of (LSF$_g$).*

Theorem 26. *(Relabelling) If u is a weak solution of (LSF$_g$) and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then $\varphi \circ u$ is a weak solution of (LSF$_g$).*

Theorem 27. *(Comparison) If u and v solve (LSF$_g$) and satisfy*

1. *$u(0,\cdot) \le v(0,\cdot)$ on M, and*
2. *$\limsup_{(t,x)\to\infty}(u - v) \le 0$,*

then $u \le v$ on $[0,T) \times M$.

Here $\limsup_{(t,x)\to\infty} f \le 0$ is automatically satisfied if M is compact; otherwise it refers to (t,x) leaving compact subsets of $[0, T-\varepsilon] \times M$ (this was built into the Evans–Spruck definition!).

Proof. This is the bulk of Ilmanen's paper; one needs to modify the arguments for the manifold setting. □

For the existence of weak solutions of (LSF$_g$) we need assumptions on (M,g). We again regularise and consider, for f smooth,

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \left(g^{ij} - \frac{\nabla^i u^\varepsilon \nabla^j u^\varepsilon}{\varepsilon^2 + |\nabla u^\varepsilon|^2} \right) \nabla^2_{ij} u^\varepsilon = 0 & \text{on } (0,T) \times M, \\ u^\varepsilon = f & \text{on } \{t = 0\}. \end{cases} \tag{LSF$_{g,\varepsilon}$}$$

Proposition 1. *When (M,g) is compact there is a unique smooth solution u^ε of (LSF$_{g,\varepsilon}$), with the estimates*

1. *$\|u^\varepsilon(t,\cdot)\|_{L^\infty} \le \|f\|_{L^\infty}$;*
2. *$\left\|\frac{\partial u^\varepsilon}{\partial t}(t,\cdot)\right\|_{L^\infty} \le \left\|\frac{\partial u^\varepsilon}{\partial t}(0,\cdot)\right\|_{L^\infty} \le C\|\nabla^2 f\|_{L^\infty}$;*

3. $\|\nabla u^\varepsilon(t,\cdot)\|_{L^\infty} \leq e^{-\lambda t}\|\nabla f\|_{L^\infty}$, *if* $\mathrm{Ric} \geq \lambda > -\infty$ *for a constant* λ

(the Ricci term comes from the Bochner formula, i.e. from commuting derivatives).

We then get:

Theorem 28. *(Existence and uniqueness in compact manifolds) Let* (M,g) *be compact. Then for any continuous* $f : M \to \mathbb{R}$ *there is a unique weak solution of (LSF*$_g$*) with initial data* f*.*

Proof. Use the above proposition as in Evans–Spruck for existence, and comparison for uniqueness (in a compact manifold, comparison holds without the limsup condition). □

Theorem 29. *(Level set flows of closed subsets of compact manifolds) For* (M,g) *compact and* $\Gamma_0 \subset M$ *closed (hence compact), there is a unique* $t \mapsto \Gamma_t$*, for all* $t \geq 0$*, which is a level set flow; i.e.* Γ_t *is the zero level set of some weak solution of (LSF*$_g$*), and it is uniquely determined by* $\{f = 0\}$ *(so any other solution with* $\{\hat{f} = 0\} = \{f = 0\}$ *has the same zero sets).*

Corollary 1. *If* $\Gamma_0 \subset \hat{\Gamma}_0 \subset M$ *are closed, then* $\Gamma_t \subset \hat{\Gamma}_t$ *for all* $t \geq 0$*.*

Remark 34. *There are level set flows with* $\Gamma_t \supseteq U \neq \emptyset$ *for all* $t \geq 0$ *(e.g. closed minimal surfaces!). (Ilmanen shows that the level set flow of any non-separating hypersurface never vanishes!)*

3.2 Non-compact manifolds

What if M is not compact?

Theorem 30. *(Short time existence and uniqueness) Let* (M,g) *be any manifold and* $\Gamma_0 \subset M$ *compact. Then:*

1. *There are* $T > 0$ *(small) and* $t \mapsto \Gamma_t$ *compact, for* $t \in [0,T)$*, satisfying (LSF*$_g$*) (there is no uniqueness!).*
2. *If* $\bigcup_{t\in[0,T)} \Gamma_t$ *is precompact (i.e. its closure is compact), then* $t \mapsto \Gamma_t$ *is unique with respect to all other solutions of (LSF*$_g$*) for which this holds.*
3. *If* $\bigcup_{t\in[0,T)} \Gamma_t$ *is precompact, then* $t \mapsto \Gamma_t$ *extends to* $[0, T+\varepsilon)$*.*

The conclusion of (3) fails in $\mathbb{R}^n \setminus B_1(0)$ for shrinking spheres (which have a finite existence time). We prove this theorem, and the next two, in Section 3.4.

Theorem 31. *(Avoidance) Let (M, g) be any manifold and $t \mapsto \Gamma_t$, $t \mapsto \hat{\Gamma}_t$ be solutions of (LSF$_g$) on $[0, T)$. Then $\Gamma_0 \cap \hat{\Gamma}_0 = \emptyset$ implies $\Gamma_t \cap \hat{\Gamma}_t = \emptyset$ for all $t \in [0, T)$, provided we know that*

$$\Gamma_t \cap \hat{\Gamma}_t \subset K \subset M \quad \text{for a fixed compact set } K. \tag{$\star$}$$

Remark 35. *When (M, g) is complete with $\mathrm{Ric} \geq \lambda > -\infty$ and Γ_0 is compact, $(\star)$ is automatic.*

Theorem 32. *(Long time existence) If (M, g) is complete with $\mathrm{Ric} \geq \lambda > -\infty$, then for compact $\Gamma_0 \subset M$ there is a unique solution $t \mapsto \Gamma_t$ of (LSF$_g$) for all $t \geq 0$, with Γ_t compact for all $t \geq 0$.*

What if Γ_0 is closed but not compact (in 'good' manifolds)? The observation is that (LSF$_g$) is *not* unique: $\{f = 0\} = \{\hat{f} = 0\}$ does *not* imply that $\Gamma_t = \{u = 0\}$ and $\hat{\Gamma}_t = \{\hat{u} = 0\}$ agree.

Example 11. *We saw this before (Figure 1.6)! Let $h^\beta(x) = (1 + |x|)^{-1-\beta}$ and $\Gamma_0^\beta = \{(x, \pm h^\beta(x)) \,|\, x \in \mathbb{R}\} \subset \mathbb{R}^2$. The flow Γ^{in}, approximated from inside by flows of compact sets, and the graphical MCF Γ^{out} of $\pm h^\beta$ (which is non-compact) satisfy $\Gamma_0^{\mathrm{in}} = \Gamma_0^{\mathrm{out}} = \Gamma_0^\beta$, but their level set flows differ!*

Definition 8. *(Biggest flow) Let (M, g) be a complete manifold with $\mathrm{Ric} \geq \lambda > -\infty$. A family of closed sets is a **biggest flow** if $\Gamma_t = \{u(t, \cdot) = 0\}$, where u satisfies (LSF$_g$) and $\lim_{(t,x)\to\infty} u(t, x) = 0$ (i.e. the sets $\{u(t, \cdot) = h\}$ are compact for $h \neq 0$).*

Theorem 33. *All level set flows of Γ_0 satisfy $\hat{\Gamma}_t \subset \Gamma_t$, where Γ_t is a biggest flow; hence the biggest flow is unique, and it exists.*

We prove this in Section 3.5.

3.3 The finite speed property

We claim that we have a "finite speed property", i.e. solutions cannot reach a fixed point 'too fast'. Fix $x_0 \in M$ and $r_0 > 0$, and let $B(r_0) = B_{r_0}(x_0)$ be a geodesic ball (implicitly r_0 is chosen so that $\overline{B_{r_0}(x_0)}$ is complete!). The Bishop comparison theorem tells us that if we suppose that $\mathrm{Ric} \geq -(n-1)\kappa g$ for some $\kappa > 0$ on $B_{r_0}(x_0)$ (κ depending on r_0 and x_0), then

$$\partial_{reg} B(r) = \partial B(r) \setminus \{\text{cut points of } x_0\}, \quad 0 < r < r_0,$$

is such that, for $y \in \partial_{reg} B(r)$,

$$H(y) \leq (n-1)\sqrt{\kappa} \coth(\sqrt{\kappa}\, r),$$

where the right hand side is the mean curvature of $\partial B_r^{\mathbb{H}(\kappa)}$, a sphere in the hyperbolic space $\mathbb{H}(\kappa)$ of constant curvature $-\kappa$. Let

$$q(t) = \frac{\cosh^{-1}(e^{(n-1)\kappa t})}{\sqrt{\kappa}}, \quad \text{so that } q(0) = 0,$$

and $r(t) = q(q^{-1}(r_0) + t_0 - t)$ for $t \le t_0 + q^{-1}(r_0)$, for a fixed $t_0 \in [0, T)$. Then

$$\frac{dr}{dt}(t) = -(n-1)\sqrt{\kappa}\coth(\sqrt{\kappa}\, r(t)) \le - \sup_{\partial_{reg} B(r(t))} H,$$

so $t \mapsto \partial B(r(t))$ (away from cut points) is moving 'faster' than MCF. Let

$$u(t, y) = q^{-1}(d(y)) - q^{-1}(r_0) + t - t_0, \quad \text{where } d(y) = \mathrm{dist}_g(y, x_0);$$

then $u(t_0, \cdot) < 0$ on the interior of $B(r_0)$ (indeed $\{u(t, \cdot) < 0\} = B(r(t))$), and by construction u is a weak supersolution of (LSF_g) away from the cut locus (check).

At cut points this is also OK: since u is continuous, it is a global weak supersolution. Indeed, let y be a cut point, with distinct minimising geodesics γ and β from y to x_0, and let $w \ne v$ be the unit vectors in $T_y M$ (in the x_0 direction) along γ and β respectively. Then $D_w d = -1$ (the directional derivative), but since y is a cut point, $D_{-w} d < 1$, by considering a point z on β and the triangle inequality: with $\hat{d} = \mathrm{dist}(\cdot, z)$ we have $d(\gamma(-t)) \le \hat{d}(\gamma(-t)) + d(z, x_0)$ for small t, with equality at $t = 0$, and differentiating gives

$$D_{-w} d \le D_{-w} \hat{d} = (-w) \cdot (-v) < 1.$$

But then $D_w u(t, y) = -(q^{-1})'(d(y)) \ne 0$, while $D_{-w} u(t, y) < (q^{-1})'(d(y))$, so u has a (concave) corner at y; if φ is smooth with $\varphi \le u$, it *cannot* touch u at y! So u is trivially a supersolution at cut points.

