`AS.110.631` : Partial Differential Equations I

Course Lecture Notes

James Guo

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Table of Contents

1 Foundations for PDEs **1**

I Introduction **1**

I.1 Partial Differential Equations 1

I.2 Function Spaces 2

I.3 Functional Analysis Preliminaries 4

2 Abstract Functional Spaces **8**

II Sobolev Space **8**

II.1 Weak Derivative 8

II.2 Sobolev Space 11

II.3 Approximation of Sobolev Functions 14

III Boundaries of Sobolev Space **18**

III.1 Global Approximation of Sobolev Space 18

III.2 Extensions of Sobolev Functions 20

III.3 Trace of Sobolev Functions 22

III.4 Sobolev Inequalities 25

IV Hölder Spaces **29**

IV.1 Hölder Space 29

IV.2 Sobolev Embeddings 31

V Compactness in Sobolev Space **37**

V.1 Reflexivity 37

V.2 Weak Compactness 37

VI Difference Quotients **41**

VI.1 Difference Quotients 41

VI.2 Weak and Classical Derivatives 42

3 Second Order Linear PDEs 44

VII Ellipticity and Existence 44
VII.1 Preliminaries and Weak Solutions . . . 44
VII.2 Ellipticity . . . 47
VII.3 The Existence Theorems . . . 49
VII.4 Fredholm Theory . . . 53
VII.5 Eigenvalues of Elliptic Operators . . . 57

VIII Regularity Theory 61
VIII.1 Intuitions on Regularity . . . 61
VIII.2 Interior Regularity . . . 61
VIII.3 Higher Interior Regularity . . . 64
VIII.4 Boundary Regularity . . . 66

4 Second Order Nonlinear PDEs 69

IX Extending from Second Order Linear PDEs 69
IX.1 Existence for a Nonlinear PDE . . . 69
IX.2 Regularity for a Nonlinear PDE . . . 71

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- The document might contain minor typos or errors. Please point out any notable error(s).

Part 1

Foundations for PDEs

I Introduction

I.1 Partial Differential Equations

Definition I.1.1. Partial Differential Equation.
A **partial differential equation** (PDE) of order $k \in \mathbb{N} := \{0, 1, 2, \cdots\}$ is an expression of the form:

$$F\big(x, u(x), Du(x), \cdots, D^k u(x)\big) = 0, \tag{1}$$

where $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ is open, $u : \Omega \to \mathbb{R}$, and $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$, since $D^m u(x)$ is the m-th derivative of $u(x)$ and is of dimension n^m.
Here, u is unknown. We solve the PDE (1) if we find all such $u : \Omega \to \mathbb{R}$. ⌟

Remark I.1.2. Systems of PDE.
We can define **systems of PDE** by considering $u : \Omega \to \mathbb{R}^M$ for some $M \in \mathbb{N}^+$. ⌟

Then, we give some examples of PDEs:

Example I.1.3.

- The **Possion equation** is:
$$-\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f,$$
where the **Laplace equation** is when $f \equiv 0$.
- The **Heat equation** is:
$$\frac{\partial u}{\partial t} - \Delta u = 0,$$
which could be called a parabolic equation.
- The **Allen-Cahn equation** is:
$$\Delta u = u(u^2 - 1),$$
and we can note that this equation is nonlinear.
- The **Minimal Surface equation** is:
$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0,$$
which involves second order derivatives and nonlinear case. ◊

For this course, we use **multi-index** notion, *i.e.*, if $\alpha \in \mathbb{N}^n$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, we write:

$$D^\alpha u = \frac{D^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad \text{where } |\alpha| = \sum_{i=1}^n \alpha_i.$$

Note that D^α here is an operator that acts on certain classes of functions.

Then, we classify PDEs in several classes:

Definition I.1.4. We say that a PDE is:

- **Linear** if it is of the form:

$$\sum_{|\alpha| \leq k} \underbrace{a_\alpha(x)}_{\text{coefficient}} D^\alpha u(x) = f(x) \qquad \text{for given } a_\alpha, f : \Omega \to \mathbb{R}.$$

  If $f \equiv 0$, we call the PDE **homogeneous**.

- **Semi-linear** if it is of the form:

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0\big(x, u(x), \cdots, D^{k-1} u(x)\big) = 0 \qquad \text{for } a\alpha, a_0 : \Omega \to \mathbb{R}.$$

- **Quasi-linear** if it is of the form:

$$\sum_{|\alpha|=k} a_\alpha\big(x, u(x), \cdots, D^{u-1} u(x)\big) D^\alpha u(x) + a_0\big(x, u(x), \cdots, D^{k-1} u(x)\big) = 0 \qquad \text{for } a\alpha, a_0 : \Omega \to \mathbb{R}.$$

- **Fully non-linear** if it depends non-linearly on its u-th derivatives, *e.g.*, $\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0$. ⌟

I.2 Function Spaces

Definition I.2.1. Let $\Omega \subset \mathbb{R}^n$ to be open, $k \in \mathbb{N}$, $p \in [1, \infty]$, we denote:

- $C^k(\Omega) = \{u : \Omega \to \mathbb{R} \mid \text{all partial derivatives of order} \leq k \text{ exist and are continuous}\}$.
- $C^k(\overline{\Omega}) = \left\{u \in C^k(\Omega) \mid \text{all partial derivatives of order} \leq k \text{ extend continuously to } \overline{\Omega}\right\}$.
- $C^\infty(\Omega) = \bigcap_{\ell \in \mathbb{N}} C^\ell(\Omega)$ and $C^\infty(\overline{\Omega}) = \bigcap_{\ell \in \mathbb{N}} C^\ell(\overline{\Omega})$.
- $C_c^k(\Omega) = \left\{u \in C^k(\Omega) \mid \overline{\{u \neq 0\}} \text{ is compact in } \Omega\right\}$.
- $C_c^\infty(\Omega) = \bigcap_{\ell \in \mathbb{N}} C_c^\ell(\Omega)$ is the class of test functions.

In fact, making the functions to be differential is very strong, and we need to have weaker condition.

- $L^p(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is Lebesgue measurable and } \|u\|_{L^p(\Omega)} < \infty\} / \sim$, where:

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p\right)^{1/p} & \text{for } p \in [1, \infty), \\ \operatorname{ess\,sup}_\Omega |u| = \inf\{s \in \mathbb{R} \mid |\{|u| > s\}| = 0\} & \text{for } p = \infty. \end{cases}$$

and $u \sim v$ if and only if $u = v$ almost everywhere.

- $L^p_{\text{loc}}(\Omega) = \bigcap_{\tilde{\Omega} \Subset \Omega} L^p(\tilde{\Omega})$, where $\tilde{\Omega} \Subset \Omega$ if and only if there is a compact set $K \subset \Omega$ such that $\tilde{\Omega} \subset K \subset \Omega$.

⌟

Example I.2.2. $\sqrt{x} \in C^\infty\big((0,\infty)\big)$, $\sqrt{x} \in C^0\big([0,\infty)\big)$, but $\sqrt{x} \notin C^1\big([0,\infty)\big)$.

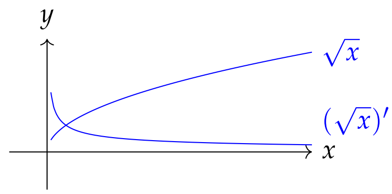

Figure I.1. Graph of $\sqrt{x}$ and its derivative.

Note that the equation ($\sqrt{x}$) extend smoothly to 0^+, but the first derivative of $\sqrt{x}$ ($1/2\sqrt{x}$) is not extending to 0^+ smoothly. ♢

By setting $\|u\|_{C^0(\overline{\Omega})} = \sup_\Omega |u|$ (supremum norm) and $\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{C^0(\overline{\Omega})}$ (C^k-norm), using the notation $\|D^\alpha u\|_* = \big\||D^\alpha u|\big\|_*$.

Then, $\big(C^k(\overline{\Omega}), \|\cdot\|_{C^k(\overline{\Omega})}\big)$ is Banach if Ω is bounded or we restrict to functions with bounded C^k-norm.

For $p \in [1,\infty]$, $\big(L^p(\Omega), \|\cdot\|_{L^p(\Omega)}\big)$ is Banach and for $p = 2$, by setting the L^2-inner product to be:

$$(u,v)_{L^2(\Omega)} = \int_\Omega u \cdot v \qquad \text{for } u, v \in L^2(\Omega).$$

Specifically, $\|u\|^2_{L^2(\Omega)} = (u,u)_{L^2(\Omega)}$, and $\big(L^2(\Omega), (\cdot,\cdot)_{L^2(\Omega)}\big)$ is Hilbert.

Lemma I.2.3. Hölder Inequality.
For $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, then if $\Omega \subset \mathbb{R}^n$ is open, $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have:

$$\|u \cdot v\|_{L^1(\Omega)} \le \|u\|_{L^p(\Omega)} \cdot \|v\|_{L^q(\Omega)}.$$

When $p = q = 2$, this is the **Cauchy-Schwartz inequality**.

The *proof* of this could utilize the **Young's inequality** ($ab \le \frac{a^p}{p} + \frac{a^q}{q}$).

Lemma I.2.4. Fundamental Lemma of Calculus of Variation (FLCV).
Suppose $\Omega \subset \mathbb{R}^n$ is open and $v \in L^1_{\text{loc}}(\Omega)$, then:

$$v = 0 \text{ almost everywhere in } \Omega \iff \int_\Omega v \cdot \varphi = 0 \text{ for every} \varphi \in C_c^\infty(\Omega).$$

The *proof* would be requiring the *mollifiers*.

Example I.2.5. Consider the following functions:

- $u(x) = e^x$ on $\mathbb{R}$, we have $u \in C^\infty(\mathbb{R})$, $\|u\|_{C^0(\mathbb{R})} = \infty$, $\|u\|_{L^p(\mathbb{R})} = \infty$, but $u \in L^1_{\text{loc}}(\mathbb{R})$.
- $v(x) = \begin{cases} 1+x, & x < 0, \\ 1-x, & x \geq 0 \end{cases}$, which can be visualized as follows:

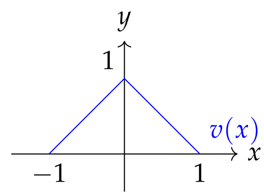

Figure I.2. Graph of $v(x)$.

we have $v \in C^0\big((-1,1)\big) \setminus \big(\bigcup_{k\geq 1}\big((-1,1)\big)\big)$, $\|v\|_{C^0\big((-1,1)\big)} = 1$, and $\|v\|_{L^p\big((-1,1)\big)} < \infty$.

- Now, consider $\epsilon > 0$ to be small and fixed, with the following functions:

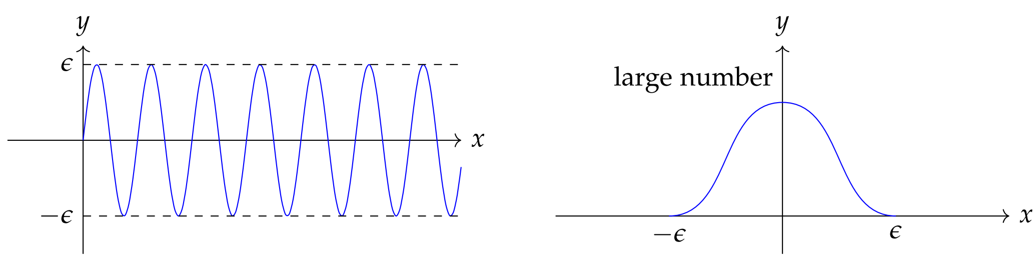

Figure I.3. Graph of two example functions.

The left function has small $\|\cdot\|_{C^0}$ and large $\|\cdot\|_{C^1}$, whereas the right function has small $\|\cdot\|_{L^p}$ and large $\|\cdot\|_{C^0}$ or $\|\cdot\|_{C^1}$ ◊

I.3 Functional Analysis Preliminaries

Recall that a **Banach space** is a complete "normed vector space."

- Some examples include $(\mathbb{R}, |\cdot|)$ or $(L^p, \|\cdot\|_{L^p})$.

A **Hilbert space** is a complete "inner product space." A Hilbert space is a Banach space.

- Some examples include $(\mathbb{R}^n, \cdot)$, in which $\cdot(a,b) = \sum_{i=1}^n a_i b_i$, or $\big(L^2, (\cdot,\cdot)_{L^2}\big)$, in which $(u,v)_{L^2} = \int uv$.

Definition I.3.1. $L : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a bounded linear map between Banach spaces X and Y if:

$$\|L\| = \sup_{\|x\|_X \leq 1} \{\|L(x)\|_Y\} < \infty.$$

⌟

Proposition I.3.2. A linear map is bounded if and only if it is continuous.

Definition I.3.3. Dual Space.
The **dual space**, denoted $(X^*, \|\cdot\|_{X^*})$, endowed with the operator norm, of $(X, \|\cdot\|_X)$ is the collection of bounded linear maps $L : (X, \|\cdot\|_X) \to (\mathbb{R}, |\cdot|)$. ⌟

Remark I.3.4.

- If $\dim(X) < \infty$, this (continuous) dual space is the algebraic dual of X. But every infinite dimensional Banach space has unbounded linear maps.
- Radon measures are dual space to $\big(C_c^0(\Omega), \|\cdot\|_{C^0(\Omega)}\big)$.

⌟

Proposition I.3.5. If $\Omega \subset \mathbb{R}^n$ is open and $p \in [1, \infty)$, then:

$$\big(L^p(\Omega)\big)^* = L^q(\Omega), \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

This "=" is an isomorphism, towing $v \in L^q(\Omega)$ to the linear map $L_v : L^p(\Omega) \to \mathbb{R}$ defined by:

$$L_v(u) = \int_\Omega u \cdot v \qquad \text{for } u \in L^p(\Omega).$$

Note that L_v is bounded since by Lemma I.2.3 Hölder inequality that:

$$|L_v(u)| = \left|\int_\Omega u \cdot v\right| \le \int_\Omega |uv| \le \|v\|_{L^p(\Omega)} \cdot \|v\|_{L^q(\Omega)},$$

and hence we have:

$$\|L_v\| \le \|v\|_{L^1(\Omega)}.$$

Remark I.3.6. Note that $L^1(\Omega) \subsetneq \big(L^\infty(\Omega)\big)^*$. ⌟

Now, consider the restriction to Hilbert space.

Theorem I.3.7. Riesz Representation Theorem.
Let $\big(\mathcal{H}, (\cdot,\cdot)\big)$ be a Hilbert space, then for each $L \in \mathcal{H}^*$ there is a unique $u \in \mathcal{H}$ such that:

$$L(v) = (u, v) \qquad \text{for any } v \in \mathcal{H}.$$

Moreover, $\ker L := \{v \in \mathcal{H} \mid L(v) = 0\} = u^\perp = \{v \in \mathcal{H} \mid (u, v) = 0\}$.

Example I.3.8. If $f : \mathbb{R}^n \to \mathbb{R}$ is linear, $f \in (\mathbb{R}^n)^*$. Consider $f(x) = a \cdot x$ for each $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$. ◊

Now, we consider a blueprint for solving PDEs.

Example I.3.9. Poisson Equation.
Let $B \subset \mathbb{R}^n$ be the unit open ball and $f \in C^\infty(\overline{B})$ be given. We want to solve the Poisson's equation on B with zero (or Dirichlet condition) on boundary.
Hence, we seek $u \in C^\infty(\overline{B})$ which solves the PDE:

$$\begin{cases} -\Delta u = f, & \text{on } B, \\ u = 0, & \text{on } \partial B. \end{cases}$$

(i) Reformulation of the equation. If we solved the equation then for each $\varphi \in \mathbb{C}_c^\infty(B)$, we would have:

$$\int_B f \cdot \varphi = -\int_B \Delta u \cdot \varphi = \int_B \nabla u \cdot \nabla \varphi,$$

since we have:

$$\operatorname{div}(\varphi \nabla u) = \nabla u \cdot \nabla \varphi + \varphi \cdot \Delta u,$$

and $\varphi\big|_{\partial B} = 0$.
Let's instead restrict to finding $u \in C^\infty(\overline{B})$ solving:

$$\begin{cases} \int_B f\varphi = \int_B \nabla u \cdot \nabla \varphi & \text{for all } \varphi \in C_c^\infty(B), \\ u = 0 & \text{on } \delta B. \end{cases} \tag{2}$$

But then we turned the original equation into (2), since if it holds, then for any $\varphi \in C_c^\infty(B)$, we have:

$$\int_B f\varphi - \int_B \nabla v \cdot \nabla \varphi = \int_B (f + \Delta v)\varphi = 0,$$

thus by Lemma I.2.4, we have:

$$f + \Delta v = 0 \text{ on } B \text{ almost everywhere.}$$

Note that we only need 1 derivative of u for (2) to make sense, even if f and ∇u are only L^p functions, *i.e.*, defined almost everywhere.

(ii) Hilbert space formulation of (2). Then we try to find some function u solving the integral equality in (2). We define on $C_c^\infty(B)$ the H_0^1 inner product as:

$$\langle u, v \rangle_{H_0^1(B)} = \int_B \nabla u \cdot \nabla v \qquad \text{for } u, v \in C_c^\infty(B).$$

This inner product induces the H_0^1 norm as:

$$\|v\|^2_{H_0^1(\Omega)} = \langle v, v \rangle_{H_0^1(B)} = \|\nabla u\|^2_{L^2(B)}.$$

If we define, for $f \in L^2(B)$, the linear map $L_f : (C_c^\infty(B), \|\cdot\|_{H_0^1(B)}) \to \mathbb{R}$ by setting:

$$L_f(\varphi) = \int_B f \cdot \varphi.$$

By Hölder's inequality (Lemma I.2.3), we have:

$$|L_f(\varphi)| = \left| \int_B f\varphi \right| \leq \int_V |f\varphi| \leq \|f\|_{L^2(B)} \cdot \|\varphi\|_{L^2(B)}.$$

Now, if we had an estimate of the form:

$$\|\varphi\|_{L^2(B)} \leq C_p \|\varphi\|_{H_0^1(B)},$$

which is known as the Poincoré inequality, then:

$$|L_f(\varphi)| \leq \|f\|_{L^2(B)} \cdot \|\varphi\|_{L^2(B)} \leq \|f\|_{L^2(B)} \cdot cC_p \|\varphi\|_{H_0^1(B)} \leq C_p \|f\|_{L^2(B)}.$$

Hence L_f is bounded.
However, $\left(C_c^\infty(B), \langle \cdot, \cdot \rangle_{H_0^1(B)}\right)$ is not a Hilbert space, so we cannot apply Riesz representation.

If it were a Hilbert space, Riesz would give a unique $u \in ?$ such that:

$$\int_B f\varphi = L_f(\varphi) = \langle u, \varphi \rangle_{H_0^1(B)} = \int_B \nabla u \cdot \nabla \varphi.$$

To get around, this, we take the Cauchy completion with respect to $\| \cdot \|_{H_0^1(B)}$ to get a Hilbert space as:

$$\left(H_0^1(B), \langle \cdot, \cdot \rangle_{H_0^1(B)}\right),$$

which turns out as a **Sobolev Space**. Hence, we would have Riesz giving a *weak solution* $u \in H_0^1(B)$:

$$\int_B f\varphi = L_f(\varphi) = \langle u, \varphi \rangle_{H_0^1(B)} = \int_B \nabla u \cdot \nabla \varphi.$$

(iii) Regularity. Given a weak solution $u \in H_0^1(B)$ to the Possion equation, it is equivalent to that:

$$\int_B f\varphi = \int_B \nabla v \cdot \nabla \varphi \qquad \text{for every } \varphi \in C_c^\infty(B),$$

which made sense for $f \in L^2(B)$, and we will show that $f \in C^k)\overline{B}$ implies that $u \in C^{k+1}(\overline{B})$ by estimating the derivative of u by the derivatives of f. (This is stronger than $f \in C^\infty$ implies $u \in C^\infty$.)

$\diamondsuit$

Part 2

Abstract Functional Spaces

II Sobolev Space

II.1 Weak Derivative

Throughout this course, we use the principle that "*strong theorems* become weak definition."

Recall when we tried to solve the Poisson equation in Example I.3.9, we used integration by parts to obtain:

$$\begin{cases} -\Delta u = f \text{ on } B \\ u = 0 \text{ on } \partial B \end{cases} \iff \begin{cases} \int \nabla u \cdot \nabla \varphi = \int f\varphi \text{ for all } \varphi \in C_c^\infty(B) \text{ on } B \\ u = 0 \text{ on } \partial B \end{cases}$$

Then, we will expand the concept of integration by parts as weak derivatives.