Lemma 3. *(Finite speed) If $r_0 > 0$ and $\mathrm{Ric} \ge -(n-1)\kappa g$, with $\kappa > 0$, on $B(r_0)$, then there are $\tau > 0$, depending on κ, and $r(t)$, with $r(t_0) = r_0$ and $r(t) > 0$ on $[t_0, t_0 + \tau]$, such that*

$$B(r_0) \cap \Gamma_{t_0} = \emptyset \implies B(r(t)) \cap \Gamma_t = \emptyset \quad \text{for } t \in (t_0, t_0 + \tau].$$

Proof. Use the u we built as a barrier, together with the comparison principle! □

Example 12. *Consider $\mathbb{R}^2$ with $g = dr^2 + r^2 e^{r^4/2}\, d\theta^2$. Its Gauss curvature is $K = -(5r^2 + r^6) < 0$, so it is complete and Cartan–Hadamard, with* no *lower bound on its curvature. Then $H_{\partial B_r} = \frac{1}{r} + r^3$, and one can check that $t \mapsto \partial B_{\sqrt{\tan(-2t)}}$ solves classical MCF for $t \in (-\frac{\pi}{4}, 0)$ (Figure 3.1), while for $t \le -\frac{\pi}{4}$ we can have $\emptyset$! (This doesn't violate avoidance!)*

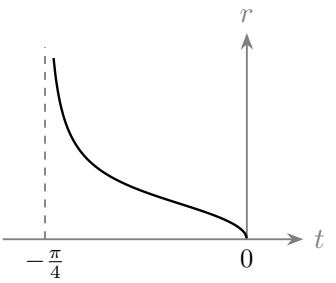


Figure 3.1. The radius $r(t) = \sqrt{\tan(-2t)}$ of circles moving by MCF in $(\mathbb{R}^2, dr^2 + r^2 e^{r^4/2} d\theta^2)$: they come in from infinity at $t = -\frac{\pi}{4}$.

So (2) in the short time existence theorem needs uniqueness in the class with $\bigcup_{t\in[0,T)} \Gamma_t$ precompact. From the level set flow point of view, the zero set is generated 'near ∞' at $t > 0$.

3.4 Proofs of the existence theorems

Lemma 4. *(Finite speed, restated) If $r_0 > 0$ and $B_{r_0}(p_0)$ is a complete geodesic ball with* $\mathrm{Ric} \geq -(n-1)\kappa g$ *for some $\kappa \geq 0$, then there are $\tau > 0$, depending on κ, and a continuous decreasing $r(t)$, with $r(t_0) = r_0$ and $r(t) > 0$ on $[t_0, t_0 + \tau]$, such that*

$$B_{r_0}(p_0) \cap \Gamma_{t_0} = \emptyset \implies B_{r(t)}(p_0) \cap \Gamma_t = \emptyset \quad \textit{for all } t \in [t_0, t_0 + \tau].$$

One can use this with the comparison principle to prove the previously stated avoidance principle (for $\Gamma_t \cap \hat{\Gamma}_t \subset K \subset M$ compact). The idea is to pick u and $\hat{u}$, with $\Gamma_t = \{u = 0\}$ and $\hat{\Gamma}_t = \{\hat{u} = 0\}$, and $u \leq \hat{u}$, so that $u < \hat{u}$ on $(0, \infty) \times \partial K$. Then $\Gamma_0 \cap \hat{\Gamma}_0 = \emptyset$ gives $u < \hat{u}$ on $\{0\} \times M$, and comparison gives $u < \hat{u}$ on $(0, T) \times K$, so $u < \hat{u}$ on $[0, T) \times M$ (as $\Gamma_t \cap \hat{\Gamma}_t \subset K$); hence $\Gamma_t \cap \hat{\Gamma}_t = \emptyset$.

We now prove the short time existence and uniqueness theorem.

Proof of Theorem 30. For (1), existence: pick a precompact open set $V \subset M$ with $\Gamma_0 \subset V$, and a continuous f with $f^{-1}(0) = \Gamma_0$ and $f \equiv 1$ on $M \setminus V$. We can then embed V into some compact manifold $\hat{M}$, and let $\hat{f}$ on $\hat{M}$ be continuous, equal to f on V and $\equiv 1$ otherwise. Existence on compact manifolds then gives a level set flow on $\hat{M}$ starting from $\hat{f}$, i.e. $\hat{u} : [0, \infty) \times \hat{M} \to \mathbb{R}$ with $\hat{\Gamma}_t = \{\hat{u}(t, \cdot) = 0\}$. Then $\hat{\Gamma}_0 \subset V$, and the finite speed lemma gives $T > 0$ such that $\hat{\Gamma}_t \subset V$ for all $t \in [0, T)$. Up to shrinking T, the finite speed lemma gives $\hat{u} = 1$ on a neighbourhood of

$\hat{M} \setminus V$ for $t \in [0,T)$, so we can transplant $\hat{u}$ back to $V \subset M$ and extend it by 1, to get a solution of (LSF_g) on M with $\{u(t,\cdot) = 0\} = \Gamma_t$ for $t \in [0,T)$.

For (2), uniqueness: if $t \mapsto \Gamma_t, \hat{\Gamma}_t$ are level set flows with $\Gamma_0 = \hat{\Gamma}_0$, and both $\bigcup_{t\in[0,T)} \Gamma_t$ and $\bigcup_{t\in[0,T)} \hat{\Gamma}_t$ are precompact, let V be an open precompact set in M containing both unions. Embedding V into a compact $\hat{M}$ again, and using uniqueness there, gives the result. (May work for connected flows.)

For (3), extending flows ('running off the edge'): if $A = \bigcup_{t\in[0,T)} \Gamma_t$ is precompact and $V \subset M$ is open and precompact with $A \subset V$, then (1) shows that we can extend Γ_t for some time depending on $\mathrm{dist}(\overline{A}, M\setminus V) > 0$ and the lower Ricci bound on $\overline{V}$. □

We then have the long time existence theorem (Theorem 32) in the following form:

Theorem 34. *If (M,g) is complete with $\mathrm{Ric} \geq -\kappa g$, $\kappa \geq 0$, then for any compact $\Gamma_0 \subset M$ there is a unique level set flow $t \mapsto \Gamma_t$ for all $t \geq 0$, with each Γ_t compact.*

Proof. The finite speed lemma holds on uniformly sized balls (which are all complete), i.e. with a fixed radius about any point. Thus all level set flows of Γ_0 are inside a compact subset of M for $t \in [0,T]$, and so we get uniqueness from (2); we then extend by a uniform amount of time (depending on the geometry), since M is complete! □

3.5 The biggest flow

Definition 9. *(Biggest flow, restated) Let (M,g) be complete with $\mathrm{Ric} \geq -\kappa g$, $\kappa \geq 0$, and let $\Gamma_0 \subset M$ be closed (it need not be compact). The **biggest flow** of Γ_0 is $t \mapsto \Gamma_t$, $t \in [0,T)$, where $\Gamma_t = \{u(t,\cdot) = 0\}$ with u solving (LSF_g) and $\lim_{(t,x)\to\infty} u(t,x) = 0$ (equivalently, $\{u(t,\cdot) = h\}$ is compact for $h \neq 0$).*

Theorem 35. *(Existence and uniqueness of biggest flows) For such (M,g) and Γ_0, the biggest flow exists, is unique, and contains all other level set flows of Γ_0.*

Proof. We first show uniqueness. If $t \mapsto \Gamma_t$ is a biggest flow and $t \mapsto \hat{\Gamma}_t$ is a level set flow of Γ_0, then we will show that $\hat{\Gamma}_t \subset \Gamma_t$ for all t, which

gives uniqueness and containment. Let $\Gamma_t^h = \{u(t,\cdot) = h\}$ for $h \neq 0$; then $t \mapsto \Gamma_t^h$ is a level set flow of compact sets, and so

$$\Gamma_0^h \cap \hat{\Gamma}_0 = \emptyset \implies \Gamma_t^h \cap \hat{\Gamma}_t = \emptyset \text{ for all } t.$$

So $u \neq h$ on $\hat{\Gamma}_t$ for any $h \neq 0$, i.e. $u = 0$ on $\hat{\Gamma}_t$, and so $\hat{\Gamma}_t \subset \{u(t,\cdot) = 0\} = \Gamma_t$!

For existence, pick a continuous f with $f^{-1}(0) = \Gamma_0$ and $f^{-1}(z)$ compact for all $z \neq 0$ (check that this can be done!). Pick $\delta > 0$ and set

$$f^\delta = \begin{cases} f - \delta & \text{if } f \geq \delta, \\ 0 & \text{if } |f| \leq \delta, \\ f + \delta & \text{if } f \leq -\delta; \end{cases}$$

then $M \setminus (f^\delta)^{-1}(0)$ is precompact by construction (since f^δ has compact support). The finite speed lemma and the existence and uniqueness theorems then give u^δ solving (LSF$_g$) on $[0,\infty) \times M$ with initial data f^δ, and one can check that $\{u^\delta = h\}$ is determined by $\{f = h + \delta\}$ when $h \geq \delta$, and by $\{f = h - \delta\}$ when $h \leq -\delta$, which are compact sets! Since $|f^\delta - f^{\delta'}| \leq |\delta - \delta'|$ everywhere, comparison with $u^{\delta'} \pm |\delta - \delta'|$ gives $|u^\delta - u^{\delta'}| \leq |\delta - \delta'|$. Sending $\delta \to 0$, we get $u^\delta \to u^0$ in C^0_{loc}, where u^0 solves (LSF$_g$), with $\{u^0(t,\cdot) = h\}$ the flow of the compact set $\{f = h\}$, which therefore remains compact for all t! $\square$

CHAPTER IV

WEAK SET FLOWS

We now define weak flows purely through the avoidance principle, as set-theoretic subsolutions of MCF (following Ilmanen [Ilm92]), and show that the maximal such flow is the biggest flow of the previous chapter.

4.1 Set-theoretic solutions of MCF

We do this in $\mathbb{R}^{n+1}$ (but it makes sense in a complete manifold (M, g) with $\mathrm{Ric} \geq -\kappa(n+1)g$ for some $\kappa \geq 0$).

Definition 10. *(Weak set flows) Let $t \in [0, \infty) \mapsto \Gamma_t \subset \mathbb{R}^{n+1}$ be a family of closed sets ($\Gamma_t = \emptyset$ is allowed). This is a **set-theoretic subsolution** of MCF, or a **weak set flow** (WSF), if for any MCF of compact sets $t \mapsto \Sigma_t$, $t \in [t_0, t_1]$ for $0 \leq t_0 \leq t_1 < \infty$,*

$$\Sigma_{t_0} \cap \Gamma_{t_0} = \emptyset \implies \Sigma_t \cap \Gamma_t = \emptyset \text{ for all } t \in [t_0, t_1]; \tag{AP}$$

i.e. if it satisfies avoidance with respect to compact smooth MCFs.

Equivalently, we may suppose that quantitative avoidance holds, i.e.

$$\mathrm{dist}(\Gamma_t, \Sigma_t) \geq \mathrm{dist}(\Gamma_{t_0}, \Sigma_{t_0}) \quad \text{for } t \in [t_0, t_1]. \tag{QAP}$$

That (QAP) implies (AP) is immediate, as $\mathrm{dist}(\Gamma_{t_0}, \Sigma_{t_0}) > 0$ if $\Sigma_{t_0} \cap \Gamma_{t_0} = \emptyset$. For (AP) implies (QAP) one can use the translation invariance of smooth MCF: if (QAP) failed at some $t' \geq t_0$, i.e. $\mathrm{dist}(\Gamma_{t'}, \Sigma_{t'}) < \mathrm{dist}(\Gamma_{t_0}, \Sigma_{t_0})$, then we translate Σ so that $\Sigma_{t'}$ touches $\Gamma_{t'}$; but then $\Sigma_{t_0} \cap \Gamma_{t_0} = \emptyset$ while $\Sigma_{t'} \cap \Gamma_{t'} \neq \emptyset$, so (AP) fails also.

Remark 36. *If $\Gamma_{t_0} = \emptyset$ and $t \mapsto \Gamma_t$ is a weak set flow, then $\Gamma_t = \emptyset$ for all $t \geq t_0$ (we can always use a sphere!); this shows why one needs some assumptions in the manifold setting.*

4.2 Properties of weak set flows

1. **Stability under closures of unions.** Let $t \mapsto \Gamma_t^i$ be a family of weak set flows and set $\Gamma_t = \overline{\bigcup_i \Gamma_t^i}$; then $t \mapsto \Gamma_t$ is also a weak set flow.

Proof. We use (QAP). Let $t \mapsto \Sigma_t$, for $t \in [t_0, t_1]$, be a smooth MCF of compact sets. If $\operatorname{dist}(\Gamma_{t_0}, \Sigma_{t_0}) \geq \delta > 0$, then $\operatorname{dist}(\Gamma^i_{t_0}, \Sigma_{t_0}) \geq \delta > 0$ for each i; but then (QAP) gives $\operatorname{dist}(\Gamma^i_t, \Sigma_t) \geq \delta > 0$ for all $t \in [t_0, t_1]$, and so $\operatorname{dist}(\Gamma_t, \Sigma_t) \geq \delta$ for such t also! □

Example 13. *Let $\Gamma^i_0 \subset \mathbb{R}^2$ be two circles of radius $1 - \frac{1}{i}$, centred at ± 1 (so at distance $\frac{2}{i}$ apart), whose space-time tracks are a pair of (round) cones. Then $\Gamma_0 = \overline{\bigcup_i \Gamma^i_0}$ is a closed set accumulating on the two unit circles centred at ± 1, which touch at the origin, and the space-time track of Γ_t fills out the space-time track of the level set flow from Γ_0. So we expect that Γ_t is contained in the level set flow of Γ_0.*

Definition 11. *(Maximal flow) For any closed $\Gamma_0 \subset \mathbb{R}^{n+1}$ there is a unique **maximal flow** $t \in [0, \infty) \mapsto \mathcal{M}_t$, where $\mathcal{M}_t$ is the closure of the union of all weak set flows $t \mapsto \hat{\Gamma}_t$ with $\hat{\Gamma}_0 \subset \Gamma_0$ (so $\mathcal{M}_0 = \Gamma_0$).*

2. **Inclusion property.** If $t \mapsto \Gamma_t$ is a weak set flow and $t \mapsto \mathcal{M}_t$ is the maximal flow of $\mathcal{M}_0$, then $\Gamma_0 \subset \mathcal{M}_0$ implies $\Gamma_t \subset \mathcal{M}_t$ for $t \geq 0$.

The avoidance principle for the level set flow/biggest flow tells us that if $t \mapsto B_t$, for $t \geq 0$, is the biggest flow of $\Gamma_0 \subset \mathbb{R}^{n+1}$ (so $B_0 = \Gamma_0$), then $t \mapsto B_t$ is a weak set flow (by the avoidance principle for LSF), and so $B_t \subset \mathcal{M}_t$, for $\mathcal{M}_t$ the maximal flow of Γ_0. In fact we will see that $\mathcal{M}_t \subset B_t$!

Theorem 36. *(Avoidance principle for weak set flows) If $t \mapsto \Gamma_t$ and $t \mapsto \hat{\Gamma}_t$ are weak set flows with $\Gamma_0 \cap \hat{\Gamma}_0 = \emptyset$, and one of Γ_0, $\hat{\Gamma}_0$ is compact, then $\Gamma_t \cap \hat{\Gamma}_t = \emptyset$ for $t \geq 0$.*

Corollary 2. *$B_t = \mathcal{M}_t$ for all $t \geq 0$, i.e. the maximal flow* is *the biggest flow (with $B_0 = \mathcal{M}_0 = \Gamma_0$).*

Proof. Let u be the solution of (LSF) with $B_t = \{u(t, \cdot) = 0\}$ and $\Gamma^h_t = \{u(t, \cdot) = h\}$ compact for $h \neq 0$ (by the definition of the biggest flow). Then $t \mapsto \Gamma^h_t$, for $t \geq 0$, are weak set flows with Γ^h_0 compact, so since $\Gamma^h_0 \cap \Gamma_0 = \emptyset$ the theorem gives $\Gamma^h_t \cap \mathcal{M}_t = \emptyset$ for all $t \geq 0$. That is, $u(t, p) \neq h$ for $h \neq 0$ and $p \in \mathcal{M}_t$, so $u(t, p) = 0$ for $p \in \mathcal{M}_t$, and thus $\mathcal{M}_t \subset B_t$. □

4.3 The avoidance principle for weak set flows

The idea of the proof of Theorem 36: we need smooth flows for the definitions, but we aren't given any. Ilmanen introduces a lemma:

Lemma 5. *($C^{1,1}$ interposition) Given disjoint closed sets $X, Y \subset \mathbb{R}^{n+1}$ with X compact, there are a $C^{1,1}$ hypersurface Q and a bounded open set $U \subset \mathbb{R}^{n+1}$ with:*

1. $X \subset U$, $Q = \partial U$ *and* $Y \subset \mathbb{R}^{n+1} \setminus U$;
2. $\operatorname{dist}(X, Q) + \operatorname{dist}(Y, Q) = \operatorname{dist}(X, Y)$.

For example, if X and Y are nested squares then Q is a rounded square between them, which is $C^{1,1}$ but not C^2. (we will prove this for Q being C^1 later.) Let's use this to prove the theorem.

Proof of Theorem 36. Say Γ_0 is compact. As $t \mapsto \Gamma_t$ and $t \mapsto \hat{\Gamma}_t$ avoid shrinking balls,

$$\Gamma_s \subset \{x \,|\, \operatorname{dist}(x, \Gamma_t) \leq (2n(s-t))^{\frac{1}{2}}\} \quad \text{for } s \geq t \geq 0$$

(so Γ_t is compact for all $t \geq 0$), and similarly we have the inclusion (but not compactness) for $\hat{\Gamma}_s$. Thus

$$\operatorname{dist}(\Gamma_s, \hat{\Gamma}_s) \geq \operatorname{dist}(\Gamma_t, \hat{\Gamma}_t) - 2(2n(s-t))^{\frac{1}{2}}.$$

If $\operatorname{dist}(\Gamma_{t_0}, \hat{\Gamma}_{t_0}) > 0$, then the $C^{1,1}$ interposition lemma gives a $C^{1,1}$ hypersurface Q with

$$\operatorname{dist}(\Gamma_{t_0}, Q) + \operatorname{dist}(Q, \hat{\Gamma}_{t_0}) = \operatorname{dist}(\Gamma_{t_0}, \hat{\Gamma}_{t_0}) > 0.$$

The idea is to flow Q by classical MCF and use it as a barrier. There are two issues:

- a technical point: Q is only $C^{1,1}$, so the flow is not smooth at $t = t_0$;
- the flow only exists for some finite time, but we need to ensure that it lasts long enough to get information.

We mollify Q to get smooth Q^ε with

$$\operatorname{dist}(\Gamma_{t_0}, Q^\varepsilon) + \operatorname{dist}(Q^\varepsilon, \hat{\Gamma}_{t_0}) > \operatorname{dist}(\Gamma_{t_0}, \hat{\Gamma}_{t_0}) - \varepsilon$$

and $\sup_{Q^\varepsilon} |A|^2 \leq C(t_0) < \infty$, independently of ε (as the normal to Q is Lipschitz). Short time existence of MCF for Q^ε gives a smooth MCF $t \mapsto Q^\varepsilon_t$ for $t \in [t_0, t_0 + T(t_0, \varepsilon)]$, but the control of the second fundamental form shows that T is independent of ε (or at least has a lower bound independent of ε)! We also get U^ε_t with $\partial U^\varepsilon_t = Q^\varepsilon_t$ and $\Gamma_{t_0} \subset U^\varepsilon_{t_0}$, so the definition of a weak set flow gives

$$\Gamma_t \subset U^\varepsilon_t \quad \text{and} \quad \hat{\Gamma}_t \subset \mathbb{R}^{n+1} \setminus U^\varepsilon_t \quad \text{for all } t \in [t_0, t_0 + T(t_0)],$$

and so $\operatorname{dist}(\Gamma_t, \hat{\Gamma}_t) \geq \operatorname{dist}(\Gamma_t, Q^\varepsilon_t) + \operatorname{dist}(Q^\varepsilon_t, \hat{\Gamma}_t)$. Thus, by (QAP),

$$\begin{aligned} \operatorname{dist}(\Gamma_t, \hat{\Gamma}_t) &\geq \operatorname{dist}(\Gamma_{t_0}, Q^\varepsilon_{t_0}) + \operatorname{dist}(Q^\varepsilon_{t_0}, \hat{\Gamma}_{t_0}) \\ &\geq \operatorname{dist}(\Gamma_{t_0}, \hat{\Gamma}_{t_0}) - \varepsilon \quad \text{for all } t \in [t_0, t_0 + T(t_0)]. \end{aligned}$$

Sending $\varepsilon \searrow 0$ gives $\operatorname{dist}(\Gamma_t, \hat{\Gamma}_t) \geq \operatorname{dist}(\Gamma_{t_0}, \hat{\Gamma}_{t_0})$, and then iterating (with a clopen argument) gives $\operatorname{dist}(\Gamma_t, \hat{\Gamma}_t) \geq \operatorname{dist}(\Gamma_0, \hat{\Gamma}_0)$ for all $t \geq 0$. □

Remark 37. *Hershkovits–White observed that C^1 regularity is enough for a uniform existence time (so Q being C^1 is enough!). Why? On a small scale any C^1 hypersurface has almost Euclidean area; then Huisken's monotonicity shows that this property is preserved under the MCF from smoothings of the C^1 hypersurface, which gives a uniform existence time by Brakke regularity (see Section 4.5).*

4.4 C^1 interposition

We will prove a C^1 interposition result (following Hershkovits–White).

Theorem 37. *(C^1 interposition) Let $X, Y \subset \mathbb{R}^{n+1}$ be closed, with $r = \frac{\operatorname{dist}(X,Y)}{2} > 0$ and $\{p \,|\, \operatorname{dist}(p, X) = r\}$ compact (e.g. if X is compact). Then there is a compact C^1 hypersurface M separating X and Y with $\operatorname{dist}(X, M) = \operatorname{dist}(Y, M) = r$ (Figure 4.1).*

This should also hold in a manifold, with appropriate assumptions.