Definition II.1.1. Weak Derivatives.
If $\Omega \subset \mathbb{R}^n$ is open, $u \in L^1_{\text{loc}}(\Omega)$, and $\alpha \in \mathbb{N}^k$, then $v \in L^1_{\text{loc}}(\Omega)$ is a **weak derivative** for u corresponding to α if:

$$\int_\Omega u \cdot D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v\varphi \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We then write $D^\alpha u = v$ almost everywhere on Ω. ⌟

In other words, we can say that integration by parts holds for u and v.

Remark II.1.2. If $u \in C^k(\Omega)$, then $D^\alpha u$, in the usual sense, is a weak derivative of u corresponding to α for $|\alpha| \leq k$. ⌟

Lemma II.1.3. Properties of Weak Derivatives.
If $\Omega \subset \mathbb{R}^n$ is open, $\alpha \in \mathbb{N}^k$, and $u, \tilde{u} \in L^1_{\text{loc}}(\Omega)$, then:

(i) Weak derivatives are unique almost everywhere, so $D^\alpha u$ is unambiguous as an $L^1_{\text{loc}}(\Omega)$ function.

(ii) Taking weak derivatives is a linear operator, *i.e.*:

$$D^\alpha(u + \lambda \tilde{u}) = D^\alpha u + \lambda D^\alpha \tilde{u}.$$

Proof. (i) Suppose $v, \tilde{v}$ are weak derivative of u, by definition:

$$(-1)^{|\alpha|} \int_\Omega v \cdot \varphi = \int_\Omega u \cdot D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega \tilde{v} \cdot \varphi$$

for any test function $\varphi \in C_c^\infty(\Omega)$, so we have:

$$\int_\Omega v \cdot \varphi = \int_\Omega \tilde{v} \cdot \varphi \qquad \text{for any } \varphi \in C_c^\infty(\Omega).$$

Hence, we equivalently have:

$$\int_\Omega (v - \tilde{v}) \cdot \varphi = 0 \qquad \text{for any } \varphi \in C_c^\infty(\Omega),$$

and so by the fundamental lemma of calculus on variation (Lemma I.2.4), $v - \tilde{v} = 0$ almost everywhere.

(ii) Linearity follows from the linearity of *integration*. □

Now, we will see some examples of weak derivatives, in which they are badly behaving, but not too badly.

Example II.1.4. Consider $|x| \in C^0(-1,1) \setminus C^1(-1,1)$.

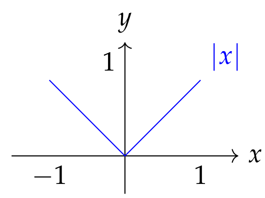

Figure II.1. Graph of $|x|$.

Away from 0, we have $|x|$ differentiable with the derivative that:

$$v(x) = \begin{cases} -1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Here, we can illustrate it as:

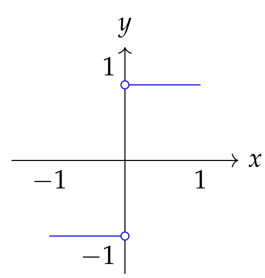

Figure II.2. Graph of $v(x)$, candidate of the weak derivative of $|x|$.

Let's now verify that v is the first weak derivative of u. For any $\varphi \in C_c^\infty(-1,1)$, we have:

$$\begin{aligned}\int_{-1}^1 |x| \cdot \varphi'(x) &= -\int_{-1}^0 x\varphi'(x) + \int_0^1 x\varphi'(x) \\ &= \left(\Big[-x\varphi(x) \Big]_{-1}^0 + \int_{-1}^0 \varphi(x) \right) + \left(\Big[x\varphi(x) \Big]_0^1 - \int_0^1 \varphi(x) \right) \\ &= \int_{-1}^0 \varphi(x) - \int_0^1 \varphi(x) = -\left(-\int_{-1}^0 \varphi(x) + \int_0^1 \varphi(x) \right) = -\int_{-1}^1 v(x) \cdot \varphi(x),\end{aligned}$$

and hence v is the weak derivative of $|x|$.
Then, we could see if $D^2|x| = Dv$ exists weakly, as we saw $D|x| = v$. But if w was a weak derivative of v, since if $\phi \in C_c^\infty(0,1)$ then:

$$\int_0^1 w(x)\phi(x) = \int_{-1}^1 w(x)\phi(x) = -\int_{-1}^1 v(x)\phi'(x) = -\int_0^1 v(x)\phi(x) = -\int_0^1 \phi'(x) = \phi(0) - \phi(1) = 0,$$

by the fundamental theorem of calculus, and since ϕ vanishes on the boundary. Again, by the fundamental lemma on calculus of variation, $w = 0$ almost everywhere on $(0,1)$.
Similarly, we can choose $\phi \in C_c^\infty(-1,0)$, then $w = 0$ almost everywhere on $(-1,0)$. Thus, w is almost everywhere on $(-1,1)$.
Now, if we choose some $\varphi \in C_c^\infty(-1,1)$ with $\varphi(0) \neq 0$, then we have:

$$0 = \int_{-1}^1 w(x)\varphi(x) = -\int_{-1}^1 v(x)\varphi'(x) = -2\int_0^1 \varphi'(x) = -2\int_0^1 \varphi'(x) = 2\varphi(0) \neq 0,$$

and this leads to a contradiction. Hence w is not weak derivative of v. Thus v is not weakly differentiable.

Aside, v do have a weaker derivative, called a distributional derivative, which turns out to be the dirac delta function with a infinite point mass at 0.

◊

Hence, we would have Lipchitz function to be weakly differentiable, but not step functions.

There are continuous functions on $(0,1)$ which are classically/strongly differentiable almost everywhere, but not weakly differentiable.

Example II.1.5. Cantor/Devil's Staircase.
Consider the ternary cantor set $\mathcal{C}$, on each level construction, we set the middle $1/3$ part to middle of the left and right endpoints with middle connected with straight lines.

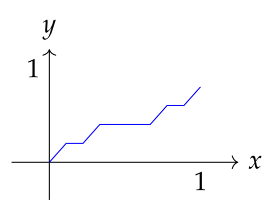

Figure II.3. Graph of Cantor Staircase equation at level 2.

Thus, the function is continuous and monotone, while is it differentiable almost everywhere as 0, but it is not weakly differentiable. ◊

Another example is corresponding to the spike function with respect to the Sobolev space.

Example II.1.6. If we let $\gamma \in (0,\infty)$, let $u : B \to \mathbb{R}$ by setting $u(x) = |x|^{-\gamma}$ on $B \setminus \{0\}$ and $u(0) = 0$, and this γ tells me how spike this function is.

In particular, we can show that $u \in L^1(B)$ if and only if $\gamma < n$ for $B \subset \mathbb{R}^n$.

When away from 0, we have:

$$\frac{\partial}{\partial x_i}|x|^{-\gamma} = -\gamma \cdot \frac{x_i}{|x|^{\gamma+1}} \qquad \text{for } i = 1, \cdots, n.$$

Noting that $\left|\frac{\partial}{\partial x_i}u\right| \le \gamma \cdot \frac{1}{|x|^{\gamma+1}}$, which leads to $\frac{\partial}{\partial x_i}u \in L^1(B)$ which is if and only if $\gamma + 1 < n$.
Hence, if $\varphi \in C_c^\infty(B)$, by the divergence theorem, we have:

$$\int u\frac{\partial \varphi}{\partial x_i} = \lim_{\epsilon \to 0}\left(\int_{B\setminus B_\epsilon} u\frac{\partial \varphi}{\partial x_i}\right) = \lim_{\epsilon \to 0}\left(\int_{\partial B_\epsilon} u\varphi \cdot v_i - \int_{B\setminus B_\epsilon} \frac{\partial u}{\partial x_i}\varphi\right).$$

Now, we have:

$$\left|\int_{\partial B_\epsilon} u\varphi v_i\right| \le \int_{\partial B_\epsilon}|u\varphi| \le \|\varphi\|_{L^\infty(B)}\int_{\partial B_\epsilon}|x|^{-\gamma} = \|\varphi\|_{L^\infty(B)}\epsilon^{-\gamma}\cdot\omega_{n-1}\epsilon^{n-1}.$$

Hence, if $n - 1 - \gamma > 0$, or equivalently $n > \gamma + 1$, then:

$$\int_{\partial B_\epsilon} u\varphi v_i \to 0 \qquad \text{as } \epsilon \to 0.$$

So provided $\gamma + 1 < n$, then we have:

$$\int_B u\frac{\partial \varphi}{\partial x_i} = -\int_B \frac{\partial u}{\partial x_i}\varphi \qquad \text{for any } \varphi \in C_c^\infty(B).$$

Therefore, $-\gamma\frac{x_i}{|x|^{\gamma+2}}$ is the weak derivative of $|x|^{-\gamma}$ if $\gamma + 1 < n$. ♢

Remark II.1.7. This is precise by the dominated convergence theorem. ⌟

II.2 Sobolev Space

Definition II.2.1. If $\Omega \subset \mathbb{R}^n$ is open, $k \in \mathbb{N}$, $p \in [1, \infty]$, we then define the **Sobolev space** as:

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \text{ such that all weak derivatives of } u \text{ up to order } k \text{ exists, and are in } L^p(\Omega)\}.$$

With the norm defined as:

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k}\|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}, & \text{for } p \in [1,\infty), \\ \sum_{|\alpha|\le k}\|D^\alpha u\|_{L^\infty(\Omega)}, & \text{for } p = \infty. \end{cases}$$

⌟

Note that we can have $\|f\|_* := \big\||f|\big\|_*$ for $f \in \Omega \to \mathbb{R}^M$.

Remark II.2.2. $W^{0,p}(\Omega) = L^p(\Omega)$ for $p \in [1, \infty]$ with $\|\cdot\|_{W^{0,p}(\Omega)} = \|\cdot\|_{L^p(\Omega)}$. ⌟

Theorem II.2.3. If $\Omega \subset \mathbb{R}^n$ is open, $k \in \mathbb{N}$, and $p \in [1, \infty]$, then $W^{k,p}(\Omega)$ is Banach.

Proof. We genuinely consider the case when $k = 1$, and $k > 1$ follows by the same reasoning.
We define an inclusion map:

$$I : W^{1,p}(\Omega) \to \prod_{i=1}^{n+1} L^p(\Omega), \qquad u \mapsto \left(u, D^1 u, \cdots, D^n u\right)$$

for $u \in W^{1,p}(\Omega)$ where $D_k u = D^\alpha u$ for $\alpha = e_k = (0, \cdots, 0, 1, 0, \cdots, 0)$ in which 1 is at the k-th space.

It should be easy to observe that this map I is linear. We will see that $I\left(W^{1,p}(\Omega)\right)$ is closed in $\prod_{i=1}^{n+1} L^p(\Omega)$, which itself is a Banach space, hence $W^{1,p}(\Omega)$ will be seen to be closed, *i.e.*, I is an embedding.

If $(u_i, D_1 u_i, \cdots, D_n u_i) \to (v_0, v_1, \cdots, v_n) \in \prod_{i=1}^{n+1} L^p(\Omega)$, then we have that $u_i \to v_i$ and $D_u v_i \to v_u$ in $L^p(\Omega)$ for $k = 1, \cdots, n$. We now show that $v_u = D_u v_0$ so that $(v_0, v_1, \cdots, v_n) = (v_0, D_1 v_0, \cdots, D_n v_0) \in I\left(W^{1,p}(\Omega)\right)$.
By Hölder's inequality, if $\eta_i \to \eta$ in $L^p(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$, then:

$$\int_\Omega \eta_i \cdot \varphi \to \int_\Omega \eta \cdot \varphi \qquad \text{as } i \to \infty.$$

Hence, by the above, for $\varphi \in C_c^\infty(\Omega)$, we have:

$$\int_\Omega v_0 D_u \varphi = \lim_{i \to \infty} \int_\Omega u_i D_u \varphi = -\lim_{i \to \infty} \int_\Omega D_u v_i \varphi = -\int_\Omega v_k \varphi.$$

So by definition, $D_k v_0 = v_k$. Hence $I\left(W^{1,p}(\Omega)\right) \subset X$ is closed, thus as I is an embedding, we have that $W^{1,p}(\Omega)$ is Banach. □

Definition II.2.4. If $\Omega \subset \mathbb{R}^n$ is open, we set:

$$\mathcal{H}^k(\Omega) = W^{k,2}(\Omega),$$

which are Hilbert spaces with respect to the inner produce:

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\alpha| \le k} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u D^\alpha v.$$

⌟

Recall in Example I.3.9, we defined a space without weak derivatives, but we can then define another class of functions.

Definition II.2.5. If $\Omega \subset \mathbb{R}^n$ is open, $k \in \mathbb{N}$, and $p \in [1, \infty]$, then:

$$W_0^{k,p}(\Omega) = \overline{C_c^\infty(\Omega)} \text{ in } (W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}),$$

and set $H_0^k = W_0^{k,2}(\Omega)$. ⌟

Remark II.2.6. $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$, but later we will see that $u \in W_0^{k,p}(\Omega)$ if and only if $u \in W^{k,p}(\Omega)$ and $u = 0$ almost everywhere on $\partial\Omega$ (a.e. for $\partial\Omega$ is smooth enough) in some sense (trace). ⌟

Again, if we consider our old example in Example I.2.5, we will be using mollifiers to construct a sequence of $W^{1,p}(\Omega)$ that converges.

Example II.2.7. Recall in Example II.1.4, we consider that $|x|$ on $(-1, 1) \subset \mathbb{R}$ has one weak derivative, but not two, which lay in $L^p\big((-1, 1)\big)$ for any $p \in [1, \infty]$. Hence $|x| \in W^{1,p}\big((-1, 1)\big) \setminus W^{2,p}\big((-1, 1)\big)$.
Similarly, in $|x|^{-\gamma}$ on $B \subset \mathbb{R}^n$ from Example II.1.6, it has $L^1(B)$ weak derivative if $(\gamma + 1) < n$, so $|x|^{-\gamma} \in W^{1,p}(B)$ if $(\gamma + 1)p < n$.
By modifying this example, we can define:

$$u(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} |x - x_i|^{-\gamma} \in W^{1,p}(B),$$

where $\{x_i\}$ is dense in B (for example, $\mathbb{Q}^n \cap B$) if $(\gamma + 1)p < n$.

In fact, $|x|^{-\gamma} \in L^{\frac{np}{n-p}}(B)$ since $(\gamma + 1)p < n$ but $\frac{np}{n-p} > p$, so being in $W^{1,p}(\Omega)$ gives more integrability.

This is a special instance of Sobolev embedding. ♢

If $n = 1$, $|x|^{-\gamma}$ fails to be in $W^{1,p}(B)$ for any $\gamma \in (0, \infty)$. In fact, we can characterize $W^{1,p}$ if $n = 1$ and $p \in [1, \infty]$ in a clean way.

Definition II.2.8. Absolute Continuity.
A function $u : [a, b] \to \mathbb{R}$ is called absolutely continuous if it is differentiable almost everywhere on (a, b) with $u' \in L^1\big((a, b)\big)$ and for each $x \in [a, b]$, we have:

$$u(x) = u(a) + \int_a^x u'(s)ds.$$

⌟

Remark II.2.9. u' is then the weak derivative of u by the fundamental lemma of calculus on variation, since $u\varphi$ is absolutely continuous for $\varphi \in C_c^\infty\big((a, b)\big)$.
Hence, u being absolutely continuous and $u' \in L^p\big((a, b)\big)$ implies that $u \in W^{1,p}\big((a, b)\big)$. ⌟

Theorem II.2.10. If $(a, b) \subset \mathbb{R}$ and $p \in [1, \infty]$, then $u \in W^{1,p}\big((a, b)\big)$ if and only if $u = \overline{u}$ almost everywhere on (a, b) for $\overline{u}$ being absolutely continuous and $\overline{u}' \in L^p\big((a, b)\big)$.

Proof. ($\Longleftarrow$:) Remark II.2.9.
($\Longrightarrow$:) We define:

$$\tilde{u}(x) = \int_a^x u'(s)ds,$$

hence $\tilde{u}$ is absolutely continuous. Then $\tilde{u}$ and u have the same weak derivative, so $(\tilde{u} - u)' = 0$ almost everywhere in (a, b), and hence $u = \tilde{u} + C$ almost everywhere on (a, b) for some $C \in \mathbb{R}$. Hence we set $\overline{u} = \tilde{u} + C$ to conclude. □

II.3 Approximation of Sobolev Functions

We would need a class of new tools to have Sobolev function approximated.

Remark II.3.1. It is true that if $u \in L^p(\Omega)$ where $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^n$ is open, that u can be approximated by some sequence of $\{u_i\}_{i \in \mathbb{N}^+} \subset C_c^0(\Omega) \cap L^p(\Omega)$, *i.e.*, $\|u - u_i\|_{L^p(\Omega)} \to 0$ as $i \to \infty$.

- With $p = \infty$, the issues is with approximating an indicator function over an open interval. ⌟

To improve the above to a sequence $\{u_i\}_{i \in \mathbb{N}^+} \subset C_c^\infty(\Omega) \cap L^p(\Omega)$, we need a new tool.

Definition II.3.2. Mollifier.
A **mollifier** is a nonnegative test function $\eta \in C_c^\infty(\mathbb{R}^n)$ such that:

$$\int_{\mathbb{R}^n} \eta = 1.$$

A **standard mollifier**, $\eta \in C_c^\infty(B)$ is defined by:

$$\eta(x) = \begin{cases} c_n \exp\left(\frac{1}{4|x|^2 - 1}\right), & \text{if } |x| < \frac{1}{2}, \\ 0, & \text{if } |x| \geq \frac{1}{2}, \end{cases}$$

where c_n is chosen so that $\int_B \eta = 1$. ⌟

Remark II.3.3. Let η be defined as the standard mollifier:

- η is radially symmetric (*i.e.*, $\eta(x) = \eta(y)$ if and only if $|x| = |y|$).
- η is not real analytic. ⌟

Often, we "rescale" η and for $\epsilon > 0$ we set:

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right) \in C_c^\infty(B_\epsilon), \tag{3}$$

where B_ϵ is the ball of radius ϵ centered at $0 \in \mathbb{R}^n$.

Proposition II.3.4. η_ϵ is also a mollifier, *i.e.*, $\int_{\mathbb{R}^n} \eta_\epsilon = 1$.

Definition II.3.5. If $\Omega \subset \mathbb{R}^n$ is open and $\epsilon > 0$, we let:

$$\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}.$$

For $v \in L^1_{\text{loc}}(\Omega)$, we define the **mollification of u** at scale ϵ to be:

$$u_\epsilon(x) = (u * \eta_\epsilon)(x) = \int_{B_\epsilon(x)} \eta_\epsilon(x-y)u(y)dy = \int_{B_\epsilon(0)} \eta_\epsilon(y)u(x-y)dy,$$

so that $u_\epsilon = u * \eta_\epsilon \in C^\infty(\Omega_\epsilon)$. ⌟

Remark II.3.6.

- One does not need the standard mollifier to define (3).
- To see why $u_\epsilon \in C^\infty(\Omega_\epsilon)$, we note that:

$$D^\alpha(\eta_\epsilon * u)(x) = \int_\Omega D^\alpha \eta_\epsilon(x-y)u(y)dy, \qquad \text{for } x \in \Omega_\epsilon.$$

- $f * g$ is the convolution of f and g.

⌟

Lemma II.3.7. If $\Omega \subset \mathbb{R}^n$ is open, and $u \in L^1_{\text{loc}}(\Omega)$, then:

(i) $u_\epsilon \to u$ almost everywhere in Ω as $\epsilon \searrow 0$.

(ii) If $u \in C^0(\Omega)$, $u_\epsilon \to u$ uniformly on compact subsets of Ω.

(iii) If $u \in L^p(\Omega)$ for $p \in [1, \infty)$, $u_\epsilon \to u$ in L^p on compact subsets of Ω, *i.e.*, in $L^p_{\text{loc}}(\Omega)$.