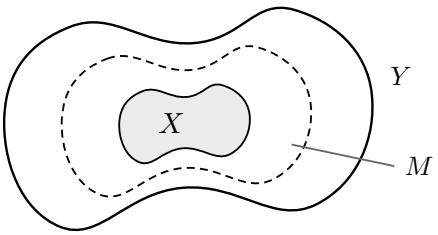


Figure 4.1. C^1 interposition (schematic): a compact C^1 hypersurface M separating X from Y, at distance $r = \frac{1}{2}\operatorname{dist}(X, Y)$ from both.

Remark 38. *Ilmanen showed that it holds with M being $C^{1,1}$.*

Proof. There is $\delta > 0$ such that $\{p \,|\, r - \delta \leq \operatorname{dist}(p, X) \leq r + \delta\}$ is compact; let

$$X' = \{p \,|\, \operatorname{dist}(p, X) \leq r - \delta\} \quad \text{and} \quad Y' = \{p \,|\, \operatorname{dist}(p, X) \geq r + \delta\}.$$

Then set

$$A = \{p \,|\, \operatorname{dist}(p, X) \leq r\} = \{p \,|\, \operatorname{dist}(p, X') \leq \delta\},$$
$$B = \{p \,|\, \operatorname{dist}(p, Y') \leq \delta\},$$

$Z = A \cap B \neq \emptyset$, and $U = \mathbb{R}^{n+1} \setminus (A \cup B)$. The claim is that Z is a $C^{1,1}$ hypersurface, never closer than r to the two sets that it separates (so if $U = \emptyset$, then Z is what we want!).

Pick $z \in Z$, and let C_1^z and C_2^z be line segments of shortest length joining z to X' and Y' respectively. We claim that $C_1^z \cup C_2^z$ has the shortest length between X' and Y', so that C_1^z and C_2^z are parallel. Indeed, since $\operatorname{dist}(z, X'), \operatorname{dist}(z, Y') \leq \delta$ by the definitions of A and B,

$$2\delta \geq \operatorname{dist}(z, X') + \operatorname{dist}(z, Y') \geq \operatorname{dist}(X', Y') \geq 2\delta,$$

so $\operatorname{dist}(X', Y') = \operatorname{dist}(z, X') + \operatorname{dist}(z, Y')$, and the claim holds. This also shows that $z \mapsto C_i^z$ is unique (otherwise there would be a non-smooth minimising path between X' and Y'!) and continuous. Thus $z \mapsto \nu_z$, the unit vector parallel to C_1^z (pointing away from X'), is continuous, and ν_z is the normal to Z at z, so Z is C^1 (the proof even shows that Z is locally $C^{1,1}$). If $U = \emptyset$ we are done, since Z has the distance property!

If $U \neq \emptyset$, then U is open and precompact; let $h : U \to \mathbb{R}$ minimise the Dirichlet energy $\int_U |\nabla h|^2$ with

$$h = -1 \text{ on } \partial A \setminus B \quad \text{and} \quad h = +1 \text{ on } \partial B \setminus A$$

(or use Perron's method). Assuming such a minimiser exists, h is harmonic on U ($\Delta h = 0$), and thus smooth. Moreover, h is continuous on $\overline{U} \setminus Z$: if $p \in \partial U \setminus Z$ and $q \in X' \cup Y'$ is closest to p, then $B_\delta(q) \subset U^c$ and $p \in \partial B_\delta(q)$, so $h = \pm 1$ near p, and so h is continuous on $\overline{U} \setminus Z$. (It is not immediate why one *can* minimise, but one can cut out balls around $\partial U \cap Z$ and solve, then shrink the balls; or use Perron's method on the same regions.)

Let $c \in (-1, 1)$ be a regular value of h and set

$$M = Z \cup h^{-1}(c),$$

where Z is C^1 (or $C^{1,1}$) with normal ν_z, and $h^{-1}(c)$ is smooth with normal $\frac{\nabla h}{|\nabla h|}$ (pointing from A to B). We now claim that M is C^1, for which it suffices to show that the normal is C^0; i.e. we want to show that if $p_i \in U$ and $p_i \to z \in Z \cap \partial U$, then $\frac{\nabla h(p_i)}{|\nabla h(p_i)|} \to \nu_z$. Let $B_{r_i}(q_i) \subset U$ be the largest ball in U containing p_i (with $q_i \in U$, which may not be equal to p_i!) and set

$$U_i = \frac{U - q_i}{r_i} \supset B_1(0), \quad \hat{p}_i = \frac{p_i - q_i}{r_i}, \quad \text{and} \quad h_i(x) = h(r_i x + q_i),$$

so that $h_i : U_i \to (-1, 1)$ is harmonic. We claim that

$$U_i \to \text{Slab} = \{-1 < x \cdot \nu_z < 1\}.$$

Then, up to a subsequence, $h_i \to h_\infty$ in C^∞_{loc} with $\hat{p}_i \to p_\infty \in \overline{B_1(0)}$, where h_∞ is harmonic on the slab and ± 1 on its boundary pieces, so h_∞ is linear with $h_\infty(x) = \nu_z \cdot x$. Hence

$$\frac{\nabla h(p_i)}{|\nabla h(p_i)|} = \frac{\nabla h_i(\hat{p}_i)}{|\nabla h_i(\hat{p}_i)|} \to \frac{\nabla h_\infty(p_\infty)}{|\nabla h_\infty(p_\infty)|} = \nu_z.$$

□

4.5 Uniform existence times

Let's see why C^1 interposition is enough for the avoidance principle for weak set flows.

Theorem 38. *(White [Whi05]) There are $\varepsilon = \varepsilon(n) > 0$ and $C = C(n) > 0$ so that if $t \in [0, T) \mapsto \Sigma_t \subset \mathbb{R}^{n+1}$ is a smooth MCF with*

$$\frac{\mathcal{H}^n(\Sigma_0 \cap B_r(x_0))}{\omega_n r^n} < 1 + \varepsilon \quad \text{for all } x_0 \in \mathbb{R}^{n+1},\ r \in (0, r_0), \qquad (\circledast)$$

where $\omega_n r^n = \mathcal{H}^n(B^n_r(0))$, then for all $p_0 \in \Sigma_{t_0}$, with $0 < t_0 < \min(T, r_0^2)$, we have

$$|A_{\Sigma_{t_0}}|^2(p_0) < \frac{C}{t_0}.$$

This is a smooth version of Brakke's regularity theorem, and for static flows (Σ_0 a minimal hypersurface) it is a smooth version of Allard's regularity theorem. (Brakke and Allard regularity are hard in general; White's result is easier since we assume smoothness a priori.)

The point is that the estimate on the curvature is independent of T (the time of existence). So if we had $T < r_0^2$, we would get uniform estimates up to T, and $\Sigma_t \to \Sigma_T$ in C^∞, with Σ_T smooth with uniform curvature bounds. If Σ_T is compact we can then use short time existence to get a smooth flow $t \mapsto \Sigma_t$ on $[0, r_0^2)$. Also, if Σ is a C^1 closed hypersurface then there is some r_0, depending on Σ, such that $(\circledast)$ holds for $0 < r < r_0$; and if the Σ_i are smooth with $\Sigma_i \to \Sigma$ in C^1, then $(\circledast)$ holds for the Σ_i for $0 < r < \frac{r_0}{2}$, for $i \geq 1$ sufficiently large. We can then pass the flows of the Σ_i to the limit, to give a smooth flow out of Σ for a uniform time!

Remark 39. *It is interesting to know what the optimal ε is. The argument of White shows that it is essentially determined by self-shrinkers, i.e. solutions of $\vec{H}_\Sigma = -\frac{x^\perp}{2}$, which are exactly those Σ for which $t \mapsto \sqrt{-t}\,\Sigma$, $t < 0$, is a MCF (singular at $t = 0$, unless Σ is a plane). In fact, it is better to use*

a slightly different measure of 'flatness', namely the ***Gaussian density ratio*** *(or entropy)*

$$\Theta(\Sigma, x_0, r) = (4\pi r^2)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x-x_0|^2}{4r^2}}\, d\mathcal{H}^n(x).$$

White observes that the optimal ε should then be determined by finding the non-flat self-shrinker $\Sigma \subset \mathbb{R}^{n+1}$ with the smallest Gaussian density ratio. For a self-shrinker, Huisken's monotonicity gives

$$\sup_{x_0 \in \mathbb{R}^{n+1},\, r>0} \Theta(\Sigma, x_0, r) = \Theta(\Sigma, 0, 1) = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}}\, d\mathcal{H}^n,$$

i.e. we want to minimise this quantity among self-shrinkers Σ that are not flat. We might hope that the sphere is the optimal shrinker; this is true for $n = 1$ by Abresch–Langer [AL86] (in any codimension), and for $n = 2$ the sphere is optimal by Bernstein–Wang (in codimension one). All other cases are open.

CHAPTER V

TOPOLOGY CHANGE FOR THE BIGGEST FLOW

Following White [Whi95], we show that the biggest flow can only change the topology of the complement in restricted ways: path components cannot merge, loops cannot appear out of nowhere, and certain cycles at the final time cannot become trivial.

5.1 The topological theorem

Let $\mathcal{M} \subset \mathbb{R}^+ \times N$ be closed, where $\mathbb{R}^+ = [0, \infty)$ and N is an $(n+1)$-manifold; i.e. $\mathcal{M}$ is a subset of space-time. Let $W = \mathcal{M}^c$ be its space-time complement, and for $\tau \geq 0$ let

$$W[\tau] = \{(t, x) \in W \,|\, t = \tau\} \quad \text{and}$$
$$W[0, T] = \{(t, x) \in W \,|\, 0 \leq t \leq T\} = \bigcup_{0 \leq \tau \leq T} W[\tau].$$

Theorem 39. *(White) Let $\mathcal{M} = \{(t, x) \,|\, x \in \mathcal{M}_t\}$ be the space-time track of the biggest flow $t \mapsto \mathcal{M}_t$ of $\mathcal{M}_0 \subset N$ (so $\mathcal{M}_t$ is n-dimensional in the $(n+1)$-manifold N). Then, for general N:*

1. *The inclusion $i_0 : W[0] \hookrightarrow W[0, T]$ induces an isomorphism $(i_0)_* : H_0(W[0]) \to H_0(W[0, T])$.*
2. *Any loop in $W[0, T]$ is homotopic to a loop in $W[0]$, i.e. $(i_0)_* : \pi_1(W[0]) \to \pi_1(W[0, T])$ is surjective.*

Moreover, if N is complete with Ricci curvature bounded below, then:

3. *If $i_T : W[T] \hookrightarrow W[0, T]$ is the inclusion, then $(i_T)_* : H_{n-1}(W[T]) \to H_{n-1}(W[0, T])$ and $(i_T)_* : H_n(W[T]) \to H_n(W[0, T])$ are injective (and (3) applies to* any *weak flow!).*

Remark 40. *(1) says that path components cannot coalesce under the flow. If the space-time track looked like two tubes merging into one (Figure 5.1,*

left), then $W[0]$ would have 3 components but $W[0,T]$ only 2; but this is inconsistent with the avoidance principle. If instead one tube splits into two (Figure 5.1, right), then $W[0]$ and $W[0,T]$ both have 2 components, which is fine, even though $W[T]$ has 3.