Proof. The theorem relies on the Lebesgue differentiation theorem, which says that if $u \in L^1_{\text{loc}}(\Omega)$, then:

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - u(x)|dy \to 0 \qquad \text{as } r \to 0 \text{ for almost every } x \text{ in } \Omega.$$

□

Lemma II.3.8. Local Approximation.
If Ω is open, $n \in \mathbb{N}$, and $p \in [1, \infty)$, and $u \in W^{k,p}(\Omega)$, then:

(i) $\|u_\epsilon\|_{W^{k,p}(\Omega)} \leq \|u\|_{W^{k,p}(\Omega)}$,

(ii) $u_\epsilon \to u$ in $W^{k,p}$ on compactly contained subsets of Ω, *i.e.*, $W^{k,p}_{\text{loc}}(\Omega)$.

Proof. (i) By the definition of mollifiers, Fubinni and Hölder after establishing the following. We show that for $\alpha \in \mathbb{N}^\alpha$ with $|\alpha| \leq k$, we have:

$$D^\alpha(u_\epsilon) = \eta_\epsilon * (D^\alpha u) = (D^\alpha u)_\epsilon, \tag{4}$$

then the $L^1_{\text{loc}}(\Omega)$ convergence follows by the previous lemma part 3.
To check (4), for each $x \in \Omega_\epsilon$, we have:

$$\begin{aligned}
D^\alpha(u_\epsilon)(x) &= \int_\Omega D^\alpha_x \eta_\epsilon(x-\alpha)u(y)dy \\
&= (-1)^{|\alpha|}\int_\Omega D^\alpha_y \eta_\epsilon(x-y)u(y)dy = (-1)^{|\alpha|}(-1)^{|\alpha|}\int_\Omega \eta_\epsilon(x-y)D^\alpha_y u(y)dy.
\end{aligned}$$

□

Theorem II.3.9. Interior Approximation.
If $\Omega \subset \mathbb{R}^n$ is open, $k \in \mathbb{N}$, $p \in [1, \infty)$, and $u \in W^{k,p}(\Omega)$, then there is a sequence $\{u_i\} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $u_i \to u$ in $W^{k,p}(\Omega)$.

Proof. We first find a sequence of bounded open sets, $\{\Omega_i\} \Subset \Omega$ such that $\Omega = \bigcup_{i\geq 1}\Omega_i$ and $\Omega_i \Subset \Omega_{i+1}$ for $i \geq 1$, with $\Omega_0 = \emptyset$, which is a compact exhaustion of Ω.

For bounded cases, Evans uses $\Omega_i = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 1/i\}$ as a concrete construction.

Setting $\tilde{\Omega}_i = \Omega_{i+1} \setminus \overline{\Omega_i}$ for $i \geq 1$, we see that $\{\tilde{\Omega}_i\}$ forms a locally finite open cover of Ω, *i.e.*, $\Omega = \bigcup_{i\geq 1}\tilde{\Omega}_i$.
Therefore, we can find a **partition of unity** subordinate to the $\{\tilde{\Omega}_i\}$, *i.e.*, a sequence $\{\mathcal{P}_i\} \subset C^\infty_c(\Omega)$ with $\operatorname{supp}(\mathcal{P}_i) \Subset \tilde{\Omega}_i$ for $0 \leq \mathcal{P}_i \leq 1$ and $\sum_{i=1}^\infty \mathcal{P}_i(x) = 1$ for each $x \in \Omega$.
Noting that $\mathcal{P}_i \cup W^{k,p}(\Omega)$ which equals 0 on $\Omega \setminus \tilde{\Omega}_i$ with $\operatorname{supp}(\mathcal{P}_i u) \Subset \tilde{\Omega}_i$ for each $i \geq 1$.
For each $\epsilon > 0$, we can find a sequence $\{\epsilon_i\} \to 0$ being small enough, such that for $i \geq 2$:

$$v_i = \eta_{\epsilon_i} * (\mathcal{P}_i u) \in C^\infty_c(\Omega_{i+1} \setminus \overline{\Omega_{i-2}})$$

is such that $\|v_i - (\mathcal{P}_i u)\|_{W^{k,p}(\Omega)} \leq \frac{\epsilon}{2^i}$.
For $x \in \Omega$ we then let:

$$v(x) = \sum_{i=2}^\infty v_i(x) \in C^\infty(\Omega) \cap W^{k,p}(\Omega).$$

Then, $\|v - r\|_{W^{k,p}(\Omega)} \leq \|\sum_{i=2}^\infty (u\mathcal{P}_i - v_i)\|_{W^{k,p}(\Omega)} \leq \sum_{i=1}^\infty \frac{\epsilon}{2^i} = \epsilon$, we then set $u_i = v^{\epsilon_i}$ for each ϵ_i. □

Remark II.3.10. The $\{U_i\}_{i=1}^\infty \subset W^{k,p}(\Omega)$ in particular ensures the control of weak derivatives near boundary of Ω in the L^p sense. ⌟

Lemma II.3.11. Chain Rule.
If $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$, $\Omega \subset \mathbb{R}^n$ is open, and $u \in L^1_{\text{loc}}(\Omega)$ which is weakly differentiable in Ω,

then $f \circ u \in L^1_{\text{loc}}(\Omega)$ and has weak derivatives:

$$D_i(f \circ v) = (f' \circ u) D_i u \qquad \text{for } i = 1, \cdots, n.$$

Moreover, if $u \in W^{1,p}_{\text{loc}}(\Omega)$, then $(f \circ u) \in W^{1,p}_{\text{loc}}(\Omega)$ for $p \in [1, \infty]$.

Proof. Show $(f \circ u) \in L^1_{\text{loc}}(\Omega)$ directly, then show approximations converge locally. $\square$

III Boundaries of Sobolev Space

III.1 Global Approximation of Sobolev Space

Definition III.1.1. Boundary Regularity.
If $\Omega \subset \mathbb{R}^n$ is open and bounded, $k \in \mathbb{N}$, then $\partial\Omega$ is said to be $\boldsymbol{C^k}$ **regular** if for each $x \in \partial\Omega$, there is some $r > 0$ and a function $\gamma \in C^k(\mathbb{R}^{n-1})$ such that (up to relabeling the coordinate axes) we have:

$$\Omega \cap B_r(x) = \{y \in B_r(x) : y > \gamma(y_1, \cdots, y_{n-1})\}.$$

In particular, $\partial\Omega \cap B_r(x) = \{y \in B_r(x) : y = \gamma(y_1, \cdots, y_{n-1})\}$. ⌟

We can illustrate the graph as follows:

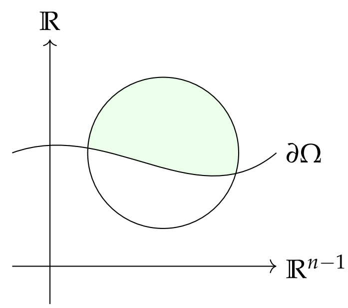

Figure III.1. The region above the curve (shaded).

Remark III.1.2. Consider the above definition:

- This works for **Lipschitz/Hölder** γ.
- This, equivalently, says that $\partial\Omega$ is an embedded $(n-1)$-manifold of class C^k.

With this definition, it rules out cusps, immersions/self-intersection, slitted domains, or polygonal domains from candidate of γ.

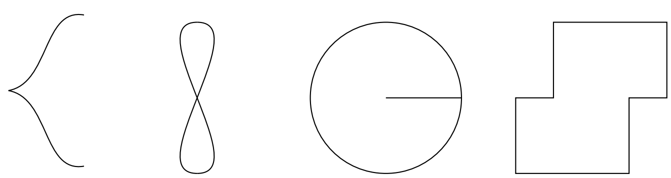

Figure III.2. Examples of cusps, immersions/self-intersection, slitted domains, or polygonal domains (from left to right). ⌟

Theorem III.1.3. Global Approximation.
If $\Omega \subset \mathbb{R}^n$ is open and bounded, with $\partial\Omega$ is C^1 regular (or Lipschitz), $k \in \mathbb{N}$, $p \in [1, \infty)$, and $u \in W^{k,p}(\Omega)$, then there is a sequence $\{u_i\}_{i=1}^{\infty} \in C^\infty(\overline{\Omega})$ such that $u_i \to u$ in $W^{k,p}(\Omega)$.

Example III.1.4. Consider on $S = B_1(0) \setminus \{x_1 > 0, x_2 = 0\} \subset \mathbb{R}^2$, then we have $f(r, \theta) = \theta \in W^{1,2}(S)$, and we cannot achieve the regularity as requested. ◊

Proof of Theorem III.1.3. First, if $x \in \partial\Omega$, as $\partial\Omega$ is C^1, there is some radius $r_x > 0$ such that:

$$\Omega \cap B_{r_x}(x) = \{y \in B_{r_x}(x) \mid y_n > \gamma(y_1, \cdots, y_{n-1})\}$$

of some $\gamma_x \in C^1(\mathbb{R}^{n-1})$.
We set $V_x = \Omega \cap B_{r_x/2}(x)$, and for each $y \in V_x$ and $\lambda, \epsilon > 0$, we set:

$$y_\epsilon = y + \lambda \cdot \epsilon \cdot e_n,$$

and for fixed $\lambda > 0$ large enough, we have that:

$$B_\epsilon(y_\epsilon) \subset B_{r_x}(x)$$

for all $y \in V_x$ and ϵ sufficiently small. We then set:

$$u^\epsilon(y) = u(y_\epsilon) \qquad \text{on } V_x,$$

i.e., translate u by $\lambda\epsilon e_n$, and let:

$$v_x^{\tilde{\epsilon}} - \eta_{\tilde{\epsilon}} * u^\epsilon \in C^\infty(V_x) \qquad \text{for all } \tilde{\epsilon} \in (0, \epsilon).$$

Now, we show that, as $\epsilon, \tilde{\epsilon} \to 0$, that:

$$v_x^{\tilde{\epsilon}} \to u \qquad \text{in } W^{k,p}(V_x).$$

To see this, we note that by the triangle inequality:

$$\|v_x^{\tilde{\epsilon}} - u\|_{W^{k,p}(v_x)} \leq \|v_x^{\tilde{\epsilon}} - u^\epsilon\|_{W^{k,p}(v_x)} + \|u^\epsilon - u\|_{W^{k,p}(v_x)}.$$

By the continuity of translation in L^P for $p \neq \infty$, we have that $u^\epsilon \to u$ in $W^{k,p}(V_x)$ as $\epsilon \searrow 0$.
Now, for a given tolerance $\delta > 0$, we choose $\epsilon > 0$ small enough to ensure $\|u^\epsilon - u\|_{W^{k,p}(v_x)} > \frac{\delta}{2}$. Then, for this fixed $\epsilon > 0$, we choose $\tilde{\epsilon} > 0$ sufficiently small so that as $v_x^{\tilde{\epsilon}} = \eta_{\tilde{\epsilon}} * u^\epsilon \to u^\epsilon$ in $W^{k,p}(V_x)$ as $\tilde{\epsilon} \to 0$, which is extending to be 0 outside Ω, then resulting in $\|v_x^{\tilde{\epsilon}} - u^\epsilon\|_{W^{k,p}(v_x)} < \frac{\delta}{2}$ also.
Therefore, since $\delta > 0$ is arbitrary, $v_x^{\tilde{\epsilon}} \to u$ in $W^{k,p}(v_x)$.
As $\partial\Omega$ is closed and bounded (as Ω is bounded), it is compact, and so for some $\{x_1, \cdots, x_N\} \subset \partial\Omega$ we have that:

$$\partial\Omega \subset \bigcup_{i=1}^N B_{\frac{r_x}{2}}(x_i) \qquad \text{for } r_{x_i} \text{ as above.}$$

By the previous reasoning, for any $\delta > 0$, we can find $v_i \in C^\infty(\overline{V_{x_i}})$ for $i = 1, \cdots, N$ such that $\|v_i - u\|_{W^{k,p}(V_{x_i})} < \delta$. We apply local/interior approximation to some $V_0 \subset \Omega$ that is open with $\Omega = V_0 \cup V_{x_1} \cup \cdots \cup V_{x_N}$ to get $v_0 \in C^\infty(\overline{V_0})$ with $\|v_0 - u\|_{W^{k,p}(v_0)} < \delta$.
We can then choose a partition of unity subordinate to $\{V_0, V_{x_1}, \cdots, V_{x_N}\}$ on $\overline{\Omega}$, call it $\{\mathcal{P}_i\}_{i=0}^N$, *i.e.*, $\text{supp}(\mathcal{P}_i) \Subset V_{x_i}$ for $i = 1, \cdots, N$, $\text{supp}(\mathcal{P}_0) \Subset V_0$, and $\mathcal{P}_i \in C^\infty(\overline{B + r_{x_i}(x_i)})$ for $i = 1, \cdots, N$, $\mathcal{P}_0 \in C^\infty(\overline{V_0})$, and for each $x \in \overline{\Omega}$, $\sum_{i=1}^N \mathcal{P}_i(x) = 1$.
We then define for this $\delta > 0$ that:

$$v = \sum_{i=0}^N \mathcal{P}_i \cdot v_i \in C^\infty(\overline{\Omega}).$$

Since we can write:

$$u = \sum_{i=0}^{N} \mathcal{P}_i \cdot u \qquad \text{on } \Omega.$$

Hence, we can use the product rule for weak derivatives to see that for $\alpha \in \mathbb{N}^n$ with $|\alpha| \le u$, we have:

$$\|D^\alpha v - D^\alpha u\|_{L^p(\Omega)} \le \sum_{i=0}^{N} \|D^\alpha(\mathcal{P}_i v_i) - D^\alpha(\mathcal{P}_i u)\|_{L^p(V_i)} \le C_k \sum_{i=1}^{N} \|v_i - u\|_{W^{k,p}(V_i)} \le C_k(N+1)\delta,$$

where C_k is a constant depending on the supremum of the derivative of the $\{\mathcal{P}_i\}_{i=0}^N$ of order at most k. Hence, for each $\delta > 0$, we can find $v \in C^\infty(\overline{\Omega})$ with $\|v - u\|_{W^{k,p}(\Omega)} < \tilde{\delta}$. Then, we let $u_i = v$ for $\tilde{\delta} = \frac{1}{i}$, then $\{u_i\}_{i=1}^\infty \subset C^\infty(\overline{\Omega})$ with $u_i \to u$ in $W^{k,p}(\Omega)$. □

III.2 Extensions of Sobolev Functions

Then, we think about how to extend the Sobolev functions to the whole domain.

Example III.2.1. Consider $u = 1 \in W^{k,p}\big((-1,1)\big)$ fo all $u \in \mathbb{N}$ and $p \in [1,\infty]$.

- Of course, we can extend u to $\mathbb{R}$ by setting $u = 0$ on $\mathbb{R} \setminus (-1,1)$. However, this extension does not lie in $W^{k,p}(\mathbb{R})$ for any $k \ge 1$.

Hence, we would need to ◊

Theorem III.2.2. Extension on $W^{1,p}$.
If $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ are open, bounded, with $\partial\Omega \in C^1$, $\Omega \Subset \tilde{\Omega}$, and $p \in [1,\infty)$, then there is a bounded linear map (called the **extension operator**):

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n),$$

such that for each $u \in W^{1,p}(\Omega)$ the following hold:

(i) $E(u)\big|_\Omega = u$ almost everywhere in Ω,

(ii) $\operatorname{supp}\big(E(u)\big) = \overline{\{u \neq 0\}} \subset \tilde{\Omega}$, *i.e.*, $u = 0$ almost everywhere on $\mathbb{R}^n \setminus \tilde{\Omega}$,

(iii) $\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$ for a constant $C > 0$ depending only on $\Omega, \tilde{\Omega}$, n, and p. (Note that C does not depend on u.)

Example III.2.3. Contiued from Example III.2.1...
Consider the original equation, we can consider $\tilde{\Omega} = (-2,2)$ so we have $E(u) \in W^{1,p}(\mathbb{R})$. ◊

Remark III.2.4. This works for $W^{k,p}$ and $p = \infty$ but the construction is more delicate, provided we assume that $\partial\Omega$ is in C^k.

In fact, this really shows the existence of an extension operator:

$$\hat{E} : W^{k,p}(\Omega) \to W_0^{k,p}(\tilde{\Omega})$$

if $\partial\Omega$ is C^k. ⌟

Proof of Theorem III.2.2. Let's first prove it in the case that $\partial\Omega = \{x_n = 0\}$ in a ball, and $u \in C^\infty(\overline{\Omega})$.

(i) **Step 1:** Flat $\partial\Omega$.
Fix $x \in \partial\Omega$ and suppose that in $B = B_r(x)$ for some $r > 0$, we have:

$$B^+ := B \cap \{y \in \mathbb{R}^n \mid y \geq 0\} \subset \Omega,$$
$$B^+ := B \cap \{y \in \mathbb{R}^n \mid y \leq 0\} \subset \mathbb{R}^n \setminus \Omega.$$

Since $u \in C^\infty(\overline{\Omega})$ is defined on B^+, we reflect u in $\{x_n = 0\}$ and for $y \in \partial B^+ = \partial B^-$, let:

$$\overline{u}(y) = \begin{cases} u(y), & \text{if } y \in B^+, \\ -3u(y_1, \cdots, y_{n-1}, -y_n) + 4u(y_1, \cdots, y_{n-1}, -\frac{y_n}{2}) & \text{if } y \in B \setminus B^+, \end{cases}$$

then $\overline{u} \in C^1(B)$ (check by direct computation) and $\|\overline{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}$, as C is not defined on u.

(ii) **Step 2:** Staighten $\partial\Omega$
If the $\partial\Omega$ is not $\{x_n = 0\}$, since the $\partial\Omega \in C^1$, we can find some diffeomorphism (C^1 map with inverse), such that if $y \in B_r(x)$ for some $r > 0$:

$$\Phi_x(y) = \big(y_1, \cdots, y_{n-1}, y_n - \gamma(y_1, \cdots, y_{n-1})\big).$$

If $\gamma_x \in C^1(\mathbb{R}^{n-1})$ with $\partial\Omega \cap B_r(x) = \{y \in B_r(x) \mid y_n > \gamma(y_1, \cdots, y_n)\}$. So $\partial\Omega \cap B_r(x)$ maps to $\{x_n = 0\}$ with Φ_x.
And since Φ_x is invertible, there is an open set $x \in W \subset B_r(x)$ with $\Phi_x(W) = B_s(z)$ and $\Phi_x(\Omega \cap W)$ where $x = \Phi_x(x) \in \mathbb{R}^{n-1} \times \{0\}$ and $s > 0$.
We then set $u' = u \circ \Phi_x^{-1}$ on $B_s(z)$.
Hence, $u' = C^1\big(\overline{B_s(z)}\big)$ so we can apply step 1 to extend u' to a function $\overline{u'} \in C^1\big(\overline{B_s(z)}\big)$ with:

$$\|\overline{u'}\|_{W^{1,p}\left(B_s(z)\right)} \leq C\|u'\|_{W^{1,p}\left(B_s(z)\right)}.$$

Then, $\overline{u'} \circ \Phi_x \in C^1(W)$ and as Φ_x is C^1, the chain rule gives:

$$\|\overline{u}\|_{W^{1,p}(W)} \leq \tilde{C}\|u\|_{W^{1,p}(W\cap\Omega)} \leq \tilde{C}\|u\|_{W^{1,p}(\Omega)}$$

(iii) **Step 3:** Patching up.
As $\partial\Omega$ is compact, there are finitely many $\{x_1, \cdots, x_n\} \subset \partial\Omega$ for which we have open sets $\{W_{x_1}, \cdots, W_{x_n}\} \subset \mathbb{R}^n$ with $x_i \in W_i$ for each $i = 1, \cdots, N$ and such that $\partial\Omega \subset \bigcup_{i=1}^N W_{x_i}$, *i.e.*, by **step 2** for each $x \in \partial\Omega$.
To each x_i, we have an associated local boundary extension, $\overline{u_i} \in C^1(W_i)$, $\overline{u} = u$ on $W_i \cap \Omega$ almost everywhere.

Now, we choose some $\Omega_0 \Subset \Omega$ as open such that:

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} W_i, \qquad \text{with } W_i = W_{x_i}.$$

Then, we fix a partition of unity subordinate to this cover, $\{\mathcal{P}_i\}_{i=0}^{N}$, so $\operatorname{supp}(\mathcal{P}_i) \Subset W_i$, and $\sum_{i=0}^{N} \mathcal{P}(x) = 1$.
We then set:

$$\overline{u} = \sum_{i=0}^{N} \mathcal{P}_i \overline{u_i} \in C^1(\mathbb{R}^n),$$

and we have:

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Thus, $\overline{u}$ satisfies (i) and (iii) if $u \in C^1(\overline{\Omega})$, by considering $\chi \cdot \overline{u}$ for some $\chi \in C_c^\infty(\tilde{\Omega})$ with $\chi = 1$ in Ω (where one way of construction is $\chi = \eta_\epsilon * \mathbb{1}_{\Omega_\epsilon}$), then $\chi \cdot \overline{u}$ satisfies (i), (ii), and (iii).

We then define $E(u) = \chi \cdot \overline{u}$ if $u \in C^1(\overline{\Omega})$, which we note is linear and bounded. ($\overline{u+v} = \overline{u} + \overline{v}$).
By global approximation, if $u \in W^{1,p}(\Omega)$, there is a sequence:

$$\{u_i\} \subset C^\infty(\overline{\Omega}) \subset C^1(\overline{\Omega}),$$

with $u_i \to u$ in $W^{1,p}(\Omega)$.
As $\{E(u_i)\} \subset W^{1,p}(\mathbb{R}^n)$ we have:

$$\|E(u_i) - E(u_j)\|_{W^{1,p}(\mathbb{R}^n)} = \|E(u_i, u_j)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_i - u_j\|_{W^{1,p}(\Omega)}.$$

as $u_i \to u$ in $W^{1,p}(\Omega)$ that $\{E(u_i)\}$ is Cauchy, hence convergent as $W^{1,p}(\tilde{\Omega})$ is Banach, and we denote $E(u) := \lim_{i\to\infty} E(u_i)$ in $W^{1,p}(\tilde{\Omega})$.
One can check that this limit is independent of the approximating sequence. Hence, $E(u)$ is well-defined for any $u \in W^{1,p}(\Omega)$ and satisfies (i), (ii), and (iii). $\square$

Remark III.2.5. If $\partial\Omega \in C^k$, we can use:

$$\sum_{i=1}^{k} c_i \cdot u\left(y_1, \cdots, y_{n-1}, -\frac{y_n}{i}\right)$$

in the flat case (step 1) where $\sum_{i=1}^{k} c_i \left(-\frac{1}{i}\right)^m = 1$ for $m = 1, 2, \cdots, k-1$. ⌟

III.3 Trace of Sobolev Functions

Let $\Omega \subset \mathbb{R}^n$ be open. If $u \in C^0(\overline{\Omega})$, then $u\big|_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ is well defined. But if $u \in L^p$, since $L^n(\partial\Omega) = 0$ (provided $\partial\Omega$ is C^1), $u\big|_{\partial\Omega}$ doesn't have a unique or canonical choice.

We want to define $u\big|_{\partial\Omega}$ for Sobolev functions:

Theorem III.3.1. Trace on $W^{1,p}$.
If $\Omega \subset \mathbb{R}^n$ is open, bounded, $\partial\Omega$ is C^1, and $p \in [1, \infty)$, then there is a bounded linear map:

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega),$$

called the **trace operator**, such that for each $u \in W^{1,p}(\Omega)$:

- $T(u) = u\big|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ almost everywhere, and
- $\|T(u)\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$, where C is independent of u.

Remark III.3.2.

- We call $T(u) \in L^p(\partial\Omega)$ the trace of u.
- If $u \in W^{k,p}(\Omega)$, $T(D^\alpha u)$ is well defined for all $|\alpha| \le k-1$.
- There is no trace operator $T : L^p(\Omega) \to L^p(\partial\Omega)$. For example, we let $\Omega = B_1(0) \subset \mathbb{R}^n$ and $u = (1 - |x|^\alpha)^{-1}$ for $\alpha p < n$.
- This also works for $p = \infty$.
- Since the trace operator is bounded linear operator, we can utilize this to show that $C_c^\infty(\Omega)$ cannot be dense in $W^{1,p}(\Omega)$.

⌟

Proof of Theorem III.3.1. We prove it locally, and then "patch up" ad for the extension operator.
Let $u \in C^1(\overline{\Omega})$, $x \in \partial\Omega$, and set $B = B_r(x)$, $\hat{B} = B_{\frac{r}{2}}(x)$ under the assumption that $\partial\Omega \cap B = \{y_n = 0\} \cap B$.
We fix $\mathcal{P} \in C_c^\infty(B)$ with $\mathcal{P} \ge 1$ on $\hat{B}$. We denote $\Gamma := \partial\Omega \cap \hat{B}$. Then, we apply the divergence theorem:

$$\begin{aligned} \int_\Gamma |u|^p dy' \le \int_{\{y_n = 0\}} \mathcal{P}|u|^p dy' &= -\int_{B^+ := B \cap \{y_n > 0\}} \frac{\partial}{\partial y_n}\left[\mathcal{P}|u|^p\right] dy \\ &= \int_{B^+} \left(\frac{\partial \mathcal{P}}{\partial y_n} \cdot |u|^p + \mathcal{P} \cdot p u^{p-1} \cdot \operatorname{sgn}(u) \cdot \frac{\partial v}{\partial y_n} \right) dy. \end{aligned}$$

Then, we use the Young's inequality that $|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}$ with $p := \frac{p}{p-1}$ and $q := p$ on $\left|u^{p-1} \frac{\partial u}{\partial y_n}\right|$:

$$\int_{B^+} \left(\frac{\partial \mathcal{P}}{\partial y_n} \cdot |u|^p + \mathcal{P} \cdot p u^{p-1} \cdot \operatorname{sgn}(u) \cdot \frac{\partial v}{\partial y_n} \right) dy \le C \int_\Omega \left(|u|^p + |Du|^p\right) dy.$$

Hence, by straightening $\partial\Omega$, we obtain that:

$$\int_\Gamma |u|^p \le C \int_\Omega \left(|u|^p + |Du|^p\right) dy \qquad \text{for any } \Gamma \subset \partial\Omega \text{ being open.}$$

By compactness of $\partial\Omega$, we have:

$$\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

So we set $T(u) = u\big|_{\partial\Omega}$ if $u \in C^1(\overline{\Omega})$, then this satisfies the two conditions.
By using the global approximation for $u \in W^{1,p}(\Omega)$, as in the proof of extension operator, we can then

define that:

$$T(u) = \lim_{i\to\infty} T(u_i) \qquad \text{if } u_i \to u \text{ in } W^{1,p}(\Omega).$$

Since approximations coverage uniformly if $u \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ which implies that $T(u) = u\big|_{\partial\Omega}$ is this case, hence satisfying the conditions. □

Note that when we have the **Dirichlet's condition**, we automatically obtain that the trace operator is zero.

Theorem III.3.3. Trace Zero in $W^{1,p}$.
If $\Omega \subset \mathbb{R}^n$ is open, bounded, $\partial\Omega \in C^1$ and $p \in [1,\infty)$, then:

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \mid T(u) = 0\}.$$

Proof. ($\subset$:) if $u \in W_0^{1,p}(\Omega)$, then there are $\{u_i\} \subset C_c^\infty(\Omega)$ with $u_i \to u$ in $W^{1,p}(\Omega)$, then as $T(u_i) = 0$ and T is bounded, then $T(u) = 0$ as well.
($\supset$:) If $u \in W^{1,p}(\Omega)$ and $T(u) = 0$. By construction, there are $\{u_i\} \subset C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ with $T(u_i) \to 0$ in $L^p(\partial\Omega)$. Again, by straightening $\partial\Omega$ and patching up, we can prove the case $u \in W^{1,p}(\mathbb{R}^n_+)$ with $\overline{\{u \neq 0\}}$ compact, where $R^\infty_+ = \{y \in \mathbb{R}^n \mid y_n > 0\}$ and $T(u) = 0$. Therefore, for $x' \in \mathbb{R}^{n-1}$ and $x_n \geq 0$, by the fundamental theorem of calculus, we have:

$$|u_i(x', x_n)| \leq |u_i(x',0)| + \left| \int_0^{x_n} \frac{\partial}{\partial x_n} u_i(x,s) ds \right|.$$

Therefore, by integration and using $(a+b)^p \leq \frac{a^p}{2} + \frac{b^p}{2}$, we have:

$$\begin{aligned}
2\int_{\mathbb{R}^{n-1}} |u_i(x', x_n)^p| dx' &\leq \int_{\mathbb{R}^{n-1}} |u_i(x',0)| + \int_{\mathbb{R}^{n-1}} + \left[\int_{\mathbb{R}^{n-1}} \int_0^{x_n} \left| \frac{\partial}{\partial x_n} u_i(x,s) \right| ds \right]^p dx \\
&\leq \int_{\mathbb{R}^{n-1}} |u_i(x',0)|^p dx' + \int_{\mathbb{R}^{n-1}} \left[\left(\int_0^{x_n} 1^{\frac{p}{p-1}} ds \right) \left(\int_0^{x_n} \left| \frac{\partial}{\partial x_n} u_i(x',s) \right|^p ds \right) \right] \\
&\leq \underbrace{\int_{\mathbb{R}^n} |u_i(x',0)|^p dx'}_{\|u_i\|^p_{L^p(\partial\Omega)} \to 0} + \int_{\mathbb{R}^{n-1}} x_n^{p-1} \int_0^{x_n} |Du_i(x',s)|^p dx'.
\end{aligned}$$

Hence, by sending $i \to \infty$, we deduced that:

$$\int_{\mathbb{R}^{n-1}} |u(x', x_0)|^p dx' \leq \frac{x_n^{p-1}}{2} \int_{\mathbb{R}^{n-1}} \int_0^{x_0} |D_u(x',s)|^p ds dx'.$$

Now, let $\mathcal{P} \in C^\infty\big([0,\infty)\big)$ with $\mathcal{P} = 1$ on $[0,1]$, $\mathcal{P} = 0$ on $[2,\infty)$ and $0 \leq p \leq 1$. We set $\mathcal{P}_m(x) = \mathcal{P}(mx_n)$ for $m \geq 1$, $x \in \mathbb{R}^n_+$ and define $w_m(x) = u(x) \cdot \big(1 - \mathcal{P}_m(x)\big)$. Then, we have that:

$$\begin{cases}
\dfrac{\partial w_m}{\partial x_n} = \dfrac{\partial u}{\partial x_n} \cdot (1 - \mathcal{P}_m) - m\mathcal{P}'(m_)u, \\
D_{x'} w_m = D_{x'} u (1 - \mathcal{P}_m).
\end{cases}$$

Hence, we estimate:

$$\begin{aligned}\int_{\mathbb{R}^n_+} |Dw_m - Du|^p &= \int_{\mathbb{R}^n_+} \left| D_{x'}u(-\mathcal{P}_m) + \frac{\partial}{\partial x_n}(-\mathcal{P}_m) - mu\mathcal{P}'(m\_) \right|^p \\ &\leq \underbrace{\frac{1}{2}\int_{\mathbb{R}^n_+} |Du|^p|\mathcal{P}_m|^p}_{\to 0 \text{ when } m\to\infty} + \frac{1}{2}m^p\|\mathcal{P}'\|^p_{C^0(\mathbb{R})}\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}} |u|^p dx' ds.\end{aligned}$$

Now, we just consider the remaining part. Note that we can set $C \geq \frac{1}{2}\|\mathcal{P}'\|^p$, so we have:

$$\begin{aligned}&\frac{1}{2}m^p\|\mathcal{P}'\|^p_{C^0(\mathbb{R})}\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}} |u|^p dx' ds \leq m^p C\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}} |u(x',s)|^p dx' ds \\ &\leq Cm^p\int_0^{\frac{2}{m}}\left[\frac{s^{p-1}}{2}\int_{\mathbb{R}^{n-1}}\int_0^s |Du(x',t)|^p dx dt\right] ds \leq Cm^p\left(\int_0^{\frac{2}{m}}\frac{s^{p-1}}{2}ds\right)\left(\int_{\mathbb{R}^n}\int_0^{\frac{2}{m}} |Du(x',t)|^p ds dt\right) \\ &= Cm^p\left(\left(\frac{2}{m}\right)^p\cdot\frac{1}{p}\right)\left(\int_0^{\frac{2}{m}}\int_{\mathbb{R}^{n-1}} |Du(x',t)|^p dx' dt\right) \leq \tilde{C}\left(\int_{\mathbb{R}^n_+} |Du|^p - \int_{\frac{2}{m}}^{\infty}\int_{\mathbb{R}^{n-1}} |Du|^p\right) \to 0\end{aligned}$$

as $m \to \infty$.
Hence, $Dw_m \to Dv$ in $L^p(\mathbb{R}^n_+)$ and as $w_m \to u$ in $L^p(\mathbb{R}^n_+)$, we have that $w_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$.
Noting that $w_m = 0$ for $x_n \in \left[0, \frac{1}{m}\right]$, by having $v_m = \eta_{\frac{1}{2m}} * w_m \in C_c^\infty(\mathbb{R}^n_+)$ with $v_m \to v$ in $W^{1,p}(\mathbb{R}^n_+)$, and hence $v \in W_0^{1,p}(\mathbb{R}^n_+)$, as desired. □

III.4 Sobolev Inequalities

We saw that $|x|^{-\gamma} \in W^{1,p}(B)$ implies that $|x|^{-\gamma} \in L^{\frac{np}{n-p}}(B)$, *i.e.*, it gained integrability as $p < \frac{np}{n-p}$ for $p \neq \infty$. This is a general principle, namely belonging to a Sobolev space means the function belongs to other (ideally more regular spaces).

However, we need to take care of the cases:

$$1 \leq p < n, \qquad p = n, \qquad \text{or} \qquad n < p \leq \infty.$$

Remark III.4.1. Gagliardo-Nirenberg-Sobolev Inequality.
For $1 \leq p < n$, we try try to see that if:

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for some $C > 0$ independent of $u \in C_c^\infty(\mathbb{R}^n)$. We see that q is pre-determined by p.
To see this, if $u \neq 0$, we set:

$$u_\lambda(x) = u(\lambda x) \text{ for } \lambda > 0 \text{ and } x \in \mathbb{R}^n.$$

Therefore, for any u_λ, we have:

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)},$$

but we have:

$$\begin{cases} \|u_\lambda\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u_\lambda|^q = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \int_{\mathbb{R}^n} \lambda^{-n}|u|^q dy = \lambda^{-n}\|u\|^q_{L^q(\mathbb{R}^n)}, \\ \|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |Du_\lambda|^p = \int_{\mathbb{R}^n} |D\big(u(\lambda x)\big)|^p dx = \int_{\mathbb{R}^n} \lambda^p |Du(\lambda x)|^p dx = \int_{\mathbb{R}^n} |Du(y)|^q dy = \lambda^{p-n}\|Du\|_{L^p(\mathbb{R}^n)}. \end{cases}$$

Hence, we have that $\lambda^{-\frac{n}{q}}\|u\|_{L^q(\mathbb{R}^n)} \leq c\lambda^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}$, which is equivalently that:

$$\|u\|_{L^2(\mathbb{R}^n)} \leq \lambda^{1-\frac{n}{p}+\frac{n}{q}}\|Du\|_{L^p(\mathbb{R}^n)}.$$

If $1-\frac{n}{p}+\frac{n}{q} \neq 0$, by sending $\lambda \nearrow \infty$ or $\searrow 0$, we get a contradiction to $u \neq 0$. Thus if the assumption holds, we have $1-\frac{n}{p}+\frac{n}{q} = 0$, this is equivalently $q = \frac{np}{n-p}$. ⌟

Definition III.4.2. Sobolev Conjugate.
For $p \in [1,n)$, the **Sobolev conjugate** of p is:

$$p^* = \frac{np}{n-p},$$

so that we have $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and $p^* > p$. ⌟

Theorem III.4.3. Gagliardo-Nirenberg-Sobolev Inequality (GNS).
If $p \in [1,n)$, then there exists a constant $C > 0$ such that:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } u \in C^1_c(\mathbb{R}^n).$$

Proof. The idea is to use the fundamental theorem of calculus and the general Hölder inequality.
Let's assume $p = 1$ as $u \in C^1_c(\mathbb{R}^n)$ for each $i = 1, \cdots, n$ and $x \in \mathbb{R}^n$ we have by the fundamental theorem of calculus that:

$$|u(x)| \leq \int_{-\infty}^{x_i} \left|\frac{\partial u}{\partial x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)\right| dy_i.$$

Noting that $1^* = \frac{n\cdot 1}{n-1} = \frac{n}{n-1}$ so we have:

$$|u(x)|^{\frac{n}{n-1}} = (|u(x)|^n)^{\frac{1}{n-1}} \leq \prod_{i=1}^n \left[\int_{-\infty}^{\infty} |Du(x_1, \cdots, y_i, \cdots, x_n)| dy_i\right]^{\frac{1}{n-1}}.$$

Then, when we integrate with respect to x_1, we obtain:

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} \leq \left(\int_{-\infty}^{\infty} |Du| dy_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^n \left[\int |Du| dy_i\right]^{\frac{1}{n-1}} dx_1.$$

Consider generalized Hölder for $p_i = n-1$, then we have:

$$\sum_{i=1}^{n-1} \frac{1}{p_i} = \frac{n-1}{n-1} = 1,$$

so we have have that:

$$\int_{-\infty}^{\infty} \prod_{i=2}^{n} \left[\int |f_i| dy_i \right]^{\frac{1}{n-1}} dx_1 \leq \prod_{i=2}^{n} \left(\int_{|f|_i}^{n-1} dx_1 \right)^{\frac{1}{n-1}}.$$

We use $f_i = (\int |Du| dx_i)^{\frac{1}{n-1}}$, and with Fubinni, we have:

$$\begin{aligned}\int |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \left(\int |Du| dy_1 \right)^{\frac{1}{n-1}} \int \prod_{i=2}^{n} \left[\int |Du| dy_i \right]^{\frac{1}{n-1}} dx_1 \\ &\leq \left(\int |Du| dy_1 \right)^{\frac{1}{n-1}} \cdot \left[\prod_{i=2}^{n} |Du| dx_1 dy_i \right]^{\frac{1}{n-1}}.\end{aligned}$$

Then, we integrate with respect to x_2 to obtain that:

$$\iint |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\iint |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \int \left(\int |Du| dy_1 \right)^{\frac{1}{n-1}} \cdot \prod_{i=3}^{n} \left(\iint |Du| dx_1 dy_i \right)^{\frac{1}{n-1}} dx_2.$$

For simplicity, we use $I_i := f_i$ so again, by generalized Hölder and Fubinni, we have:

$$\iint |u|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\iint |Du| dy_1 dx_2 \right)^{\frac{1}{n-1}} \left(\iint |du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \left(\prod_{i=3}^{n} \iiint |Du| dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}}.$$

Repeat inductively, we have:

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int |Du| dx_1 \cdots dx_{i-1} dy_i dx_{i+1} \cdots dx_n \right)^{\frac{1}{n-1}} = \|Du\|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}.$$

Hence, this holds for $p = 1$ case.

Now, for any $p \in (1, n)$, of we consider $|u|^\gamma$, we go from the $p = 1$ case that:

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} \right)^{\frac{n-1}{n}} &= \left\| |u|^\gamma \right\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \left\| D|u|^\gamma \right\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} D(|u|^\gamma) = \int_{\mathbb{R}^n} \gamma \cdot |u|^{\gamma-1} |Du| \\ &\leq \gamma \|Du\|_{L^p(\mathbb{R}^n)} \cdot \left\| |u|^{\gamma-1} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)}.\end{aligned}$$

If we choose γ such that:

$$\frac{\gamma n}{n-1} = (\gamma - 1) \frac{p}{p-1} \iff \gamma = \frac{p(n-1)}{n-p} > 1.$$

Hence, we have $\frac{\gamma n}{n-1} = \cdots = \frac{np}{n-p}$, then if $\| \|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)} \neq 0$, *i.e.*, when $u \neq 0$, we can divide by $\| \|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)}$ on both sides, to obtain that:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} = \left(\int |u|^{\frac{np}{n-p}} \right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \frac{p(n-1)}{n-p} \cdot \|Du\|_{L^p(\mathbb{R}^n)}.$$

□

Remark III.4.4.

- We need $u \in C_c^1(\mathbb{R}^n)$, as for example, we have $u = 1$ fails for the inequality.