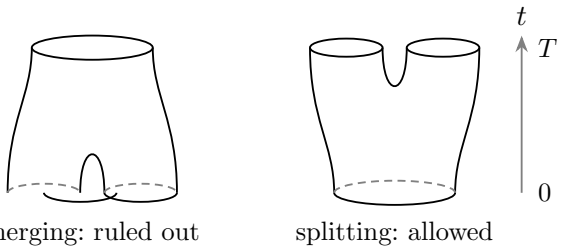


Figure 5.1. Space-time tracks of flows of curves (schematic). Left: two tubes merging, so that components of the complement coalesce, which (1) rules out. Right: one tube splitting into two, which is allowed.

Remark 41. *(2) says that loops cannot appear out of nowhere: a 'C'-shaped tube with no loops at $t = 0$ cannot become a torus, with loops, at some $t > 0$ (this is again inconsistent with the avoidance principle). Loops can however disappear: a torus can pinch and become a 'C'-shaped tube. For example, if $\mathcal{M}[0] = T^2_A$ is the Angenent torus then $\pi_1(W[0]) = \mathbb{Z}^2$ and $\pi_1(W[0,T]) = \mathbb{Z}^2$! (Here the complement has two components, inside and outside the torus, each with $\pi_1 \cong \mathbb{Z}$, and $\mathbb{Z}^2$ means one $\mathbb{Z}$ for each; the same applies to the doughnut with a disk in §5.2.)*

Let us focus on $n = 2$ (surfaces in 3-manifolds). Then (2) and (3) give that $H_1(W[0]) \to H_1(W[0,T])$ is surjective and $H_1(W[T]) \to H_1(W[0,T])$ is injective. Essentially this implies that $\dim H_1(W[T]) \leq \dim H_1(W[0])$, so if $\mathcal{M}_0$ and $\mathcal{M}_T$ are smooth, then the genus of $\mathcal{M}_0$ is at least the genus of $\mathcal{M}_T$: the genus is non-increasing! We can have the genus drop at neck-pinches, but we don't have to (the dumbbell). We can also have the genus drop at a self-shrinker of positive genus (e.g. the Angenent doughnut). (For (3), it is unclear what an example where $(i_T)_*$ is not surjective would be.)

5.2 Examples

The proofs of the topological results will not be too hard; it is more important to understand examples. The following examples show that the theorem is sharp: we can't relax from the biggest flow to other flows, and we can't say anything more general about intermediate homology.

Example 14. *(Sudden vanishing) The weak set flow*

$$\mathcal{M}_t = \begin{cases} \partial B_{\sqrt{R_0^2-2nt}} & \text{for } t \in [0, \frac{R_0^2}{4n}], \\ \emptyset & \text{for } t > \frac{R_0^2}{4n}, \end{cases}$$

is such that $W[0]$ has two components, but $W[0,T]$ has one component for $T > \frac{R_0^2}{4n}$; so it fails (1) of the theorem.

Example 15. *(Figure eight) If $\mathcal{M}[0]$ is a figure eight, then $\mathcal{M}[T]$ is fattened for $T > 0$. Here $W[0]$ has 3 components and $W[0,T]$ has 3 components (which is fine for the theorem!). But let $\mathcal{M}'(t)$ be the 'outer boundary' of $\mathcal{M}(t)$; this goes from the figure eight to a peanut-shaped curve, and then to a circle and a point. Then $W'[0]$ has 3 components while $W'[0,t]$ has 2 components for $t > 0$, which violates the theorem.*

Example 16. *(Doughnut with a disk) Rotate a circle, together with a segment joining it to the z-axis, around the z-axis, to get a torus with a disk spanning its hole. One can flow this, as a surface with triple junctions (Schulze–White)! At some time $0 < T_{cr} < T_{sing}$ one can 'delete' the central disk to get a torus, and restart the flow. Then $\pi_1(W[0]) = \mathbb{Z}$, but for $T > T_{cr}$, $\pi_1(W[0,T]) = \mathbb{Z}^2$ (a loop appears from nowhere!).*

Example 17. *(Generalised dumbbell) Fix $a \gg 1$ and let*

$$I = B_1((a,0)) \cup B_1((-a,0)) \cup ([-a,a] \times [-\varepsilon,\varepsilon]) \subset \mathbb{R}^2,$$

a planar dumbbell. For $p \geq 1$ and $q \geq 2$ write $\mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q$, with $z = (x,y) \in \mathbb{R}^{p+q}$, and set

$$\tilde{I} = \{(x,y) \,|\, (|x|,|y|) \in I\},$$

the actual dumbbell, which is $C^{1,1}$ and mean convex for $a \gg 1$ and $\varepsilon \ll 1$ (globally, after smoothing). (Since $|x| \geq 0$, the ball $B_1((-a,0))$ never contributes; and for $p \geq 2$ each 'lobe' $\{(|x|-a)^2 + |y|^2 \leq 1\}$ is the unit tube around $\mathbb{S}^{p-1}(a) \times \{0\}$, so it is $\mathbb{S}^{p-1} \times B^{q+1}$ rather than a ball, while for $p = 1$ it is the two balls at $x = \pm a$.) By work of White, the flow of $\tilde{I}$ only has neck-pinches. We have $\mathcal{M}_0 \cap \{x = 0\} \neq \emptyset$, but there is $T > 0$ with $\mathcal{M}_T \cap \{x = 0\} = \emptyset$ while $\mathcal{M}_T$ is not empty (put an Angenent doughnut around the neck, and balls in the lobes!). Then

$$W[T] \supset S = \partial B_a^p(0) \times \{0\},$$

which does not *bound anything in $W[T]$. But mean convexity gives that the inside of $W[0]$ contains the inside of $W[T]$, which contains S, so*

$S = \partial R$ for some R in the inside of $W[0]$. So, for $0 \leq k < n-1$, $H_k(W[T]) \to H_k(W[0,T])$ is not *injective (which shows we can't say more about general topology).*

Similarly, let S' be the union of $\{0\}^p \times B^q_{a+2}(0)$ and a hemisphere, a q-dimensional submanifold. Then S' is disjoint from the inside of $W[T]$, but has non-empty intersection with the inside of $W[0]$; and S' does not bound any R in $W[T]$. So for $2 \leq k \leq n$, $H_k(W[0]) \to H_k(W[0,T])$ is not *surjective, since $[S'] \neq 0$ and nothing in $W[0]$ maps to it.*

5.3 Flows with boundary

To set up the proofs we need flows with boundary.

Definition 12. *(Classical flows) Fix a Riemannian manifold (N, g) and let M be any n-manifold, with or without boundary. We say that*

$$\mathcal{M} = \{(t,x) \,|\, t \in [a,b],\ x = f(t,p) \text{ for some } p \in M\}$$

is a ***classical flow*** *of MCF if $f : [a,b] \times M \to N$ is continuous, one-to-one, smooth on $(a,b) \times (M \setminus \partial M)$, and $\left(\frac{\partial f}{\partial t}\right)^{\perp} = \vec{H}_{f(t,M)}$. Its* ***heat boundary*** *is*

$$\partial\mathcal{M} = \{(t, f(t,x)) \,|\, t = a \text{ and } x \in M,\ \text{or } t \in [a,b] \text{ and } x \in \partial M\},$$

i.e. the initial time slice together with the space-time track of the boundary.

Theorem 40. *(Avoidance for classical flows with boundary) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two classical flows. Then*

$$\mathcal{M}_1 \cap \mathcal{M}_2 \neq \emptyset \implies \partial\mathcal{M}_1 \cap \mathcal{M}_2 \neq \emptyset \text{ or } \partial\mathcal{M}_2 \cap \mathcal{M}_1 \neq \emptyset.$$

In fact, if $X \in \mathcal{M}_1 \cap \mathcal{M}_2$, then there is a connected $K \subset \mathcal{M}_1 \cap \mathcal{M}_2$ with $X \in K$, and some $X' \in K \cap (\partial\mathcal{M}_1 \cup \partial\mathcal{M}_2)$.

Proof. The maximum principle (check). □

Theorem 41. *(Short time existence for flows with boundary)*

1. *If $M \subset N$ is a $C^{1,1}$ (or C^1, as we saw) compact embedded hypersurface and $t \in \mathbb{R}$, then there is a classical flow $\mathcal{M}$ with $\partial\mathcal{M} = \{t\} \times M$ (the usual short time existence).*

2. *If $U \subset [0,\infty) \times N$ is open and U' is the union of all classical flows with heat boundary in U, then U' is open (by the continuous dependence on initial data).*

Proof. Check! □

Definition 13. *(Weak flows with boundary) Fix a closed set* $\Gamma \subset [0,\infty) \times N$*. A closed subset* $\mathcal{M} \subset [0,\infty) \times N$ *is a* ***weak flow generated by*** Γ *if:*

1. $\mathcal{M}[0] = \Gamma[0]$*;*
2. *if* $\mathcal{M}'$ *is a classical flow with* $\partial\mathcal{M}' \cap \mathcal{M} = \emptyset$ *and* $\mathcal{M}' \cap \Gamma = \emptyset$*, then* $\mathcal{M}' \cap \mathcal{M} = \emptyset$*.*

The avoidance principle shows that any classical flow $\check{\mathcal{M}}$ is a weak flow generated by $\partial\check{\mathcal{M}}$.

Remark 42. *We can restrict to comparisons with classical flows* $\mathcal{M}'$ *whose heat boundary is their initial time slice,* $\partial\mathcal{M}' = \mathcal{M}'[t] \times \{t\}$*. In particular, if* $\Gamma = \Gamma_0 \times \{0\}$ *then this recovers Ilmanen's definition of a weak flow. So White's definition implies Ilmanen's, and in fact the converse holds also! To see this, suppose that* $\mathcal{M}'$ *is a weak flow in the new sense; if we fix a time slice* t*, then a smoothing* $\tilde{M}$ *of a tubular neighbourhood of* $\mathcal{M}'[t]$ *remains disjoint from* $\mathcal{M}$ *and* Γ *for some (short) time, and so* $\mathcal{M}'$ *is also. . . .*

Some further observations: let $\tau : [0,\infty) \times N \to [0,\infty) \times N$, $(t,x) \mapsto (t+\tau, x)$, be time translation by $\tau \geq 0$. Then $\mathcal{M}$ is a weak flow generated by Γ if and only if $\tau(\mathcal{M})$ is a weak flow generated by $\tau(\Gamma)$. Thus we have the **restarting property**: if $\mathcal{M}$ is a weak flow generated by Γ and $T \geq 0$, set

$$\Gamma' = \{(t,x) \in \Gamma \,|\, t \geq T\} \cup \mathcal{M}[T] \quad \text{and} \quad \mathcal{M}' = \{(t,x) \in \mathcal{M} \,|\, t \geq T\};$$

then $\mathcal{M}'$ is a weak flow generated by Γ'. We also allow **popping**: for $T > 0$ and $\mathcal{M}$ a weak flow generated by Γ,

$$\mathcal{M}' = \Gamma \cup \{(t,x) \in \mathcal{M} \,|\, 0 \leq t \leq T\}$$

is also a weak flow generated by Γ.

Theorem 42. *(Avoidance principle for weak flows) Suppose that* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are weak flows generated by* Γ_1 *and* Γ_2*, that* $\{(t,x) \in \mathcal{M}_1 \,|\, t \leq T\}$ *is compact, and that at* $T > 0$*,*

$$\operatorname{dist}(\mathcal{M}_1(T), \mathcal{M}_2(T)) < \min_{i \neq j} \operatorname{dist}(\mathcal{M}_i(T), \Gamma_j(T)).$$

Then $t \mapsto \operatorname{dist}(\mathcal{M}_1(t), \mathcal{M}_2(t))$ *is non-decreasing on* $[T, T+\varepsilon]$*, with* ε *depending on* $\mathcal{M}_1[T]$ *(which is compact).*

Proof. Use Ilmanen's result (interposing a hypersurface between $\mathcal{M}_1[T]$ and $\mathcal{M}_2[T]$, away from $\Gamma_1[T]$ and $\Gamma_2[T]$). □

There is an analogous result if N is complete with $\mathrm{Ric} \geq -(n+1)\kappa g$, etc.; one can then extend ε to be uniform, depending on the boundary.

Definition 14. *(Biggest flow with boundary) Fix a closed set $\Gamma \subset [0,\infty) \times N$, and inductively define $W_0 = \{(0,x) \notin \Gamma\}$, which is open in $\{0\} \times N$, and let W_{k+1} be the union of all classical flows $\mathcal{M}$, disjoint from Γ, with $\partial\mathcal{M} \subset W_k$. Then the W_k are all open in $[0,\infty) \times N$, with $W_k \subset W_{k+1}$; set*

$$W = \bigcup_{k \geq 0} W_k,$$

which is open in $[0,\infty) \times N$ and disjoint from Γ by construction. Then

$$\mathcal{M}(\Gamma) = ([0,\infty) \times N) \setminus W$$

*is closed, with $\Gamma \subset \mathcal{M}(\Gamma)$, and we call this the **biggest flow generated by** Γ!*

5.4 Path components

Proposition 2. *The biggest flow $\mathcal{M} = \mathcal{M}(\Gamma)$ generated by Γ is a weak set flow generated by Γ, and if $\mathcal{M}'$ is any other weak set flow generated by Γ, then $\mathcal{M}' \subset \mathcal{M}$.*

Proof. If $\mathcal{M}'$ is a weak set flow generated by Γ, then (inductively) $\mathcal{M}' \cap W_k = \emptyset$ for all $k \geq 0$, so $\mathcal{M}' \subset \mathcal{M}(\Gamma)$; this is the second statement. On the other hand, if $\mathcal{M}'$ is a classical flow with $\mathcal{M}' \cap \Gamma = \emptyset$ and $\partial\mathcal{M}' \cap \mathcal{M} = \emptyset$, then $\partial\mathcal{M}' \subset W$, so $\partial\mathcal{M}' \subset W_k$ for some $k \geq 0$ (by compactness), so $\mathcal{M}' \subset W_{k+1} \subset W$, and thus $\mathcal{M}' \cap \mathcal{M}(\Gamma) = \emptyset$. So $\mathcal{M}(\Gamma)$ is a weak set flow generated by Γ. □

Theorem 43. *Let $\mathcal{M}$ be the biggest flow of Γ. Then:*

1. *For each $X \in \mathcal{M}^c = W$ there is a time-like path $t \mapsto (t, \gamma(t))$ in W connecting X to some $Y = (0,y) \in W_0$.*
2. *If X and Y are in different connected components of W_0, then they are in different connected components of W.*

Together, (1) and (2) say that the path components are the same, which is (1) of Theorem 39.

Proof. For (1): $X \in W$ implies $X \in W_k$ for some $k \geq 0$; we induct on k. The case $k = 0$ is immediate (take the constant path). If $k > 0$ then there is a classical flow $\mathcal{M}'$ with $X \in \mathcal{M}'$ and $\partial\mathcal{M}' \subset W_{k-1}$, so we can connect X, within $\mathcal{M}'$, to a point Y on $\partial\mathcal{M}'$, going backwards in time. Then $Y \in W_{k-1}$, so we can use the inductive assumption to connect Y to W_0!

For (2), let $W_0 = W_0^X \sqcup W_0^Y$ be a decomposition of W_0 into disjoint open sets, with $X \in W_0^X$ and $Y \in W_0^Y$. If X connects to Y in W, then it does so in some W_k. Let W_k^X and W_k^Y be the construction applied to W_0^X and W_0^Y respectively. If $Z \in W_k^X \cap W_j^Y$ for some $j, k \geq 0$, then there are a classical flow $\mathcal{M}'$ with $\mathcal{M}' \cap \Gamma = \emptyset$ and $\partial\mathcal{M}' \subset W_{k-1}^X$, and a classical flow $\mathcal{M}''$ with $\mathcal{M}'' \cap \Gamma = \emptyset$ and $\partial\mathcal{M}'' \subset W_{j-1}^Y$, both containing Z. Then the avoidance principle gives $W_{k-1}^X \cap W_j^Y \neq \emptyset$ or $W_k^X \cap W_{j-1}^Y \neq \emptyset$; but then, iterating, we get that $W_0^X \cap W_\ell^Y \neq \emptyset$ or $W_\ell^X \cap W_0^Y \neq \emptyset$ for some $\ell \geq 0$, which is a contradiction (as each W_ℓ^Y meets $\{t = 0\}$ only in W_0^Y, and similarly for W_ℓ^X). Thus $W_k^X \cap W_j^Y = \emptyset$ for all j, k, so Z cannot exist! □

5.5 Loops

We now look at π_1, for a fixed closed set $\Gamma \subset [0, \infty) \times N$.

Lemma 6. *(Lack of spontaneous generation) Let $\mathcal{M}'$ be a classical flow with $\mathcal{M}' \cap \Gamma = \emptyset$ and $\partial\mathcal{M}' \subset W_k$ (so that $\mathcal{M}' \subset W_{k+1}$). Then each connected component of $\mathcal{M}' \cap W_k$ contains points of $\partial\mathcal{M}'$.*

Proof. Let U be a connected component of $\mathcal{M}' \cap W_k$, let $j \geq 0$ be the smallest number with $U \cap W_j \neq \emptyset$, and pick $X \in U \cap W_j$. Observe that $0 \leq j \leq k$. If $j = 0$ we're done, as W_0 only contains boundary points of $\mathcal{M}'$! We claim that $U \cap W_j$ contains points of $\partial\mathcal{M}'$. If not, then $j > 0$, and so there is a classical flow $\mathcal{M}^*$ with $X \in \mathcal{M}^*$, $\mathcal{M}^* \cap \Gamma = \emptyset$ and $\partial\mathcal{M}^* \subset W_{j-1}$ (by the definition of W_j). The avoidance principle (Theorem 40) then gives a connected subset K of $\mathcal{M}' \cap \mathcal{M}^*$ with $X \in K$, and a point

$$Y \in (\partial\mathcal{M}' \cup \partial\mathcal{M}^*) \cap K. \tag{$\circledast$}$$

By construction $\mathcal{M}^* \subset W_j \subset W_k$, so $K \subset U$; but as $\partial\mathcal{M}^* \subset W_{j-1}$, we have $Y \notin \partial\mathcal{M}^*$ by the choice of j (as a minimum; otherwise $W_{j-1} \cap U \supset W_{j-1} \cap K$ would contain Y). So $Y \in \partial\mathcal{M}'$ by ($\circledast$). □

Theorem 44. *Let $\mathcal{M}$ be a level set flow (the biggest flow). Then every closed loop in $W[0,T] = \{(t,x) \in \mathcal{M}^c \,|\, 0 \leq t \leq T\}$ is homotopic to a closed loop in $W[0]$; i.e. for the inclusion $i_0 : W[0] \hookrightarrow W[0,T]$, the map $(i_0)_* : \pi_1(W[0]) \to \pi_1(W[0,T])$ is surjective.*

Proof. Let $W_k[0,T] = W_k \cap W[0,T]$. Covering space theory gives a covering map $\pi : \tilde{W}[0,T] \to W[0,T]$ (with $\tilde{W}[0] = W_0$) such that a closed loop γ in $W[0,T]$ lifts to a closed loop in $\tilde{W}[0,T]$ if and only if γ is homotopic to a closed loop in $W[0]$; equivalently, $\pi_*(\pi_1(\tilde{W}[0,T])) = (i_0)_*(\pi_1(W[0]))$. From this we get a continuous lift $\check{i}_0 : W_0 \to \tilde{W}[0,T]$ with $\pi \circ \check{i}_0 = i_0$. To define this, fix a point x_0 in each component of W_0 and choose $\tilde{x}_0 \in \tilde{W}[0,T]$ with $\pi(\tilde{x}_0) = x_0$. Then for $\gamma : [0,1] \to W_0$ with $\gamma(0) = x_0$ and $\gamma(1) = y_0$ there is a unique lift $\tilde{\gamma} : [0,1] \to \tilde{W}[0,T]$ with $\tilde{\gamma}(0) = \tilde{x}_0$, and we define $\check{i}_0(y_0) = \tilde{\gamma}(1)$. If ξ is another path between x_0 and y_0, then $\xi^{-1} \circ \gamma$ is a closed path in W_0, so $\tilde{\xi}^{-1} \circ \tilde{\gamma}$ is also closed, and $\tilde{\xi}(1) = \tilde{\gamma}(1) = \check{i}_0(y_0)$.

We inductively define $\check{i}_k : W_k[0,T] \to \tilde{W}[0,T]$, the lift of the inclusion $i_k : W_k[0,T] \hookrightarrow W[0,T]$ (with $W_0[0,T] = W_0$). Given $\check{i}_{k-1}$, the lift of i_{k-1} (the base case $k = 1$ being as above), if $\mathcal{M}'$ is a classical flow disjoint from Γ with $\partial\mathcal{M}' \subset W_{k-1}$, then there is a unique lift $\psi : \mathcal{M}' \to \tilde{W}[0,T]$ with $\psi(x) = \check{i}_{k-1}(x)$ for $x \in \partial\mathcal{M}'$. (To see this, if $F : [a,b] \times M \to \mathcal{M}' \subset (0,\infty) \times N$ parametrises the classical flow, then $t \mapsto F(t,p)$ is a path with $F(a,p) \in \partial\mathcal{M}' \subset W_{k-1}$; lift these paths.)

Claim 1: $\psi(x) = \check{i}_{k-1}(x)$ for all $x \in \mathcal{M}' \cap W_{k-1}[0,T]$. Let U be a connected component of $\mathcal{M}' \cap W_{k-1}$; then both ψ and $\check{i}_{k-1}$ give lifts to $\tilde{W}[0,T]$ on U. The lemma gives a point $Y \in U \cap \partial\mathcal{M}'$, where $\psi(Y) = \check{i}_{k-1}(Y)$ by definition, so the two lifts agree on all of U (lifts agreeing at a single point agree), which gives the claim.