- The constant $C = \frac{p(n-1)}{n-p}$ is not optimal in the proof. In fact, if $p = 1$, GNS is equivalent to the isoperimetric inequality.

  If we choose $u \in C_c^1(\mathbb{R}^n)$ with $\begin{cases} u = 1 \text{ on embedding of } \Omega, \Omega_\epsilon \\ |\nabla u| \sim \frac{1}{\epsilon} \text{ on } \Omega_\epsilon \setminus \Omega, \end{cases}$ and then we ahve:

$$\|\nabla u\|_{L^1(\mathbb{R}^n)} \sim \text{Area of } \partial\Omega,$$

  or we have that:

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \sim (\text{Volume of } \Omega)^{\frac{n-1}{n}}.$$

Theorem III.4.5. GNS for $W^{1,p}$.
If $\Omega \subset \mathbb{R}^n$ is open, bounded, $\partial\Omega$ is C^1, $p \in [1, n)$, then:

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$$

and there exists a $C > 0$, depending on Ω, p, n such that:

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \qquad \text{for all } u \in W^{1,p}(\Omega).$$

Proof of sketch. We extend u to $W^{1,p}(\mathbb{R}^n)$ function, mollify it, and apply GNS for $\mathbb{R}^n$. □

Theorem III.4.6. Poincaré Inequality.
If $\Omega \subset \mathbb{R}^n$ is open, bounded, $p \in [1, n)$, and $q \in [1, p^*]$, there is a constant $c > 0$, depending on Ω, p, q, n such that:

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \qquad \text{for } u \in W_0^{1,p}(\Omega)$$

In particular, as $p \leq p^*$, which implies that $\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$.

Proof of sketch. As $u \in W_0^{1,p}(\Omega)$, this implies that $\{u_i\} \subset C_c^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^n)$, and with $u_i \to u$ in $W^{1,p}$, we can then apply GNS to u_i. □

IV Hölder Spaces

IV.1 Hölder Space

We treat the case when $p > n$.

Definition IV.1.1. Hölder Continuous.
If $\Omega \subset \mathbb{R}^n$ is open, we say that $u : \Omega \to \mathbb{R}$ is **Hölder continuous** with exponents $\gamma \in (0, 1]$ if there is a constant $C > 0$ such that:

$$|u(x) - u(y)| \leq C \cdot |x - y|^\gamma \qquad \text{for all } x, y \in \Omega.$$

When $\gamma = 1$, we have the Lipschitz condition. ⌟

Definition IV.1.2. The γ-Hölder semi-norm is defined as:

$$[u]_{C^{0,\gamma}(\overline{\Omega})} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left(\frac{|u(x) - u(y)|}{|x - y|^\gamma} \right).$$

⌟

The Hölder semi-norm is not a norm since for the constant function 1, we have:

$$[1]_{C^{0,\gamma}(\overline{\Omega})} = 0,$$

but 1 is not the zero function.

Definition IV.1.3. The γ-Hölder norm is defined using the semi-norm:

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \|u\|_{C^0(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}.$$

⌟

The Hölder space, for $k \in \mathbb{N}$, are then:

$$C^{k,\gamma}(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}) \mid \|D^\alpha u\|_{C^{0,\gamma}(\overline{\Omega})} < \infty \text{ for } |\alpha| \leq k\},$$

in which if we equip with the norm, we have:

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + \sum_{|\alpha| \leq k} [D^\alpha u]_{C^{0,\gamma}}(\overline{\Omega}).$$

Remark IV.1.4.

- $\left(C^{k,\gamma}(\overline{\Omega}), \|\cdot\|_{C^{k,\gamma}(\overline{\Omega})}\right)$ is Banach (by Arzelá-Ascoli).
- If $\gamma > 1$, then the γ-Hölder functions are constant.
- If Ω is bounded and convex, $u \in C^{k+1}(\overline{\Omega})$ implies that $u \in C^{k,\gamma}(\overline{\Omega})$ for all $\gamma \in (0, 1]$.

⌟

Example IV.1.5. If $\gamma \in (0,1]$, we let $u(x) = |x|^\gamma$ on $(-1,1)$ is $C^{0,\gamma}$ but not $C^{0,\tilde{\gamma}}$ for $\tilde{\gamma} \in (\gamma, 1]$. ◊

Theorem IV.1.6. Morrey Inequality.
If $p \in (n, \infty]$ then there is a constant $C > 0$ depending on p and n such that for $u \in C^1(\mathbb{R}^n)$ we have:

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

where $\gamma = 1 - \frac{n}{p}$ (where $\gamma = 1$ if $p = \infty$).

Proof. The idea is to bound $\|u\|_{C^0(\overline{\Omega})}$ and $[u]_{C^{0,\gamma}(\overline{\Omega})}$ by the fundamental theorem of calculus and Hölder inequality.
Fix $r > 0$, $x \in \mathbb{R}^n$, and for $\omega \in \partial B_1(0) = \mathbb{S}^{n-1}$, we have by the fundamental theorem of calculus that for $0 < s < r$, we have:

$$u(x+sw) - u(x) \leq \int_0^s \left|\frac{d}{dt}\big(u(x+tw)\big)\right| dt = \int_0^s |Du(x+tw)\cdot w|\, dt \leq \int_0^s |Du(x+tw)| dt.$$

Then, by using Fubini, we have:

$$\int_{\mathbb{S}^{n-1}} |u(x+sw) - u(x)| dS \leq \int_{\mathbb{S}^{n-1}} \int_0^s |Du(x+tw)| dt dS = \int_0^s \int_{\mathbb{S}^{n-1}} |Du(x+tw)| \frac{t^{n-1}}{t^P n-1} dS dt.$$

Now, we set $y = x + tw$ and $t = |x-y|$ (since $|w| = 1$) with $dy = b^{n-1} dS dt$, we have:

$$\int_0^s \int_{\mathbb{S}^{n-1}} |Du(x+tw)| \frac{t^{n-1}}{t^P n-1} dS dt \leq \int_{B_r(x)} |u(y) - u(x)| dy.$$

Then, by integrating from 0 to r after multiplying by $\mathbb{S}^{n-1}$, we have:

$$\begin{aligned} \int_0^r \int_{\mathbb{S}^{n-1}} |u(x+sw) - u(x)| dS s^{n-1} ds = \int_{B_r(x)} |u(y) - u(x)| dy &\leq \int_0^r s^{n-1} \int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy ds \\ &= \frac{r^n}{n} \int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy. \end{aligned} \tag{5}$$

By denoting $⨍_A f = \frac{1}{m(A)} \int_A f$, we have:

$$\begin{aligned} |u(x)| &= ⨍_{B_1(x)} |u(x)| \leq ⨍_{B_1(x)} |u(x) - u(y)| dy + ⨍_{B_1(x)} |u(y)| dy \\ &\leq \frac{m\big(B_1(0)\big)}{n} \int_{B_1(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy + ⨍_{B_1(x)} |u(y)| \\ &\leq C\|Du\|_{L^p\left(B_1(x)\right)} \cdot \left(\int_{B_1(x)} |x-y|^{(1-n)\cdot \frac{p}{p-1}}\right)^{\frac{p-1}{p}} + C\|u\|_{L^p\left(B_1(x)\right)} \leq C\|u\|_{W^{1,p}\left(B_1(x)\right)}, \end{aligned}$$

as we have $\frac{(n-1)p}{p-1} < n$ and we can consider this as a Sobolev spike, *i.e.*:

$$\|u\|_{C^0(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Now, if $x, y \in \mathbb{R}^n$, $r = |x-y|$, and we set $W := N_r(x) \cap B_r(y)$, so we have:

$$|u(x)-u(y)| = \fint_W |u(x)-u(z)|dz + \fint_W |u(z)-u(y)|dz.$$

Hence, by (5) and Hölder inequality, we have:

$$\begin{aligned}
\fint_{B_r(x)} |u(x)-u(z)|dz = \frac{C}{r^n}\int_{B_r(x)} |u(x)-u(z)|dz &\le C\int \frac{|Du(z)|}{|x-z|^{n-1}}dz \\
&\le C\|Du\|_{L^p(\mathbb{R}^n)}\left(\int_{B_r(x)} |x-z|^{(1-n)\cdot\frac{p}{p-1}}dz\right)^{\frac{p-1}{p}} \\
&\le C\|Du\|_{L^p(\mathbb{R}^n)}\left(r^{n-(n-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} = C\|Du\|_{L^p(\mathbb{R}^n)}\cdot r^{1-\frac{n}{p}}.
\end{aligned}$$

Hence, we have:

$$|u(x)-u(y)| \le C\cdot r^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)} = C|x-y|^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}.$$

So we have that:

$$[u]_{C^{0,1-\frac{n}{p}}} \le C\cdot\|Du\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

and hence we have $\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$. □

Theorem IV.1.7. Morrey for $W^{1,p}$.
If $\Omega \subset \mathbb{R}^n$ os open, bounded, $\partial\Omega$ is C^1, and $p \in (n, \infty]$, then there is a constant $c > 0$ depending on p, n, and Ω such that if $u \in W^{1,p}(\Omega)$, then $u = \tilde{u}$ almost everywhere for some $\tilde{u} \in C^{0,\gamma}(\overline{\Omega})$ where $\gamma = 1 - \frac{n}{p}$ (as well $\gamma = 1$ if $p = \infty$), and $\|\tilde{u}\|_{C^{0,\gamma}(\overline{\Omega})} \le C\|u\|_{W^{1,p}(\Omega)}$.

Proof of sketch. Same as for GNS in $W^{1,p}$, we extend and mollify/approximate u is $p < \infty$. $p = \infty$ case we will treat later. □

IV.2 Sobolev Embeddings

We have not addresses $W^{1,n}$, but note that as $p \nearrow n$, we have:

$$p^* = \frac{np}{n-p} \to \infty,$$

so we may hope that $W^{1,n} \subset L^\infty$.

Example IV.2.1. If $u(x) = \log\big(\log(1+\frac{1}{|x|})\big)$, then $u \in W^{1,n}(B) \setminus L^\infty(B)$. ◊

Remark IV.2.2. One can show that $W^{1,n} \subset \text{BMO}$ (bounded mean oscillation) by establishing the Poincaré type inequality. Hence, we have BMO replaces L^∞ if $p = n$. ⌟

Theorem IV.2.3. Sobolev Embedding.
If $\Omega \subset \mathbb{R}^n$ is open, bounded, $\partial\Omega$ is C^1, $p \in [1, \infty)$, $k \in \mathbb{N}$, and $u \in W^{k,p}(\Omega)$, then:

(i) If $k < \frac{n}{p}$, then $u \in L^q(\Omega)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ and:

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}$$

for $C > 0$ dependent on k, p, n, Ω.

(ii) If $k > \frac{n}{p}$, then $u \in C^{k-1-\lfloor \frac{n}{p} \rfloor, \gamma}(\overline{\Omega})$ where:

$$\gamma = \begin{cases} \text{any value in } (0,1), & \text{if } \frac{n}{p} \in \mathbb{N}, \\ \lfloor \frac{n}{p} \rfloor - \frac{n}{p} + 1 & \text{if } \frac{n}{p} \notin \mathbb{N}, \end{cases}$$

and $\|u\|_{C^{k-1-\lfloor \frac{n}{p} \rfloor, \gamma}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)}$

for $C > 0$ dependent on k, n, p, γ, Ω.

Remark IV.2.4. For the above theorem, when $k = 1$, then (1) is GNS and (2) is Poincaré. ⌟

Proof. If $k < \frac{n}{p}$, as $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$, so we can apply GNS inequality (Theorem III.4.3) for $W^{1,p}$ to $D^\beta u \in W^{1,p}(\Omega)$ for $|\beta| \leq k-1$ implies that:

$$\|D^\beta u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)},$$

so we have $u \in W^{k-1,p^*}(\Omega)$. Note that $p^* > p$.
We apply the GNS inequality again to obtain that $u \in W^{k-2,p^{**}}(\Omega)$, where we have:

$$\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{1}{n} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}.$$

Doing this k times, we see that:

$$u \in W^{0,p^{*\cdots*}}(\Omega) = L^{p^{*\cdots*}}(\Omega),$$

where we have $\frac{1}{p^{*\cdots*}} = \frac{1}{p} - \frac{k}{n}$, so we have that $u \in L^{\frac{1}{1/p-k/n}}(\Omega) = L^q(\Omega)$, so (1) case holds.
For the other case, consider $k > \frac{n}{p}$. If $\frac{n}{p} \in \mathbb{N}^+$, let's set $\ell = \frac{n}{p} - 1 \in \mathbb{N}$, applying GNS inequality gives us that $u \in W^{k-\ell,r}(\Omega)$ where:

$$\frac{1}{r} = \frac{1}{p} - \frac{\ell}{n} \iff r = \frac{np}{n - p\ell} = n,$$

i.e., $u \in W^{k-\ell,n}(\Omega) \subset W^{k-\ell,s}(\Omega)$ for all $s \in [1, n)$ (Bounds for Hölder inequality) since Ω is bounded.
Now, we apply GNS inequality for $W^{1,p}$ to get $D^\alpha u \in L^{s^*}(\Omega)$ for $|\alpha| \leq k - \ell - 1 = k - \frac{n}{p}$ and noting that $s \in [1, n)$ implies that $s^* \in (n, \infty)$, we have, by the Morrey inequality, for $W^{1,p}$ implies that $D^\beta u \in C^{0,1-\frac{n}{q}}(\overline{\Omega})$ for any $q \in (n, \infty), |\beta| \leq u - 1 - \frac{n}{p}$.
Eventually, if $k > \frac{n}{p}$ and $\frac{n}{p} \notin \mathbb{N}$, we repeat as before with $\ell = \lfloor \frac{n}{p} \rfloor$ implying that $u \in W^{k-\ell,r}(\Omega)$ with $r = \frac{np}{n-p\ell} > n$ for $\ell p < n$. Therefore, by Morray again, we have $D^\beta u \in C^{0,1-\frac{n}{r}}(\overline{\Omega})$ for $|\beta| \leq k - 1 - \lfloor \frac{n}{p} \rfloor$

and as $1 - \frac{n}{r} = 1 - \frac{n}{p} + \ell = \lfloor \frac{n}{p} \rfloor - \frac{n}{p} + 1$, *i.e.*:

$$u \in C^{k-1-\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor - \frac{n}{p} + 1}(\overline{\Omega}),$$

as desired. □

Remark IV.2.5. Consider $u \in W^{k,p}(\Omega)$ for all $k \geq 1$ implies that $u \in C^\infty(\overline{\Omega})$. ⌟

Then, we will be discuss about the compactness in Sobolev space.

Theorem IV.2.6. Arzelá-Ascoli.
If $K \subset \mathbb{R}^n$ is compact and $\{f_n\}_{n=1}^\infty$ is a sequence of continuous function on K such that it is:

(i) Pointwise bounded, *i.e.*, $|f_n| \leq \phi$ for some $\phi : K \to \mathbb{R}$ for all $n \geq 1$,

(ii) Equicontinuous, for $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in K$ in which $|x - y| < \delta$, then:

$$|f_n(x) - f_n(y)| < \epsilon \qquad \text{for all } n \geq 1.$$

Then $\{f_n\}$ is uniformly bounded and has a uniformly convergent subsequence.

Remark IV.2.7. If $\{f_n\} \subset C^1(\mathbb{R}^n)$ almost everywhere bounded, then $f_n \to f$ uniformly on compact sets up to a subsequence. ⌟

Definition IV.2.8. If X and Y are Banach, with $X \subset Y$, we have a **continuous embedding**, denoted:

$$X \hookrightarrow Y$$

if $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$ and $C > 0$ independent of x.
We denote $X \overset{c}{\hookrightarrow} Y$ if $X \hookrightarrow Y$ and every bounded sequence in X has a convergent subsequence in Y, in which we say X is compactly embedded in Y. ⌟

Example IV.2.9.

- By Arzelá-Ascoli (Theorem IV.2.6), $C^1(\overline{\Omega}) \overset{c}{\hookrightarrow} C^0(\overline{\Omega})$ for $\Omega \in \mathbb{R}^n$ being bounded.
- By GNS inequality (Theorem III.4.3), we have $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ for $p < n$.
- By Morrey (Theorem IV.1.6), we have $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\overline{\Omega})$ for $p > n$. ◊

Theorem IV.2.10. Rellich-Kondrachov Compactness.
If $\Omega \subset \mathbb{R}^n$ is open, bounded, $\partial\Omega$ is C^1, and $p \in [1, n)$, then:

$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \qquad \text{for } q \in [1, p^*).$$

If instead $p \in [1,\infty]$, we have:

$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^p(\Omega).$$

And, if $\partial\Omega$ is not necessarily C^1, we have:

$$W_0^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^p(\Omega).$$

Example IV.2.11. $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) = L^{\frac{np}{n-p}}(\Omega)$ is never compact.
To see this, let $\Omega = B := B_1(0) \subset \mathbb{R}^n$, and $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\|u\|_{W^{1,p}(\Omega)} > 0$. In particular, $u \in W^{1,p}(\mathbb{R}^n)$ by extension, with $u = 0$ on $\mathbb{R}^n \setminus B$. By scaling, for $\lambda \in (0,1]$, we have that:

$$u_\lambda(x) = u\left(\frac{x}{\lambda}\right) \qquad \text{so we have} \qquad \begin{cases} \|u_\lambda\|_{L^{p^*}} = \lambda^{\frac{n}{p^*}}\|u\|_{L^{p^*}(B)} > 0, \\ \|Du_\lambda\|_{L^p(B)} = \lambda^{\frac{n}{p}-1}\|Du\|_{L^p(B)}. \end{cases}$$

Then, we have $\lambda^{-\frac{n}{p^*}}u_\lambda \in W_0^{1,p}(B)$ with $\|\lambda^{-\frac{n}{p^*}}u_\lambda\|_{L^{p^*}(B)} = \|u\|_{L^{p^*}(B)} > 0$, then $(\lambda^{-\frac{n}{p^*}}u_\lambda)$ is bounded in $W^{1,p}(B)$.
Note that $\lambda^{-\frac{n}{p}}u_\lambda \to 0$ almost everywhere in B as $\lambda \to 0$ as $\frac{x}{\lambda} \notin B$ for some small λ.
Hence, if $\lambda^{-\frac{n}{p^*}}u_\lambda \to v$ in $L^{p^*}(B)$, then $v \equiv 0$. But $\|\lambda^{-\frac{n}{p^*}}u_\lambda\|_{L^{p^*}(B)} > 0$ so $\|v\|_{L^{p^*}(B)} > 0$. Therefore, it has no convergent subsequence, so $W^{1,p}(B) \hookrightarrow L^{p^*}(B)$ is not compact. $\Diamond$

Proof of Theorem IV.2.10. Fix $q \in [1, p^*)$, and since Ω is bounded:

$$W^{1,p}(\Omega) \hookrightarrow L^1(\Omega).$$

(If $\{u_i\} \subset W^{1,p}(\Omega)$ is bounded, by the extension theorem (since $\partial\Omega$ is C^1), we can assume $u_i \in W^{1,p}(\tilde{\Omega})$ for some $\Omega \Subset \tilde{\Omega}$ with:

$$\sup_i \|u_i\|_{W^{1,p}(\tilde{\Omega})} < +\infty.$$

Hence, we consider $u_i^\epsilon = \eta_\epsilon * u_i$ for $\epsilon > 0$ sufficiently small (determined by $\tilde{\Omega}$), so that $\operatorname{supp}(u_i^\epsilon) \Subset \tilde{\Omega}$.
If $u_i \in C_c^1(\tilde{\Omega})$ for some $i \geq 1$, then for $x \in \tilde{\Omega}$, we have:

$$\begin{aligned} u_i^\epsilon(x) - u_i(x) &= \int_{B_\epsilon(x)} \eta_\epsilon(x-y)\big(u_i(y) - u_i(x)\big)dy \\ &= \int_{B_1(0)} \eta(z)\big(u_i(x-\epsilon z) - u_i(x)\big)dz \\ &= -\epsilon\int_{B_1(0)} \eta(z)\left[\int_0^1 \nabla u_i(x-\epsilon tz)zdt\right]dz \qquad \text{(FTC)}. \end{aligned}$$

By Fubinni, we have:

$$\int_{\tilde{\Omega}} |u_i^\epsilon(x) - u_i(x)|dx \leq \epsilon\int_{B_1(0)} \eta(z)\left[\int_0^1\int_{\tilde{\Omega}} |Du_i(x-\eta tz)z|dxdt\right]dz \leq \epsilon\|Du_i\|_{L^1(\tilde{\Omega})}.$$

By global approximation, for $u_i \in W^{1,p}(\tilde{\Omega})$, we have:

$$\|u_i^\epsilon - u_i\|_{L^1(\tilde{\Omega})} \leq \epsilon \|Du_i\|_{L^1(\tilde{\Omega})} \leq \epsilon \cdot c\|Du_i\|_{L^[(\tilde{\Omega})]} \leq \epsilon \cdot C \cdot \sup_i \|u_i\|_{W^{1,p}(\tilde{\Omega})}.$$

Thus, we have $u_i^\epsilon \to u_i$ in $L_1(\tilde{\Omega})$ as $\epsilon \to 0$ uniformly with i. By the interpolation of L^p spaces, if $\lambda \in (0,1)$, with $\frac{1}{q} = \lambda + \frac{1-\lambda}{p^*}$, then:

$$\begin{aligned}\|u + i^\epsilon - u_i\|_{L^q(\tilde{\Omega})} &\leq \|u_i^\epsilon - u_i\|^\lambda_{L^1(\tilde{\Omega})} \cdot \|u_i^\epsilon - u_i\|^{1-\lambda}_{L^{p^*}(\tilde{\Omega})} \leq C\|u_i^\epsilon - u_i\|^\lambda_{L^1(\tilde{\Omega})}\|Du_i^\epsilon - Du_i\|^{1-\lambda}_{L^p(\tilde{\Omega})} \\ &\leq C\|u_i^\epsilon - u_i\|^\lambda_{L^1(\tilde{\Omega})}\left[\|Du_i^\epsilon\|_{L^p(\tilde{\Omega})} + \|Du_i\|_{L^p(\tilde{\Omega})}\right]^{1-\lambda},\end{aligned}$$

and as $\sup_i \|u_i\|_{W^{1,p}(\tilde{\Omega})} < \infty$, then we have $u_i^\epsilon \to u_i$ in $L^q(\tilde{\Omega})$ as $\epsilon \to 0$ uniformly in i.
Hence, if $x \in \tilde{\Omega}$, we have:

$$|u_i^\epsilon(x)| \leq \int_{B_\epsilon(x)} \eta_\epsilon(x-y)|u_i(y)|dy \leq \epsilon^{-n}\|u_i\|_{L^1(\tilde{\Omega})}.$$

Therefore, we have $\{u_i^\epsilon\}$ being uniformly bounded on $\tilde{\Omega}$, so:

$$|Du_i^\epsilon(x)| \leq \int_{B_\epsilon(x)} |D\eta_\epsilon(x-y)||u_i(y)|dy \leq C\epsilon^{-(n+1)}.$$

Hence, $\{u_i^\epsilon\}$ has uniformly bounded C^1 norm, so by the mean value theorem, we have $\{u_i^\epsilon\}$ being equicontinuous. Thus, by Arzelá-Ascoli, there is up to a subsequence that we have uniform convergence. So, if $\delta > 0$, there is some $N \geq 1$ with:

$$\|u_{i_j}^\epsilon - u_{i_k}^\epsilon\|_{L^q(\tilde{\Omega})} < \frac{\delta}{3} \qquad \text{for } j,k \geq N.$$

Also, as $u_i^\epsilon \to u_i$ in $L^q(|\tilde{om})$, we have both:

$$\begin{cases} \|u_{i_j}^\epsilon - u_{i_j}\|_{L^q(\tilde{\Omega})} < \frac{\delta}{3}, \\ \|u_{i_k}^\epsilon - u_{i_k}^\epsilon\|_{L^q(\tilde{\Omega})} < \frac{\delta}{3}, \end{cases} \qquad \text{for } j,k \geq \tilde{N}.$$

Hence, by the triangle inequality, we have:

$$\|u_{i_j} - u_{i_k}\|_{L^q(\tilde{\Omega})} \leq \|u_{i_j} \mp u_{i_j}^\epsilon \mp u_{i_k}^\epsilon - u_{i_k}\|_{L^q(\tilde{\Omega})} < \delta.$$

By choosing $\delta = \frac{1}{\ell}$ for $\ell \geq 1$, we have $\{u_{i_j}\}$ being Cauchy in $L^2(\tilde{\Omega})$, and hence the space is complete. As $L^1(\tilde{\Omega})$ is complete, we obtain a convergent subsequence of $\{u_i\}$.
Hence, we have $W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$.
Noting that $p < p^*$ and $p^* \to \infty$ as $p \to n^-$, so we're done if $p \in [1,n)$, we just need to consider the case in which $p \geq n$.

- If $p^* > n$, then $W^{1,n}(\Omega) \hookrightarrow W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^n)(\Omega)$.

For $p \in (n,\infty]$, we have the Morrey and Arzelá-Ascoli giving us that:

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\overline{\Omega}) \overset{c}{\hookrightarrow} L^p(\Omega).$$

If $\partial\Omega$ is not C^1, as in the first part, choose $\Omega \Subset \tilde{\Omega}$ with $\tilde{\Omega}$ bounded, $\partial\tilde{\Omega}$ is c^1 to apply the above and get:

$$W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,p}(\tilde{\Omega}) \overset{c}{\hookrightarrow} L^p(\Omega)$$

for $p \in [1, \infty]$. □

Remark IV.2.12. Also, this theorem fails if Ω is unbounded, as we can translate test functions to infinity. ⌟

V Compactness in Sobolev Space

V.1 Reflexivity

Recall that if X is Banach, X^* is the space of all bounded linear functions on X, *i.e.*, $L : X \to \mathbb{R}$ which is bounded (or *continuous*).

Note that X^* is also a Banach space with respect to the operator norm $\| \cdot \|_{\mathrm{op}}$.

Then $(X^*)^*$ is the *double* dual consisted of bounded linear functionals on X^*. Hence, if $x \in X$ and $L \in X^*$, then $L(x) \in \mathbb{R}$. Then, we define $x^{**} : X^* \to \mathbb{R}$ by setting:

$$x^{x**}(L) := \mathrm{ev}_x(L) = L(x).$$

Therefore, also, we have $\|x^{**}\|_{**} = \|x\|_X$.

Thus, $\Lambda : X \to (X^*)^*$ defined by $\Lambda(x) = x^{**}$ is a bounded linear injection/embedding (which we can interpret it as $X \subset X^{**}$).

Definition V.1.1. If $\Lambda : X \to (X^*)^*$ is a linear isomorphism, we say that X is reflexive, *i.e.*, it Λ is surjective. ⌟

Remark V.1.2. Hilbert space is *reflexive* by the Riesz representation theorem. ⌟

Example V.1.3. $(L^p)^* = L^q$ if $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Hence, L^p is reflexive. However, since $L^1 \subsetneq (L^\infty)j$ so L^1, L^∞ are not reflexive. ♢

Remark V.1.4. There is example of $X \simeq X^{**}$ but not being reflexive, *e.g.* see the James' space. ⌟

Remark V.1.5. $W^{k,p}$ is also reflexive for $p \in (1, \infty)$. ⌟

V.2 Weak Compactness

Recall that $\dim(X) < \infty$ if and only if the unit ball in X is compact.

To resolve this lack of compactness when $\dim(X) = \infty$, we introduce weak compactness.

Definition V.2.1. Weak Compactness.
If X is Banach and $\{x_n\} \subset X$, we say that $x_n \rightharpoonup x$ or $\{x_n\}$ **weakly converges** to $x \in X$ if:

$$L(x_n) \to L(x) \qquad \text{for all } L \in X^*.$$

⌟

Remark V.2.2. If $x_n \to x$ in X, then $x_n \rightharpoonup x$ since:

$$|L(x_n) - L(x)| \leq \|L\|_{\text{op}} \cdot \underbrace{\|x_n - x\|_X}_{\to 0} \to 0$$

for every $L \in X^*$.
If $x_n \rightharpoonup x$, then $\{x_n\}$ is bounded. The proof uses **principle of uniform boundedness** and implies, by the **Hahn-Banach**, that:

$$\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X,$$

with lower semi-continuity with respect to weak convergence. ⌟

Example V.2.3. In a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$, **Riesz representation theorem** implies that $x_n \rightharpoonup x$ if and only if $(x_n, y) \to (x, y)$ for each $y \in \mathcal{H}$.
Thus, in $H_0^1(\Omega)$, if $f_n \rightharpoonup f$ in $\mathcal{H}_0^1$, then:

$$(f_n, g)_{H_0^1(\Omega)} \to (f, g)_{H_0^1(\Omega)} \qquad \text{for } g \in H_0^1(\Omega),$$

which is equivalently:

$$\int_\Omega Df_n \cdot Dg \to \int_\Omega Df \cdot Dg \qquad \text{for all } g \in H_0^1(\Omega).$$

◊

Example V.2.4. If $p \in [1, \infty)$ as $(L^p)^* = L^q$ for $\frac{1}{p} + \frac{1}{q} = 1$, so we have $f_n \rightharpoonup f$ in L^p if and only if $\int f_n \cdot g \to \int f \cdot g$ for all $g \in L^q$. ◊

Theorem V.2.5. Weak Compactness.
If X is a reflexive Banach space, every bounded sequence in X has a weakly convergent subsequence.

Proof. Consequence of Banach-Alaoglu theorem. □

Example V.2.6. This applies in $(H^k, \langle\cdot,\cdot\rangle_{H^k})$ which will be useful in PDEs.

- In $L^2((0,1))$ the bounded sequence $\{\sin(nx)\}_{n=1}^\infty$ has no (strongly) convergent subsequence, but $\sin(nx) \rightharpoonup 0$.

Then, we can use this to (weakly) solve the Poisson's equation (again) using weak compactness by the **direct method** (of the calculus of variations).
Recall we want to solve the PDE:

$$\begin{cases} -\Delta u = f, & \text{on } B, \\ u = 0, & \text{on } \partial B, \end{cases} \qquad \text{for some } u \in C^\infty(\overline{B}).$$

We similarly (from Example I.3.9) want to solve that:

$$\int_B \nabla u \cdot \nabla \varphi = \langle u, \varphi \rangle_{H_0'(B)} = \int_B f\varphi$$

for all $\varphi \in C_c^\infty(B)$ and for some $u \in H_0^1(B)$.
Now, suppose we solve for $f \in L^2(B)$, we can define a new concept.

Definition V.2.7. Dirichlet Energy.
We define the **Dirichlet energy** $E : H_0^1(B) \to \mathbb{R}$ by:

$$E(u) = \int_B \left(\frac{1}{2} |\nabla u|^2 - fu \right) \qquad \text{for } u \in H_0^1(B).$$

We will minimize E over $H_0^1(B)$ to solve.
By Hölder, we have:

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2(B)}^2 - \int_B fu \geq \frac{1}{2} \|\nabla u\|_{L^2(B)}^2 - \|f\|_{L^2(B)} \|u\|_{L^2(B)}.$$

Then, by the Poincaré inequality, we have:

$$E(u) \geq \frac{1}{2} \|\nabla u\|_{L^2(B)}^2 - C_p \|f\|_{L^2(B)} \cdot \|\nabla u\|_{L^2(B)} \sim \frac{x^2}{2} - ax = x\left(\frac{x}{2} - a\right).$$

By assimilating this with quadratic, we have the minimum at a with value $-\frac{a^2}{2}$, and thus the equation is bounded below by:

$$E(u) \geq x\left(\frac{x}{2} - a\right) \geq -\frac{C_p^2 \|f\|_{L^2(B)}}{2} > -\infty.$$

Therefore, $E(u)$ is bounded below on $H_0^1(B)$.
So, consider $\{u_n\} \subset H_0^1(B)$ such that:

$$\lim_{n\to\infty} E(u_n) = \inf_{u \in H_0^1(B)} E(u) \geq -\infty.$$

Note that this infimum is $< \infty$ as $E(0) = 0$. Thus, eventually in n, $E(u_n) < 1$, therefore:

$$\frac{1}{2} \|\nabla u_n\|_{L^2(B)}^2 - c_p \|f\|_{L^2(B)} \cdot \|\nabla u_n\|_{L^2(B)} - 1 < 0,$$

so note that $\frac{1}{2}x^2 - ax - 1 < 0$ has roots given by:

$$y_\pm = c_p \|f\|_{L^2(B)} \pm \sqrt{2c_p \|f\|_{L^2(B)}^2 + 2} < \infty,$$

and hence, we have:

$$\|u_n\|_{H_0^1(B)} = \|\nabla u_n\|_{L^2(B)} < y_+ < \infty \qquad \text{for large } n \geq 1.$$

Thence, we have $\{u_n\}$ bounded in $H_0^1(B)$ so weak compactness implies that $u \rightharpoonup u \in H_0^1(B)$ for some u (up to a subsequence, not relabeled). By lower semi-continuity of norm under weak convergence, we have:

$$\|u\|_{H_0^1(B)} \leq \liminf_{n\to\infty} \|u_n\|_{H_0^1(B)},$$

and so we have $\int_B |\nabla u|^2 \leq \liminf_{n\to\infty} \int_B |\nabla u_n|^2$.
By the Rellich-Kondrachov, $H_0^1(B) \overset{c}{\hookrightarrow} L^2(B)$, so we have $u_n \to u$ in $L^2(B)$.
Therefore, we have $\int f u_n \to \int f u$ as $n \to \infty$:

$$\int |f u_n - f u| = \int |f||u_n - u| \leq \|f\|_{L^2} \|u_n - u\|_{L^2}.$$

Thus, we have that:

$$E(u) = \int_B \left(\frac{1}{2}|\nabla u|^2 - f u\right) \leq \liminf_{n\to\infty} \int_B \left(\frac{1}{2}|\nabla u| - f u_n\right) = \liminf_{n\to\infty} E(u_n) = \inf_{\tilde{u}\in H_0^1(B)} E(\tilde{u}),$$

so we have u minimizes E on $H_0^1(B)$.

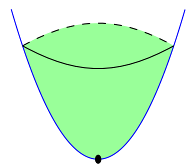

Figure V.1. The minimizer of the energy.

Now, if $\varphi \in H_0^1(B)$ and $t \in \mathbb{R}$ with $g(0) \leq g(t)$, so we have $E(u) \leq E(u + t\varphi)$ since $u + t\varphi \in H_0^1(b)$.

As a function $t \mapsto E(u + t\varphi) = g(t)$, and we have:

$$g(t) = \int \frac{1}{2}|\nabla(u + t\varphi)|^2 - f(u + t\varphi) = E(u) + t\left(\int_B \left(\nabla u \cdot \nabla \varphi - \int \varphi\right)\right) + \frac{t^2}{2}\int_B |\nabla \varphi|^2.$$

Then g is smooth in t, so we have:

$$\begin{cases} g'(t) = 0 + \int_B (Du \cdot D\varphi - f\varphi) + t\int_B |\nabla\varphi|^2, \\ g''(t) = \int |\nabla \varphi|^2 \geq 0. \end{cases}$$

But we have $g'(0) = \int_B (\nabla u \cdot \nabla \varphi - f\varphi)$ and $g'(0) = 0$. Therefore, we have:

$$\int_B \nabla u \cdot \nabla \varphi = \int_B f\varphi \qquad \text{for all } \varphi \in H_0^1(B).$$

$\Diamond$

VI Difference Quotients

VI.1 Difference Quotients

Definition VI.1.1. Difference Quotients.
If $\Omega \subset \mathbb{R}^n$ is open, $\tilde{\Omega} \Subset \Omega$, and $u : \Omega \to \mathbb{R}$, then the i-th difference quotient of u (for $i = 1, \cdots, n$) of size h, where $0 < |h| < \operatorname{dist}(\tilde{\Omega}, \partial\Omega)$ is:

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h},$$

where e_i is the i-th basis vector.
Specifically, we write $D^h u(x) = \big(D_1^h u(x), \cdots, D_n^h u(x)\big)$. ⌟

Remark VI.1.2. This is a *discrete* derivative and if $u \in C_c^1(\Omega)$, then $D_i^h u \to D_i u$ uniformly as $h \to 0$. ⌟

Lemma VI.1.3. Properties of D_i^h.
If $\Omega \subset \mathbb{R}^n$ is open, $\tilde{\Omega} \Subset \Omega$, $u, v : \Omega \to \mathbb{R}$, and $0 < |h| < \operatorname{dist}(\tilde{\Omega}, d\Omega)$, then for each $i = 1, \cdots, n$, we have:

(i) For all $x \in \tilde{\Omega}$:

$$D_i^h(uv)(x) = u(x) D_i^h v(x) + D_i^h u(x) v(x + he_i).$$

(ii) If $\frac{1}{p} + \frac{1}{q} = 1$, $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ and $\operatorname{supp}(v) \subset \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 2h\}$, then:

$$\int_\Omega (D_i^h u) v = -\int_\Omega u \cdot D_i^{-h} v.$$

Lemma VI.1.4. D_i^h to D.
If $\Omega \subset \mathbb{R}^n$ is open, $\tilde{\Omega} \Subset \Omega$ and $0 < |h| < \frac{1}{2} \operatorname{dist}(\tilde{\Omega}, \partial\Omega)$, then:

(i) If $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega)$, we have:

$$\|D^h u\|_{L^p(\tilde{\Omega})} \leq C \|Du\|_{L^p(\Omega)}$$

for a constant $C > 0$ dependent on n and p.

(ii) If $p \in (1, \infty)$ and $u \in L^p(\Omega)$ with $\|D^h u\|_{L^p(\tilde{\Omega})} \leq M$ for some $M > 0$ and all such h as above. Therefore, $u \in W^{1,p}(\tilde{\Omega})$ and $\|Du\|_{L^p(\tilde{\Omega})} \leq M$.

Remark VI.1.5. Case (ii) fails for $p = 1$. We consider the Heaviside function:

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Then, we have:

$$D^h(x) = \begin{cases} \frac{1}{h}, & \text{for } x \in [-h, 0), \\ 0, & \text{otherwise.} \end{cases}$$

If $h > 0$, we have that $\|D^h H\|_{L^1(\mathbb{R})} = 1$, but H has no weak derivative as we have shown that $H \notin W^{1,1}(\mathbb{R})$ from Example II.1.4. ⌟

Proof. (i) If $u \in C^1(\Omega)$, then for $x \in \tilde{\Omega}$, the fundamental theorem of calculus gives us that:

$$u(x + he_i) - u(x) = \int_0^h \frac{d}{dt}(u(x + te_i))dt.$$

Hence, we have:

$$|D_i^h u(x)| \le \int_0^h \frac{1}{h}\big|D_i u(x + te_i)\big|dt \le \frac{h^{\frac{1}{q}}}{h}\left(\int_0^h |D_i u(x + te_i)|^p\right)^{\frac{1}{p}}.$$

Therefore, we have:

$$\|D_i^h u\|^p_{L^p(\tilde{\Omega})} \le \frac{1}{h}\int_{\tilde{\Omega}}\left(\int_0^h |D_i u(x + te_i)|^p dt\right)dx \le \frac{1}{h}\int_0^h \|D_i u\|_{L^p(\Omega)}dt = \|D_i u\|_{L^p(\Omega)}.$$

Then, we have:

$$\|D^h u\|_{L^p(\tilde{\Omega})} = \int_{\tilde{\Omega}}\left(\sum_{i=1}^n |D_i^h u(x)|^2\right)^{\frac{p}{2}} dx \le \int_{\tilde{\Omega}}\left(\sum_{i=1}^n |D_i^h u(x)|\right)^p dx \le \sum_{i=1}^n \|D_i^h u\|^p_{L^p(\tilde{\Omega})} \le n\|Du\|_{L^p(\Omega)}.$$

Hence, $\|D^h u\|_{L^p(\tilde{\Omega})} \le n\|Du\|_{L^p(\Omega)}$ for all $u \in C^1(\Omega) \cap L^p(\Omega)$.
By approximation, we have the statement hold for $u \in W^{1,p}(\Omega)$.