Claim 2: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be classical flows disjoint from Γ, with $\partial\mathcal{M}_i \subset W_{k-1}$, and let ψ_1, ψ_2 be the corresponding lifts. If $x \in \mathcal{M}_1 \cap \mathcal{M}_2$, then $\psi_1(x) = \psi_2(x)$. By the avoidance principle there is a connected $K \subset \mathcal{M}_1 \cap \mathcal{M}_2$ containing x and some point $y \in \partial\mathcal{M}_1 \cup \partial\mathcal{M}_2 \subset W_{k-1}$, so Claim 1 gives $\psi_1(y) = \psi_2(y) = \check{i}_{k-1}(y)$, and hence $\psi_1(x) = \psi_2(x)$ by the uniqueness of lifts.

Hence there is a well defined extension of $\check{i}_{k-1}$ to a lift $\check{i}_k : W_k[0,T] \to \tilde{W}[0,T]$, defined by $x \mapsto \psi(x)$, where $\mathcal{M}'$ is a classical flow with $\partial\mathcal{M}' \subset W_{k-1}$, $\mathcal{M}' \cap \Gamma = \emptyset$ and $x \in \mathcal{M}'$, and ψ is the lift from before (this is well defined by Claim 2, and $\check{i}_k|_{W_{k-1}[0,T]} = \check{i}_{k-1}$ by Claim 1). By gluing, we get a lift $\check{i} : W[0,T] \to \tilde{W}[0,T]$; hence every closed loop γ in $W[0,T]$ can be lifted via $\check{i}$ to a closed loop in $\tilde{W}[0,T]$, i.e. γ is homotopic to a closed loop in $W[0]$! $\square$

5.6 Homology at the final time

Suppose we're working in a complete Riemannian manifold (N^{n+1}, g) with $\mathrm{Ric}_g \geq -\kappa g$ ($\kappa \geq 0$), e.g. $\mathbb{R}^{n+1}$.

Theorem 45. *Let $\mathcal{M}$ be a weak flow generated by a closed, compact set $\mathcal{M}_0 \subset N$ at time 0. If C is an $(n-1)$-cycle (e.g. an $(n-1)$-dimensional submanifold; specifically, a polyhedral cycle) in $W[T] = \{(T, x) \,|\, (T, x) \notin \mathcal{M}\}$, and C is homologically trivial in $W[0, T]$ (i.e. $C = \partial P$ is the boundary of something), then C is also homologically trivial in $W[T]$. Namely, for the inclusion $i : W[T] \hookrightarrow W[0, T]$, the map $i_* : H_{n-1}(W[T]) \to H_{n-1}(W[0, T])$ is injective.*

Proof. Let $C = \partial P$, for P an n-chain in $W[0, T]$, and let Γ be the support of P, the closed set associated to P. It is clear that we can modify P so that $\Gamma \cap \{t = 0\} = \emptyset$. Let $\mathcal{M}'$ be the level set flow generated by Γ; the assumptions on N ensure that $\mathcal{M}'[0, T]$ is compact. Set

$$\varepsilon = \min_{0 \leq t \leq T} \mathrm{dist}(\Gamma(t), \mathcal{M}(t)) > 0,$$

and thicken up $\mathcal{M}'[T]$ to get a compact set K in $W[T]$ with smooth boundary, so that $\mathcal{M}'[T] \subset K \subset W[T]$. Let $\Pi : [0, T] \times N \to \{T\} \times N$, $(t, x) \mapsto (T, x)$, be the (continuous) projection, and let $P' = \Pi_\# P$; then

$$\partial P' = \partial \Pi_\# P = \Pi_\# \partial P = \Pi_\# C = C,$$

so P' 'fills in' C in $\{T\} \times N$. If P' was supported in $W[T]$, then we would be done!

Claim: up to modification, $\mathrm{Spt}(P') \subset \mathcal{M}'$*, so that it is disjoint from* $\mathcal{M}[T]$*, and lies in* $W[T]$. Note that this claim implies the result. Let γ be a loop in $(\{T\} \times N) \setminus K$, so that γ lies in $(\mathcal{M}')^c$. Then our previous result (Theorem 44) shows that γ is homotopic, in $W'[0, T]$, to a closed loop γ_0 in $W'[0]$. But $W'[0] = N$, since $\Gamma \cap \{t = 0\} = \emptyset$! This means that the algebraic intersection number of γ and P is zero (the linking number of γ_0 is zero, since $\Gamma \cap \{t = 0\} = \emptyset$, and this number is a homotopy invariant, so γ and P have intersection number zero). Assign multiplicities to the regions of $(\{T\} \times N) \setminus (K \cup P')$ so that the multiplicity changes by α when we cross a face of P' of multiplicity α. We should worry about the consistency of this choice, but it is all OK, since any loop in the complement of K has algebraic intersection number zero with P'. This gives an $(n+1)$-chain Q in the complement of K with $\partial Q = Q' + Q''$, where $Q' \subset P' \setminus K$ and $Q'' \subset \partial K$; modifying P' by replacing $P' \setminus K$ with Q'' gives an n-chain with support inside K and boundary C. □

Theorem 46. *Let $\mathcal{M}$ be a weak flow generated by $\mathcal{M}_0$ at time 0. If a polyhedral n-cycle C in $W[T]$ bounds an $(n+1)$-chain in $W[0,T]$, then C bounds an $(n+1)$-chain in $W[T]$; i.e. $i_* : H_n(W[T]) \to H_n(W[0,T])$ is injective.*

Proof. Let Γ be the support of P, where $\partial P = C$ (with $\Gamma \subset W[0,T]$ and, again after modification, $\Gamma \cap \{t = 0\} = \emptyset$), and let $\mathcal{M}'$ be the level set flow of Γ (again $\mathcal{M}'[0,T]$ is compact), with $\operatorname{dist}(\mathcal{M}', \mathcal{M}) \geq \varepsilon > 0$. Let $P' = \Pi_\# P$, so that $\partial P' = C$ and P' lies in $\{T\} \times N$. We claim that $\operatorname{Spt}(P') \subset \mathcal{M}' \subset W[T]$. To see this, let $(T, x) \in \operatorname{Spt}(P') \setminus \operatorname{Spt}(C)$ and consider the path $t \mapsto (t, x)$ for $0 \leq t \leq T$. This path intersects P with algebraic multiplicity $\alpha \neq 0$ (whatever the multiplicity of P' at x is!). Thus we cannot connect (T, x) to $(0, x)$ by a path disjoint from Γ (or in the same path component), and this implies that $(T, x) \in \mathcal{M}'$. So $\operatorname{Spt}(P') \setminus \operatorname{Spt}(C) \subset \mathcal{M}'$, and since $C = \partial P \subset \operatorname{Spt}(P) = \Gamma$, we have $C \subset \mathcal{M}'$ also, so that $\operatorname{Spt}(P') \subset \mathcal{M}'$. □

This is the end of the material from Jacob's lectures.

CHAPTER VI

WHITE'S REGULARITY THEOREM

These are the closing talks of the course, given by Kahnrad Braxton, on White's local regularity theorem [Whi05].

6.1 Regular flows

We work in space-time $\mathbb{R}^{N,1}$, with points $X = (x,t)$ and the parabolic norm

$$\|X\| = \|(x,t)\| = \max\{|x|, |t|^{\frac{1}{2}}\}.$$

For $\lambda > 0$ let $D_\lambda : \mathbb{R}^{N,1} \to \mathbb{R}^{N,1}$, $(x,t) \mapsto (\lambda x, \lambda^2 t)$, be the parabolic dilation, and let $\tau : \mathbb{R}^{N,1} \to \mathbb{R}$, $(x,t) \mapsto t$, be the time function.

Definition 15. *(Regular flows) Let $\mathcal{M} \subset \mathbb{R}^{N,1}$ be a (proper) C^1 embedded submanifold of dimension $m+1$. If τ has no critical points on $\mathcal{M}$, then $\mathcal{M}$ is a **fully regular flow** of spatial dimension m (e.g. a minimal surface, or a static plane, times $\mathbb{R}$ or an open interval). A **regular flow** is a set of the form $\{X \in \mathcal{M} \,|\, \tau(X) \leq T\}$, for $\mathcal{M}$ fully regular and $T \in (-\infty, \infty]$.*

Example 18. *$\mathcal{M} = \{(x,t) \,|\, x \in M,\ t \in (a,b]\}$, for M minimal, is regular but not fully regular.*

If $\mathcal{M}$ is a regular flow, then for each $t \in \mathbb{R}$ the spatial slice

$$\mathcal{M}(t) = \{x \in \mathbb{R}^N \,|\, (x,t) \in \mathcal{M}\}$$

is a C^1 embedded submanifold of $\mathbb{R}^N$.

If M is a smooth manifold of dimension m, $I = (a,b)$ or $(a,b]$, and $F : M \times I \to \mathbb{R}^N$ is smooth with each $F(\cdot,t) : M \to \mathbb{R}^N$ an embedding, then

$$\mathcal{M} = \{(F(x,t),t) \,|\, x \in M,\ t \in I\}$$

is the space-time set traced out by F. If $\mathcal{M}$ is any regular (smooth) flow and $X \in \mathcal{M}$, then there exist a space-time neighbourhood U of X and a smooth map F as above such that

$$\mathcal{M} \cap U = \{(F(x,t),t) \,|\, x \in M,\ t \in I\},$$

a **local parametrisation** of $\mathcal{M}$ by F. By the fundamental existence and uniqueness theorem for ODEs, one can suppose that $\frac{\partial F}{\partial t}(x,t)$ is perpendicular to $F(M,t)$ at $F(x,t)$.

Definition 16. *(Proper flows) If $\mathcal{M}$ is a regular flow and $U \subset \mathbb{R}^{N,1}$ is open, then we say that $\mathcal{M}$ is a **proper flow in** U if $\mathcal{M} = \overline{\mathcal{M}} \cap U$.*

Example 19. *The shrinking spheres $\mathcal{M}(t) = \mathbb{S}^{N-1}(\sqrt{-2(N-1)t}) \subset \mathbb{R}^N$, $t < 0$, give*

$$\mathcal{M} = \{(x,t) \in \mathbb{R}^{N,1} \,|\, |x|^2 = -2(N-1)t,\ t < 0\},$$

which is proper in $\mathbb{R}^N \times (-\infty, 0)$, but not *proper in $\mathbb{R}^N \times \mathbb{R}$!*

Let $\mathcal{M}$ be a regular flow in $\mathbb{R}^{N,1}$. Then for each $X = (x,t) \in \mathcal{M}$ there is a vector $v = v(\mathcal{M}, X) \in \mathbb{R}^N$ such that v is normal to $\mathcal{M}(t)$ at x and $(v, 1)$ is tangent to $\mathcal{M}$ at X; this is the **normal velocity** of $\mathcal{M}$ at X. If $v = \vec{H}_{\mathcal{M}(t)}(x)$ for each $(x,t) \in \mathcal{M}$, then $\mathcal{M}$ is a MCF. Under the dilations D_λ, MCF is preserved, since v and $\vec{H}$ both scale like $\frac{1}{\lambda}$.