(ii) Since L^p is reflexive for $p \in (1, \infty)$, if $h_u \to 0$, as $\|D_i^{h_k} u\|_{L^p(\tilde{\Omega})} \le \|D^{h,k} u\|_{L^p(\tilde{\Omega})} \le M$. Thus, up to a subsequence (not relabeled), we have $D_i^{h_k} u \to v_i$ in $L^p(\tilde{\Omega})$.
If $\varphi \in C_c^\infty(\tilde{\Omega})$, we have that:

$$\int_{\tilde{\Omega}} u D_i \varphi = \int_\Omega u \cdot D_i \varphi = \lim_{h_k \to 0}\int_\Omega u \cdot D_i^{-h_k}\varphi = -\lim_{h_k \to 0}\int_\Omega D_i^{h_k} u \cdot \varphi = -\int_\Omega v_i \cdot \varphi.$$

Thus, $D_i u = v_i$ for $i = 1, \cdots, n$, hence $u \in W^{1,p}(\tilde{\Omega})$ and $\|Du\|_{L^p(\tilde{\Omega})} \le M$ by lower semi-continuity of norms under weak convergence. □

VI.2 Weak and Classical Derivatives

Theorem VI.2.1. $W^{1,\infty} = C^{0,1}$.
If $\Omega \subset \mathbb{R}^n$ is open, bounded, and $\partial\Omega$ is C^1, then $u \in W^{1,\infty}(\Omega)$ if and only if a version of u is Lipschitz, *i.e.*, $u \in C^{0,1}(\overline{\Omega})$.

Proof. ($\Longrightarrow$:) Morrey inequality (Theorem IV.1.6) implies that $u \in C^{0,1}(\overline{\Omega})$ if $u \in W^{1,\infty}(\Omega)$.
($\Longleftarrow$:) As we have $|u(x+he_i) - u(x)| \le \mathrm{Lip}(u) \cdot |h|$, so we have $|D_i^h u(x)| \le \mathrm{Lip}(u)$. So $\|D_i^h u\|_{L^\infty(\Omega)} \le \mathrm{Lip}(u)$.
Hence, $(\nabla_i^{h_k} u)$ is bounded in $L^2(\Omega)$ as $h_k \to 0$. So, by weak compactness for L^2, up to a subsequence (n, k related), we have $D_i^{h_k} u \rightharpoonup v_i$ for some $v_i \in L^\infty(\Omega)$.
As in the previous proof, we have $\nabla_i u = v_i$ for $i = 1, \cdots, n$. So $u \in W^{1,\infty}(\Omega)$. □

Remark VI.2.2. $C^{0,\alpha}$ does not imply weakly differentiable.
Here, we have the Cantor/Devil's staircase (Example II.1.5) is $C^{0,\alpha}$ for all $\alpha \in (0, \log_3 2)$ but has no weak derivative. ⌟

Theorem VI.2.3. Sobolev $\Longrightarrow$ Differentiable Almost Everywhere.
If $\Omega \subset \mathbb{R}^n$ is open, $p \in (n, \infty]$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then u is differentiable (classically) almost everywhere in Ω with gradient equal to its weak derivative.

Proof. By adjusting an estimate in the proof of Morrey, for $p \in (n, \infty)$ if $v \in W^{1,p}\big(B_{2r}(x)\big)$, then we have:

$$\big|v(y) - v(x)\big| \le Cr^{1-\frac{n}{p}} \left(\int_{B_{2r}(x)} |Dv(z)|^p dz \right)^{\frac{1}{p}}.$$

By the **Lebesgue Differentiation Theorem**, if we have:

$$v(y) = u(y) - u(x) - Du(x) \cdot (y - x)$$

and then we can apply the previous lemma for $r = |y - x|$. For Lebesgue points $x \in \Omega$, we have:

$$\begin{aligned} |u(y) - u(x) - Du(x)(y-x)| &\le Cr^{1-\frac{n}{p}} \left(\int_{B_{2r}(x)} |Du(x) - Du(y)|^p \right)^{\frac{1}{p}} \\ &= \tilde{C} r \left(\fint_{B_{2r}(x)} |Du(x) - Du(y)|^p \right)^{\frac{1}{p}} \to 0 \end{aligned}$$

as $y \to x$ and so u is differentiable at $x = Du(x)$.
Hence, this proves it if $p \in (n, \infty)$ But as $W^{1,\infty}_{\mathrm{loc}}(\Omega) \subset W^{1,p}_{\mathrm{loc}}(\Omega)$, which we have the result for $p = \infty$ also. □

Remark VI.2.4. Rademaeher's Theorem.
As $W^{1,\infty} = C^{0,1}$, this implies that Lipschitz functions are differentiable almost everywhere. ⌟

Part 3

Second Order Linear PDEs

VII Ellipticity and Existence

VII.1 Preliminaries and Weak Solutions

Definition VII.1.1. Second Order Divergence Form Operator.
If $\Omega \subset \mathbb{R}^n$ is open, we define a **second order divergence form** (partial differential) operator L to be:

$$Lu = -\sum_{i,j=1}^{n} D_j(a^{i,j} D_i u) + \sum_{i=1}^{n} b^i D_i u + cu,$$

where $u, a^{i,j}, b^i, c : \Omega \to \mathbb{R}$ are the coefficients of L. ⌟

Conceptually, $a^{i,j}$ is the diffusion term, b^i is the transport term, and c is the constant term.

Remark VII.1.2. Einstein Summation Convention.
We often use **Einstein summation convention** to sum over repeated indices (i, j), for example:

$$Lu = -D_j(a^{i,j} D_i u) + b^i D_i u + cu.$$

In particular, the divergence form refers to the term $D_j(a^{i,j} D_i u)$ and if we had $a^{i,j} D_j D_i u$ would be called **non-divergence form**. These coincide if $a^{i,j}$ is differentiable. ⌟

Example VII.1.3. Possion Equation.
If we set $a^{i,j} = \delta_{i,j} = \begin{cases} 1, & \text{when } i = j, \\ 0, & \text{when } i \neq j, \end{cases}$ and $b^i = c = 0$, then we have:

$$L = -\Delta = -\operatorname{div}(\nabla u).$$

Hence, we have retrieved the problem from Example I.3.9. ♢

To make sense of the weak solutions to $\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$, we need a space for f to lie in. This will be $H^{-1} = (H^1_0)^*$.

Definition VII.1.4. H^{-1}.
If $\Omega \subset \mathbb{R}^n$ is open, we set:

$$H^{-1}(\Omega) = \left(H^1_0(\Omega)\right)^*,$$

and we write $\langle \cdot, \cdot \rangle$ to denote the pairing between H^{-1} and H^1_0. We define the H^{-1} norm by setting, for

$f \in H^{-1}(\Omega)$ as:

$$\|f\|_{H^{-1}(\Omega)} = \sup\left\{\langle f, u\rangle \mid u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1\right\}.$$ ⌟

Remark VII.1.5. If $f \in L^2(\Omega)$, we then have $f \in H^{-1}(\Omega)$ by pairing, and $\langle f, u\rangle = \int_\Omega f \cdot u$. ⌟

Also, if $f_0, f_1, \cdots, f_n \in L^2(\Omega)$, we can define $f \in H^{-1}(\Omega)$ by setting:

$$\langle g, u\rangle = \int_\Omega \left(f_0 \cdot u - \sum_{i=1}^n f_i \cdot \nabla_i u\right) \qquad \text{for } u \in H_0^1(\Omega).$$

Theorem VII.1.6. Characterization of H^{-1}.
If $\Omega \subset \mathbb{R}^n$ is open, and $f \in H^{-1}(\Omega)$, then there exists $f_0, \cdots, f_n \in L^2(\Omega)$ such that:

$$\langle f, u\rangle = \int_\Omega \left(f_0 u - \sum_{i=1}^n f_i D_i u\right) \tag{6}$$

for $u \in H_0^1(\Omega)$ and:

$$\|f\|_{H^{-1}(\Omega)} = \inf_{f_0, \cdots, f_n}\left\{\left(\int_\Omega \sum_{i=0}^n |f_i|^2\right)^{\frac{1}{2}} \,\middle|\, (6) \text{ holds}\right\}$$

Proof. Consider $H_0^1(\Omega)$ with $(u, v) = \int_\Omega (u \cdot v + Du \cdot Dv)$ and the Riesz representation, there is some $v \in H_0^1(\Omega)$ such that $\langle f, u\rangle = (u, v)$ for each $v \in H_0^1(\Omega)$. Then (6) holds by setting $f_0 = u$, $f_i = D_i u$ for each $i = 1, 2, \cdots, n$.
Now, if $g_0, \cdots, g_n \in L^2(\Omega)$ with:

$$\langle f, u\rangle = \int_\Omega \left(g_0 u + \sum_{i=1}^n g_i D_i u\right) \qquad \text{and } \|u\|_{H_0^1(\Omega)} \leq 1.$$

Then, by the Cauchy-Schwartz (or Hölder inequality), we have:

$$|\langle f, u\rangle| \leq \left(\int_\Omega \sum_{i=0}^n |g_i|^2\right)^{\frac{1}{2}}.$$

Then we plug in $g_i = f_i - D_i v$ with $g_0 = f_0 = v$ for $i = 1, \cdots, n$, for $\frac{v}{\|v\|_{H_0^1(\Omega)}}$ (in which the norm is then 1), we get:

$$\left(\int_\Omega \sum_{i=0}^n |f_i|^2\right)^{\frac{1}{2}} \leq \left(\int_\Omega \sum_{i=0}^n |g_i|^2\right)^{\frac{1}{2}},$$

with equality if and only if $f_i = g_i$ for $i = 0, \cdots, n$.
Hence, by taking infimum over all such g_i for which (6) holds, which gives the result. □

Remark VII.1.7. We note that $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. ⌟

Definition VII.1.8. Weak Solution.
If $\Omega \subset \mathbb{R}^n$ is open and L is as above with coefficients in $L^\infty(\Omega)$, then the associated bilinear form $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is:

$$B(u,v) = \int_\Omega \left(a^{i,j} D_i u D_j v + (b^i D_i u)v + cuv\right)$$

for $u, v \in H^1(\Omega)$. If $f \in H^{-1}(\Omega)$, then we say that $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ on Ω if:

$$B(u,v) = \langle f, v \rangle \qquad \text{for all } v \in H_0^1(\Omega).$$

⌟

Remark VII.1.9.

- If $u \in H_0^1(\Omega) \subset H^1(\Omega)$ is a weak solution to $Lu = f$ on Ω, we also say that it is a weak solution to:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

  Note that we do not need $\partial\Omega$ to be C^1 here nor Ω to be bounded.

- U being a weak solution does not imply that Lu actually makes sense, unless a second weak derivative of u exists (and $a^{i,j}$ has one).
  However, if $a^{i,j} \in W^{1,\infty}(\Omega)$ and $u \in H^2(\Omega) \cap H^1(\Omega)$, then $a^{i,j} D_i u \in H^1(\Omega)$ so $Lu \in L^2(\Omega)$.
  Hence, by integration by parts, $B(u,v) = (Lu, v)_{L^2(\Omega)} = \int_\Omega Lu \cdot v$ for $v \in H_0^1(\Omega)$. Hence, u is a weak solution to $Lu = f$ on Ω for $f \in L^2(\Omega)$ if and only if $Lu = f$ almost everywhere on Ω by the Fundamental lemma of Calculus of variation.

⌟

Example VII.1.10. If $L = \Delta$, $A^{i,j} = \delta_{i,j}$, $b^i = c = 0$, we have:

$$B(u,v) = \int_\Omega \delta_{i,j} D_i u D_j v = \int_\Omega Du \cdot Dv = (u,v)_{H_0^1(\Omega)}.$$

On a bounded domain is equivalently H^1 inner product by the Poincaré inequalities.
Hence, $B(u,v) = \langle f, v \rangle$ for $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ which is equivalently the blueprint for Poisson equation (Example I.3.9) for $\int_\Omega DuDv = \int_\Omega fv$ for all $v \in C_c^\infty(\Omega)$. ♢

Remark VII.1.11. In general, we are going to denote that:

$$\begin{cases} Lu = -D_j(a^{i,j} D_i u) + b^i D_i u + cu, \\ B(u,v) = \int_\Omega (a^{i,j} D_i u D_j v + b^i (D_i u) v + cuv) \end{cases}$$

for $u, a^{i,j}, b^i, c : \Omega \to \mathbb{R}$. ⌟

VII.2 Ellipticity

Physically, **ellipticity** captures "diffusion" of the space in *every* direction.

Definition VII.2.1. We say that L is (strictly uniformly) elliptic if there exists some $\lambda > 0$ such that for almost every $x \in \Omega$ we have:

$$\sum_{i,j=1}^{n} a^{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^n$. ⌟

Remark VII.2.2. Hereafter, we will just say elliptic, but the literatures for this is not consistent:

- Strictly means that $\lambda > 0$ and weak ellipticity means that $\lambda \geq 0$.
- Uniformly means that $\lambda > 0$ does not depend on $x \in \Omega$.

⌟

Elliptic refers to the fact that the matrix $A = (a^{i,j})_{i,j}$, is positive definite for almost every $x \in \Omega$. Hence, if A is symmetric, then the equation:

$$\sum_{i,j=1}^{n} a^{i,j}(x)\xi_i\xi_j = 1$$

has a solution set given by an ellipse (if $a^{i,j} = a^{j,i}$).

Remark VII.2.3. If A is symmetric, ellipticity is equivalently that all eigenvalues of A are real and greater than or equal to λ. ⌟

From now on, we will always assume that A is symmetric, which is **not** a restrictive constraint, as we can interpret by writing:

$$a^{i,j} = a^{i,j}_{\text{sym}} + a^{i,j}_{\text{skew}}.$$

Then, by integration by parts, we see that:

$$\int_\Omega a^{i,j}_{\text{skew}} D_i u D_j v = 0 \qquad \text{for } u, v \in C_c^2(\Omega),$$

i.e., we have $a^{i,j}_{\text{skew}} = -a^{j,i}_{\text{skew}}$. Hence, the *skew-symmetric* part of A do not contribute to the bilinear form B. In particular, it will not affect the existence and regularity results.

Example VII.2.4. We will demonstrate a few cases of the equations:

- If we have $L = -\Delta$, $a^{i,j} = \delta_{i,j}$, we have:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \sum_{i,j=1}^{n} a^{i,j}\xi_i\xi_j = \sum_{i=1}^{n} \xi_i^2 = |\xi|^2.$$

  Hence, $-\Delta$ is elliptic with $\lambda = 1$.

- However, if $L = \Delta$, $a^{i,j} = \delta_{i,j}$, and so $\sum_{i,j=1}^n a^{i,j}\xi_i\xi_j = -|\xi|^2$, which is not elliptic.
- If $L = -\frac{\partial^2}{\partial x^2}$ on $\mathbb{R}^2$, we have $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which is no elliptic, and any function of y solves $Lu = 0$. $\diamondsuit$

Theorem VII.2.5. Boundedness and Coerzivity of B.
If L has bounded coefficients ($a^{i,j}, b^i, c \in L^\infty(\Omega)$) on $\Omega \subset \mathbb{R}^n$ being open, and B is its bilinear form, then:

(i) B is bounded on $H_0^1(\Omega)$, *i.e.*, there is $\alpha > 0$ such that for $u, v \in H_0^1(\Omega)$, we have:

$$|B(u,v)| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

(ii) If L is elliptic, then for some $\beta > 0$ and $\gamma \geq 0$, we have for all $u \in H_0^1(\Omega)$ that:

$$\beta \|u\|_{H_0^1(\Omega)}^2 \leq B(u,u) + \gamma \|u\|_{L^2(\Omega)}^2,$$

which is called the **Garding inequality**.
Moreover, if Ω is bounded, $b^i = 0$ for $i = 1, \cdots, n$ and $c \geq 0$, then B is coercive, *i.e.*, we can take $\gamma = 0$ in Garding, so that for $u \in H_0^1(\Omega)$ we have:

$$\beta \|u\|_{H_0^1(\Omega)}^2 \leq B(u,u).$$

Proof. (i) We can write:

$$B(u,v) = \int_\Omega \left(a^{i,j} D_i u D_j v + b^i (D_i u) v + cuv\right),$$

and by Hölder, we have:

$$\begin{aligned} B(u,v) \leq & \sum_{i,j=1}^n \|a^{i,j}\|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \sum_{i=1}^n \|b^i\|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ & + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ \leq & \, \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \end{aligned}$$

since we have $\|u\|_{H_0^1}^2 (|Du|^2 + u^2)$.

(ii) Let $\lambda > 0$ be the ellipticity constant for L, then:

$$\begin{aligned} \lambda \int_\Omega |Du|^2 & \leq \int_\Omega a^{i,j}(x) D_i u D_j u, \qquad \text{where } \xi = Du \\ & = B(u,u) - \int_\Omega b^i (D_i u) u - \int_\Omega cu^2 \\ & \leq B(u,v) + \sum_{i=1}^n \|b^i\|_{L^\infty(\Omega)} \int_\Omega |du| \cdot |u| + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

Since $ab \leq \frac{a^2+b^2}{2}$ (QM-AM-GM-HM inequalities), let $\epsilon > 0$ and replace:

$$a \leftarrow \sqrt{\epsilon} a \qquad b \leftarrow \frac{b}{\sqrt{\epsilon}} \qquad \text{so that } ab \leq \epsilon \frac{a^2}{2} + \frac{1}{\epsilon} \frac{b^2}{2},$$

which is often called the **Peter-Paul inequality**.
Hence, by applying the Peter-Paul, we have that:

$$\int_\Omega |Du| \cdot |u| \leq \frac{\epsilon}{2}\int_\Omega |Du|^2 + \frac{1}{2\epsilon}\int_\Omega |u|^2.$$

Then, if we set $\epsilon \leq \frac{\lambda}{\sum_{i=1}^n \|b^2\|_{L^\infty(\Omega)}}$ if some $b^i \neq 0$, then we have:

$$\sum_{i=1}^n \|b^i\|_{L^\infty(\Omega)} \int_\Omega |Du| \cdot |u| \leq \frac{\lambda}{2}\int_\Omega |Du|^2 + \tilde{\gamma}\int_\Omega |u|^2.$$

Hence, we can get to:

$$\lambda \int_\Omega |Du|^2 \leq B(u,u) + \frac{\lambda}{2}\int_\Omega |Du|^2 + \tilde{\gamma}\|u\|^2_{L^2(\Omega)} + \|c\|_{L^\infty}(\Omega)\|u\|^2_{L^2(\Omega)}.$$

Now, we have:

$$\frac{\lambda}{2}\|u\|^2_{H_0^1(\Omega)} = \frac{\lambda}{2}\left(\int_\Omega |Du|^2 + \|u\|^2\right) \leq B(u,u) + \gamma\|u\|^2_{L^2(\Omega)},$$

where $\gamma = \tilde{\gamma} + \|c\|_{L^\infty(\Omega)} + \frac{\lambda}{2}$, so we can set $\beta = \frac{\lambda}{2}$.
If Ω is bounded, $b^i \equiv 0$, and $c \geq 0$, then the inequality becomes:

$$\lambda \int_\Omega |Du|^2 \leq B(u,u) - \int_\Omega cu^2 \leq B(u,u).$$

On bounded Ω, we have the Poincaré inequality implying that $\|u\|_{H_0^1(\Omega)} \sim \|Du\|_{L^2(\Omega)}$, so by above, we have:

$$\beta\|u\|^2_{H_0^1(\Omega)} \leq B(u,u) \qquad \text{for } \beta \text{ depending on } \lambda, c_p, \text{ and } \Omega.$$ □

VII.3 The Existence Theorems

Recall the "Riesz representation theorem" to represent a linear functional by inner product with a vector, we are concerning the bilinear, and wish to represent it as the inner product with a linear functional.

Theorem VII.3.1. Lax-Milgram Theorem.
If $(\mathcal{H}, (\cdot,\cdot))$ is a Hilbert space and $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bounded coercive (*i.e.*, $B(u,u) \geq \beta\|u\|^2_{\mathcal{H}}$) bilinear form, and $f \in \mathcal{H}^*$, then there is a unique $u \in \mathcal{H}$ such that:

$$B(u,v) = \langle f, v\rangle \qquad \text{for all } v \in \mathcal{H}.$$

Remark VII.3.2. We do not assume symmetry of B in the Lax-Milgram Theorem. ⌟

If B, associated to L, was bounded, symmetric, and coercive, then B would be an inner product equivalent to the usual H^1 inner product. So $(H_0^1(\Omega), B)$ would be a Hilbert space, to which we could apply Riesz representation to find weak solution.