6.2 The local regularity scale

Definition 17. *(Local $C^{2,\alpha}$ norm) Let $\mathcal{M}$ be a regular smooth flow (of spatial dimension m) and $X \in \mathcal{M}$; we define $K_{2,\alpha}(\mathcal{M}, X)$ as follows. If $X = 0 \in \mathcal{M}$ and we can rotate $\mathcal{M}$ to $\mathcal{M}'$ so that $\mathcal{M}' \cap B^{N,1}$ is contained in the graph of a function $u : B^{m,1} \to \mathbb{R}^{N-m}$ with parabolic $C^{2,\alpha}$ norm at most 1, then we say that $K_{2,\alpha}(\mathcal{M}, 0) \leq 1$. Moreover,*

$$K_{2,\alpha}(\mathcal{M}, 0) = \inf\{\lambda > 0 \,|\, K_{2,\alpha}(D_\lambda \mathcal{M}, 0) \leq 1\},$$

and we then set $K_{2,\alpha}(\mathcal{M}, X) = K_{2,\alpha}(\mathcal{M} - X, 0)$.

If $K_{2,\alpha}(\mathcal{M}, \cdot)$ is uniformly bounded on a compact $K \subset \mathcal{M}$, then $\mathcal{M}$ is a $C^{2,\alpha}$ flow in K.

Theorem 47. *(Arzelà–Ascoli for flows) For $i \geq 1$, let $\mathcal{M}_i$ be proper $C^{2,\alpha}$ flows in U_i, with $\mathcal{M}_i \to \mathcal{M}$ and $U_i^c \to U^c$ as sets. Suppose also that the $K_{2,\alpha}(\mathcal{M}_i, \cdot)$ are uniformly bounded, as $i \to \infty$, on compact subsets of U. Then $\mathcal{M}' = \mathcal{M} \cap U$ is a proper $C^{2,\alpha}$ flow in U, and the convergence $\mathcal{M}_i \to \mathcal{M}$ is locally C^2 (parabolically). In particular, if $X_i \in \mathcal{M}_i$ converge to $X \in \mathcal{M}'$, then $v(\mathcal{M}_i, X_i) \to v(\mathcal{M}', X)$, $\vec{H}(\mathcal{M}_i, X_i) \to \vec{H}(\mathcal{M}', X)$, and*

$$K_{2,\alpha}(\mathcal{M}', X) \leq \liminf_{i \to \infty} K_{2,\alpha}(\mathcal{M}_i, X_i).$$

Let $d(X, U) = \inf\{\|X - Y\| \,|\, Y \in U^c\}$; then $d(X,U)\, K_{2,\alpha}(\mathcal{M}, X)$ is scale invariant (check). If $\mathcal{M}$ is a proper regular smooth flow in U, we set

$$K_{2,\alpha;U}(\mathcal{M}) = \sup_{X \in \mathcal{M}} d(X, U)\, K_{2,\alpha}(\mathcal{M}, X).$$

6.3 Gaussian density

Definition 18. *(Gaussian density) Let $\mathcal{M}$ be a regular flow in $\mathbb{R}^{N,1}$ with spatial dimension m. For $X = (x,t) \in \mathbb{R}^{N,1}$ and $r > 0$, the* ***Gaussian density ratio*** *is*

$$\Theta(\mathcal{M}, X, r) = \int_{\mathcal{M}(t-r^2)} (4\pi r^2)^{-\frac{m}{2}} \exp\left(-\frac{|y-x|^2}{4r^2}\right) d\mathcal{H}^m(y),$$

and the ***Gaussian density*** *is $\Theta(\mathcal{M}, X) = \lim_{r\to 0} \Theta(\mathcal{M}, X, r)$ (which exists by Huisken's monotonicity). Similarly, if $\mathcal{M}$ is proper in $\mathbb{R}^{N,1}$ we also have*

$$\Theta(\mathcal{M}, \infty) = \lim_{r\to\infty} \Theta(\mathcal{M}, X, r),$$

for any X.

If $\mathcal{M}$ is a proper MCF in $\mathbb{R}^N \times (a, 0)$ and $\tau(X) > a$, then $\Theta(\mathcal{M}, X, r)$ is a non-decreasing function of $r \in (0, \sqrt{\tau(X) - a})$.

Proposition 3. *If $\mathcal{M} \neq \emptyset$ is a smooth proper MCF in $\mathbb{R}^{N,1}$, then $\Theta(\mathcal{M}, \infty) \geq 1$, with equality if and only if $\mathcal{M}$ is a static plane, of the form*

$$\mathcal{M} = H \times (-\infty, T]$$

for an affine plane $H \subset \mathbb{R}^N$ and $T \in (-\infty, \infty]$.

Proof. See Stone's paper [Sto94]. □

6.4 White's regularity theorem

Theorem 48. *(White's regularity theorem) For $\alpha \in (0,1)$ there are $\varepsilon, C > 0$, depending on N, m and α, such that if $\mathcal{M}$ is a smooth proper MCF in U, with spatial dimension m, and*

$$\Theta(\mathcal{M}, X, r) < 1 + \varepsilon \quad \text{for all } X \in \mathcal{M},\ r \in (0, d(X, U)),$$

then $K_{2,\alpha;U}(\mathcal{M}) \leq C$.

Proof. We argue by contradiction: let $\bar{\varepsilon} = \inf\{\varepsilon > 0 \,|\, \text{the theorem fails}\}$, and let $\varepsilon_i \to \bar{\varepsilon}^+$. We then have a sequence of flows $\mathcal{M}_i$ in U_i such that:

1. $\Theta(\mathcal{M}_i, X, r) \leq 1 + \varepsilon_i$ for all $X \in \mathcal{M}_i$ and $r \in (0, d(X, U_i))$;
2. $K_{2,\alpha;U_i}(\mathcal{M}_i) \to \infty$.

We can assume that $K_{2,\alpha;U_i}(\mathcal{M}_i) = S_i \to \infty$, and fix $X_i \in \mathcal{M}_i$ such that

$$d(X_i, U_i)\, K_{2,\alpha}(\mathcal{M}_i, X_i) > \tfrac{1}{2} S_i.$$

By translation invariance we may assume that $X_i = 0$, and by dilation that $K_{2,\alpha}(\mathcal{M}_i, 0) = 1$, so that $d(0, U_i) \to \infty$ by (2). Now, for each $X \in \mathcal{M}_i$,

$$d(X, U_i)\, K_{2,\alpha}(\mathcal{M}_i, X) \leq S_i < 2d(0, U_i)\, K_{2,\alpha}(\mathcal{M}_i, 0) = 2d(0, U_i),$$

and thus

$$\begin{aligned} K_{2,\alpha}(\mathcal{M}_i, X) &\leq \frac{2d(0, U_i)}{d(X, U_i)} \leq \frac{2d(0, U_i)}{d(0, U_i) - \|X\|} \\ &= 2\left(1 - \frac{\|X\|}{d(0, U_i)}\right)^{-1} < C < \infty, \end{aligned}$$

which is uniformly bounded on compact subsets! Thus Arzelà–Ascoli gives subsequential convergence of the $\mathcal{M}_i$, locally, to a smooth MCF that is proper in all of $\mathbb{R}^{N,1}$; call it $\mathcal{M}$. Then (from the convergence)

$$\Theta(\mathcal{M}, X, r) \leq 1 + \bar{\varepsilon}.$$

Supposing $\bar{\varepsilon} = 0$, monotonicity gives (by Proposition 3) that $\mathcal{M}$ is of the form

$$\mathcal{M} = \mathbb{R}^m \times \{0\}^{N-m} \times (-\infty, T], \quad T \in [0, \infty].$$

By C^2 convergence, there are $r_i \to \infty$ such that $\mathcal{M}_i \cap (B^m_{r_i}(0) \times B^{N-m}_{r_i} \times (-r_i, r_i))$ is the graph of a function

$$u_i : B^m_{r_i}(0) \times I_i \to \mathbb{R}^{N-m}, \quad I_i = (-r_i, r_i) \cap (-\infty, T_i),$$

with $\|u_i\|_{C^2} \to 0$. The u_i satisfy the non-parametric MCF equation, so that

$$\frac{\partial}{\partial t} u_i - \Delta u_i = f_i, \quad \text{where} \quad f_i = -\sum_{j,k=1}^{m} \frac{D_j u_i D_k u_i}{1 + |Du_i|^2} D_j D_k u_i$$

(if $m = N - 1$). We note that $f_i \to 0$ in $C^{0,\alpha}$ on compact sets, since the u_i are uniformly bounded in $C^{2,\alpha}$ on compact sets and tend to 0 in C^2 on compact sets: indeed,

$$\frac{D_j u_i D_k u_i}{1 + |Du_i|^2} \to 0 \text{ in } C^1_{loc} \quad \text{and} \quad D_j D_k u_i \to 0 \text{ in } C^0_{loc},$$

while the $D_j D_k u_i$ are bounded in $C^{0,\alpha}$. Lastly, Schauder estimates for the heat equation imply that $\|u_i\|_{C^{2,\alpha}} \to 0$ on compact subsets of $\mathbb{R}^{N,1}$. But we have $K_{2,\alpha}(\mathcal{M}_i, 0) = 1$, a contradiction! So $\bar{\varepsilon} > 0$. □

Proposition 4. *Let $\bar{\varepsilon}$ be the infimum of the $\varepsilon > 0$ for which the theorem fails. Then:*

1. *There are no proper smooth MCFs with $1 < \Theta(\mathcal{M}, \infty) < 1 + \bar{\varepsilon}$ (static planes have $\Theta(\mathcal{M}, \infty) = 1$).*
2. *There is a flow with $\Theta(\mathcal{M}, \infty) = 1 + \bar{\varepsilon}$.*
3. *The theorem fails at $\Theta(\mathcal{M}, \infty) = 1 + \bar{\varepsilon}$ (i.e. for $\varepsilon = \bar{\varepsilon}$).*

Proof. For (1), if $\mathcal{M}$ is a smooth proper MCF in $\mathbb{R}^{N,1}$ with $\Theta(\mathcal{M}, \infty) < 1 + \bar{\varepsilon}$, then $\Theta(\mathcal{M}, \infty) < 1 + \varepsilon < 1 + \bar{\varepsilon}$ for some ε, and so monotonicity gives $\Theta(\mathcal{M}, X, r) < 1 + \varepsilon$ for all $X \in \mathcal{M}$ and $r > 0$. So the theorem (with $U = \mathbb{R}^{N,1}$, so that $d(X, U) = \infty$) gives $K_{2,\alpha}(\mathcal{M}, X) = 0$, and so $\Theta(\mathcal{M}, X, r) = 1$. For (2), some MCF fails the theorem, so as $\bar{\varepsilon}$ is an infimum there has to exist one! (3) follows similarly to (2). □

6.5 Comments on Stone's paper

Let M_0 be a smooth, compact, embedded hypersurface without boundary in $\mathbb{R}^{N+1}$. We say that M_0 has a **type I** blowup if there is some $C_0 < \infty$ such that

$$\max_{M_t} |A|^2(t) \leq \frac{C_0}{2(T-t)}.$$

Are tangent flows unique? Some results in this direction:

- [Sto94]: type I, mean convex.
- [Whi03]: mean convex, codimension one.
- [Sch14]: compact tangent flows are unique (in Euclidean space, for a closed, multiplicity one, embedded tangent flow).
- [CM15]: cylindrical tangent flows, generic singularities are unique with no mean convexity assumption; in particular this covers mean convex flows.

End of the course.

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