Proof. Note that if $v \in \mathcal{H}$, then the map $B(u, \cdot) : \mathcal{H} \to \mathbb{R}$ in which $v \mapsto B(u, v) \in \mathbb{R}$ is bounded, since B is bounded, *i.e.*, there is $\alpha > 0$ such that:

$$|B(u,v)| \leq \alpha \|u\|_{\mathcal{H}} \cdot \|v\|_{\mathcal{H}}.$$

Hence, we have $B(v, \cdot) \in \mathcal{H}^*$.
By the Riesz representation, we get that for each $u \in \mathcal{H}$, some $A(u) \in \mathcal{H}$ such that:

$$B(u,v) = \big(A(u), v\big)_{\mathcal{H}} \qquad \text{for all } v \in \mathcal{H}.$$

Consider $A : \mathcal{H} \to \mathcal{H}$ which maps $u \mapsto A(u)$, we now show that:

(i) ***A* is linear and bounded**: If $a, b \in \mathbb{R}$, $u, \tilde{u} \in \mathcal{H}$, for $v \in \mathcal{H}$, we have:

$$\begin{aligned}\big(A(Au_b\tilde{u}), v\big) &= B(au + b\tilde{u}, v) = aB(u,v) + bB(\tilde{u}, v) \\ &= a\big(A(u), v\big) + b\big(A(\tilde{u}), v\big) = \big(aA(u) + bA(\tilde{u}), v\big),\end{aligned}$$

so we have $\big(A(au + b\tilde{u}) - \big(aAu(u) + bA(\tilde{u})\big), v\big) = 0$ for all $v \in \mathcal{H}$, hence, we have:

$$A(au + b\tilde{u}) = aA(u) + bA(\tilde{u}),$$

and so A is linear.
Now, if $u \in \mathcal{H}$, we have:

$$\|A(u)\|_{\mathcal{H}}^2 = \big(A(u), A(u)\big) = B(u, A(u)) \leq \alpha \|u\|_{\mathcal{J}} \cdot \|A(u)\|_{\mathcal{H}}.$$

Note that if $A(u) = 0$, we have $\|A(u)\|_{\mathcal{H}} = 0$ so it is bounded. If not, divide by $\|A(u)\|_{\mathcal{H}}$, so we have $\|A(u)\|_{\mathcal{H}} \leq \alpha \|u\|_{\mathcal{H}}$, so A is bounded.

(ii) ***A* is bijective**: By coercivity of B, for $u \in \mathcal{H}$, we have:

$$\beta \|u\|_{\mathcal{H}}^2 \leq B(u,u) = \big(A(u), u\big) \leq \|A(u)\|_{\mathcal{H}} \cdot \|u\|_{\mathcal{H}},$$

hence we have $\beta\|u\|_{\mathcal{H}} \leq \|Au\|_{\mathcal{H}}$, so A is injective as if $u \neq v$, then $\|u - v\|_{\mathcal{H}} > 0$, and then we have $\|A(u) - A(v)\| > 0$. Hence, $A(u) \neq A(v)$.
Moreover, we see that A has closed range, since if $A(u_n) \to v$ in $\mathcal{H}$, since:

$$\beta\|u_n - u_m\|_{\mathcal{H}} \leq \|A(u_n) - A(u_m)\|_{\mathcal{H}},$$

which implies that $\{u_n\}$ is Cauchy, hence is convergent in $\mathcal{H}$, *i.e.*, $u_n \to u$ in $\mathcal{H}$, but since A is bounded (*i.e.*, continuous) so $A(u_n) \to A(u)$, so $v = A(u)$.
If A was not surjective, we could find some element $w \in \big(A(\mathcal{H})\big)^{\perp} \setminus \{0\}$. This follows since $A(\mathcal{H})$ is closed so $\mathcal{H} = A(\mathcal{H}) \oplus \big(A(\mathcal{H})\big)^{\perp}$.
Again, by coercivity:

$$0 < \beta\|w\|_{\mathcal{H}} \leq B(w,w) = \big(A(w), w\big) = 0,$$

and $0 < 0$, which is a contradiction, implying that A is surjective.

(iii) **Define $u \leftrightsquigarrow f \in \mathcal{H}^*$**: Again, by the Riesz representation theorem, for $f \in \mathcal{H}^*$, there is some $w \in \mathcal{H}$

such that:

$$\langle f, v\rangle = (w, v) \qquad \text{for all } v \in \mathcal{H}.$$

Since $A : \mathcal{H} \to \mathcal{H}$ is a bijection, there is a unique $v \in \mathcal{H}$ such that $A(v) = w$. Then, by the construction of A:

$$B(u, v) = \big(A(u), v\big) = (w, v) = \langle f, v\rangle.$$

□

Theorem VII.3.3. Existence and Uniqueness, I.
If $\Omega \subset \mathbb{R}^n$ is open, L is elliptic with coefficients in $L^\infty(\Omega)$, then there is some $\gamma \geq 0$ such that if $\mu \geq \gamma$ and $f \in H^{-1}(\Omega)$, there exists a unique *weak solution* $u \in H_0^1(\Omega)$ to the PDE:

$$\begin{cases} Lu + \mu u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof. Let $\gamma \geq 0$ be from the **Garding's inequality** from Theorem VII.2.5, *i.e.*, $\beta\|u\|^2_{H_0^1(\Omega)} \leq B(u, u) + \gamma\|u\|^2_{L^2(\Omega)}$.
For $\mu \geq \gamma$, we define for $u, v \in H^1(\Omega)$ that:

$$(u, v) = B(u, v) + \mu(v, u)_{L^2(\Omega)},$$

which is the associated operator to $L_u v = Lu + \mu v$.
Hence, we have:

$$\beta\|u\|^2_{H_0^1(\Omega)} \leq B_\mu(u, u) \qquad \text{for all } u \in H_0^1(\Omega),$$

which implies that B_μ is coercive.
Hence B_μ is bounded as B is (since the coefficients are bounded).
Hence, by Lax-Milgram (Theorem VII.3.1), for each $H^{-1}(\Omega) = \big(H_0^1(\Omega)\big)^*$, there is a unique $u \in H_0^1(\Omega)$ such that:

$$B_\mu(u, v) = \langle f, v\rangle \qquad \text{for all } v \in H_0^1(\Omega),$$

which is equivalently that u is the unique weak solution to:

$$\begin{cases} Lu + \mu u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

□

Example VII.3.4. Poisson Equation.
Again, consider $L = -\Delta$ on $B \subset \mathbb{R}^n$, by the Poincaré inequality, we have:

$$\|u\|_{H_0^1(\Omega)} \leq B(u, u) = \int_B |Du|^2,$$

where $\gamma = 0$. So there exists unique weak solution to the differential equation:

$$\begin{cases} -\Delta u + \mu u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

for all $f \in H^{-1}(\Omega)$ and $\mu \geq 0$. $\Diamond$

Remark VII.3.5. Additionally, if $Lu = D_j(a^{i,j} D_i u) + cu$, $c \geq 0$, $a^{i,j} c \in L^\infty(\Omega)$, we again get unique weak solutions to ($b^i = 0$):

$$\begin{cases} Lu + \mu v = f, & \text{on } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \qquad \text{for all } f \in H^{-1}(\Omega), \mu \geq 0.$$

In particular for

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

with $\gamma = 0$ in Garding inequality. ⌟

However, existence and uniqueness is not always guaranteed. We can, in fact, find counter examples with purely ODEs:

Example VII.3.6. Let $\Omega = (0, \pi) \subset \mathbb{R}$ and the differential equation as:

$$\begin{cases} -u'' - u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where we have $f \in H^{-1}(\Omega)$.
We then have $Lu = -u'' - u$, so $a^{1,1} = 1$, $b^1 = 0$, and $x = -1$ are all in $\mathcal{L}^\infty(\Omega)$ (and it is even smooth).
But if we set $f = 0$, both $u = 0$ and $u = \sin(x)$ solves the differential equation, so there is no unique solution.
Moreover, if $f(x) = \sin(x) \in \mathcal{L}^2(\Omega) \subset H^{-1}(\Omega)$, and $u \in H_0^1(\Omega)$ is a weak solution, then we have:

$$B(u, f) = \langle f, f \rangle = \int_0^\pi f^2 = \int_0^\pi \sin^2(x) \neq 0,$$

since we have that:

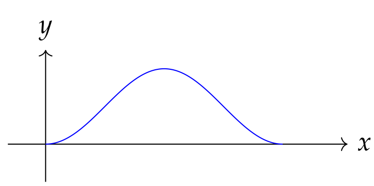

Figure VII.1. Graph of $\sin^2(x)$ *on* $[0, \pi]$.

On the other hand, we have:

$$B(u, f) = \int_0^\pi -u'f' + \int_0^\pi uf = \int_{-}^{\pi} uf'' + uf = \int_0^\pi u\big(-\sin(x) + \sin(x)\big) = 0.$$

Hence, this is a contradiction, so there is no weak solutions to the differential equation if $f = \sin(x)$. $\Diamond$

More generally, if $f \in H^{-1}(\Omega)$ and u is the weak solution, we have:

$$u + A\sin(x) \qquad \text{also being a weak solution,}$$

i.e., we have $\sin(x) \in \ker L$ if and only if $L\big(\sin(x)\big) = 0$.

VII.4 Fredholm Theory

We saw that for some $\mu > 0$ sufficiently large, we have that the operator $L_\mu : H_0^1(\Omega) \to H^{-1}(\Omega)$, defined by:

$$L_\mu u = Lu + \mu u$$

was a bounded linear bijection, which sends $u \in H_0^1(\Omega)$ to $B_\mu(u, \cdot) \in H^{-1}(\Omega)$.

Moreover, $L_m u^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is also bounded as:

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq B_\mu(u,u) = B_\mu(v,\cdot)(u) \leq \|B_\mu(u,\cdot)\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)}.$$

Therefore, we have $\beta\|u\|_{H_0^1(\Omega)} \leq \|B_\mu(u,\cdot)\|_{H^{-1}(\Omega)}$ noting that $L_\mu(u) = B_\mu(u,\cdot)$, this is equivalently:

$$\|L_\mu^{-1}\big(B_\mu(u,\cdot)\big)\|_{H_0^1(\Omega)} \leq \frac{1}{\beta}\|B_\mu(u,\cdot)\|_{H^{-1}(\Omega)},$$

we have that L_μ^{-1} being bounded. Furthermore, by Rellich-Kondrachov, we have $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ and $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, so:

$$L_\mu^{-1} : L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \xrightarrow{L_\mu^{-1}} H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega),$$

and hence, we have $L_\mu^{-1} : L^2(\Omega) \to L^2(\Omega)$ is bounded and compact. As a sanity check, we have:

$$L_\mu^{-1}(f) = u \iff u \text{ weakly solves } L_\mu(u) = f.$$

If we want general existence for weak solutions to the PDE:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

for $f \in L^2(\Omega)$. Consider it equivalently as $B(u,v) = \int_\Omega fv$ for all $v \in H_0^1(\Omega)$, which is the same as $B(u,v) + \mu\int_\Omega uv = \int_\Omega fv + \mu\int_\Omega uv$ for some $\mu \geq \gamma$ from Garding inequality.
Equivalently, we have $B_\mu(u,v) = \int_\Omega (f + \mu u)u$, so we have $B_\mu(u,\cdot) = (f + \mu u, \cdot)_{L^2(\Omega)}$ in $H^{-1}(\Omega)$. Therefore, we have:

$$L_\mu u = f + \mu u \text{ in the } \textit{above}, \qquad \text{so } u = L_\mu^{-1}(f + \mu u).$$

This is then equivalent to say that $\big(\mathrm{Id} - \mu L_\mu^{-1}\big)u = L_\mu^{-1}(f)$.

Then, we have weakly solving the differential equation is the same as solving $\big(\mathrm{Id} - \mu L_\mu^{-1}\big)u = L_\mu^{-1}(f)$.

Abstractly, at the level of linear algebra, we want to solve equations of the form:

$$(\mathrm{Id} - K)u = v \qquad \text{for } K : \mathcal{H} \to \mathcal{H}$$

as a compact, bounded, and linear map on a Hilbert space.

Definition VII.4.1. Adjoint Operator.
If $\big(\mathcal{H}, (\cdot,\cdot)\big)$ is a real Hilbert space, $A : \mathcal{H} \to \mathcal{H}$ is bounded and linear, then it's **adjoint** $A^* : \mathcal{H} \to \mathcal{H}$ is defined by setting:

$$(Au, v) = (u, A^*v) \qquad \text{for all } u, v \in \mathcal{H}.$$

We say that A is symmetric if $A = A^*$, *i.e.*, $(Au, v) = (u, Av)$ for all $u, v \in \mathcal{H}$. ⌟

Remark VII.4.2.

- The adjoint exists by the Riesz representation theorem, while also being bounded and linear, which $(A^*)^* = A$.
- If the dimension of $\mathcal{H}$ is finite, then A^* has matrix corresponding to the transpose of A, and $A^\intercal = (a^{j,i})_{i,j}$ if $A = (a^{i,j})_{i,j}$.
- If $K : \mathcal{H} \to \mathcal{H}$ is bounded, linear, and compact, then so is $K^* : \mathcal{H} \to \mathcal{H}$ (since for bounded sequence $u_n \rightharpoonup u$, hence $K(u_n) \to K(u)$, and so $K^*(u_n) \to J^*(u)$ by definition). ⌟

In finite dimension, all linear maps are bounded and compact, and from linear algebra, if $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear, we have $L(u) = v$ for $u, v \in \mathbb{R}^n$ is solvable or $\ker(L^\intercal) \neq \{0\}$ (but not both).

This does not hold in infinite dimensions though.

Example VII.4.3. Consider $K : L^2\big((0,1)\big) \to L^2\big((0,1)\big)$ sending $f \in L^2\big((0,1)\big)$ to:

$$(Kf)(x) = \int_0^x f(t)dt \in L^2\big((0,1)\big)$$

being compact (by Arzelá-Ascoli).
But, consider $\ker(K^*) = \{0\}$ and $g \in \operatorname{im} K$ implies that g is continuous, but not every $L^2\big((0,1)\big)$ function is continuous, so K is not invertible.
Hence, the analogy does not hold if the dimension is infinity. ♢

Theorem VII.4.4. Fredholm Alternative.
If $K : \mathcal{H} \to \mathcal{H}$ is linear, bounded, and compact on a real Hilbert space $\mathcal{H}$, then precisely one of the following two possibilities hold:

(i) For each $h \in \mathcal{H}$, the equation:

$$\big(\mathrm{Id} - K\big)u = h$$

has a unique solution.

(ii) The equation:

$$(\mathrm{Id} - K)u = 0$$

has a nonzero solution.

Moreover, the space of such solutions is finite dimensional, equal to the dimension of the solutions to:

$$(\mathrm{Id} - K^*)u = 0.$$

Finally, $(\mathrm{Id} - K)u = h$ is solvable if and only if $h \in \big(\ker(\mathrm{Id} - K^*)\big)^{\intercal}$.

Then, we consider the adjoint of L.

Definition VII.4.5. If $\Omega \subset \mathbb{R}^n$ is open and bounded, L is elliptic, with coefficients in $L^\infty(\Omega)$ with $b^i \in W^{1,\infty}(\Omega)$ for $i = 1, \cdots, n$, then the **formal adjoint**, L^*, of L, is defined as:

$$L^* v = -D_i(a^{i,j} D_j v) - b^i D_i v + (c - D_i b^i)v.$$

⌟

Remark VII.4.6. One can note the following properties of L^*:

- L^* is elliptic, since $a^{i,j}$ are the same.
- $b^i \in W^{1,\infty}(\Omega)$ for $i = 1, \cdots, n$, we have that for $u, v \in H_0^1(\Omega)$, then:

  $$B(u,v) = \int_\Omega \big(a^{i,j} D_i u D_j v + (b^i D_i u)v + cuv\big) = \int_\Omega \big(a^{i,j} D_i u D_j v + D_i(b^i v)u + (c - D_i b^i)uv\big) = B^*(v,u).$$

  Hence, B^* is bilinear form associated to L^* and $B(u,v) = B^*(v,u)$.
- This is a *formal adjoint* is that while it does not map a Hilbert space to itself, it has the "form" of a an adjoint since if $a^{i,j}, b^i \in W^{1,\infty}$ for $i, j = 1, \cdots, n$, and $u, v \in H^2(\Omega) \cap H_0^1(\Omega)$, then:

  $$\langle Lu, v\rangle_{L^2(\Omega)} = B(u,v) = B^*(v,u) = \langle u, L^* v\rangle_{L^2(\Omega)}.$$

  Hence, L^* locks line an adjoint with these assumptions.

⌟

Recall that $L_\mu^{-1} : L^2(\Omega) \to L^2(\Omega)$ was bounded, compact, linear for $\mu \geq 0$ large enough (from Garding inequality), But Garding then holds for L^*, so:

$$L_\mu^* = (L + \mu)^*(u) = L^*(u) + \mu(u).$$

Noting here that $\mu^* = \mu$ since $(\mu, v)_{L^2(\Omega)} = (u, \mu v)_{L^2(\Omega)}$. Note that since $H_\mu(u,v) = B_\mu^*(v,u)$, by applying our first existence result for L^*, we obtain unique weak solution to

$$\begin{cases} L^* u + \mu u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

for this choice of $\mu \geq 0$ (from Garding inequality).

Arguing as before, consider $(L_\mu^*)^{-1}$. $L^2(\Omega) \to L^2(\Omega)$ is bounded, compact, linear map. Hence, if $f \in L^2(\Omega)$ and $v \in H_0^1(\Omega)$, then:

$$\begin{cases} B_\mu\big(L_\mu^{-1}(f), v\big) = (f, v)_{L^2(\Omega)}, \\ B_\mu^*\big(((L_\mu^{-1}(f), v\big) = (f, v)_{L^2(\Omega)}, \end{cases}$$

hence, for all $f, g \in L^2(\Omega)$m, that:

$$\big(g, (L_\mu^*)^{-1}(f)\big)_{L^2(\Omega)} = B_\mu\big(L_\mu^{-1}(g), (L_\mu^*)^{-1}(f)\big) = B_\mu^*\big((L_\mu^*)^{-1}(g), L_\mu^*(f)\big).$$

Theorem VII.4.7. Existence for Elliptic PDEs II.
If $\Omega \subset \mathbb{R}^n$ is open and bounded, and L is elliptic with coefficient in $a^{i,j}, c \in L^\infty(\Omega)$ and $b^i \in W^{1,\infty}(\Omega)$ for $i, j = 1, \cdots, n$, then precisely one of the following holds:

(i) For each $f \in L^2(\Omega)$, there is a unique weak solution $v \in H_0^1(\Omega)$ to the PDE:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

(ii) There is a weak solution $u \in H_0^1(\Omega) \setminus \{0\}$ to the homogeneous PDE:

$$\begin{cases} Lu = 0, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, the space of weak solutions to the homogeneous PDE is finite dimensional, and equal to the dimension for the space of weak solutions, $v \in H_0^1(\Omega)$, to:

$$\begin{cases} L^*v = 0, & \text{on } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

Finally:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

has a weak solution for $f \in L^2(\Omega)$ if and only if $(f, v)_{L^2(\Omega)} = 0$ for all $v \in H_0^1(\Omega)$ is weak solution to:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof. For large enough $\mu \geq 0$, we have a unique weak solution $w \in H_0^1(\Omega)$ to:

$$\begin{cases} L^*w = f, & \text{on } \Omega, \\ w = 0, & \text{in } \partial\Omega, \end{cases}$$

for each $f \in L^2(\Omega)$.
Now, weak solutions to the first condition is equivalently $(\mathrm{Id} - \mu L_\mu^*)u = L_\mu^{-1}(f)$, for $L_\mu^{-1} : L^2(\Omega) \to L^2(\Omega)$ being bounded, compact linear. Setting:

$$K = \mu L_\mu^{-1} \qquad \text{and} \qquad h = L_\mu^{-1}(f),$$

we have by Fredholm alternative that (i) holds ot (ii) holds.
Moreover, we note that $K^* = (\mu L_\mu^{-1})^* = \mu (L_\mu^*)^{-1}$.
For the final assert, the first equation being weakly solvable for:

$$f \in L^2(\Omega) \iff (\mathrm{id} - K)u = h \text{ is solvable} \iff h \in \big(\ker(\mathrm{Id} - K^*)\big)^\perp,$$

and thus, for all $v \in H_0^1(\Omega)$ weakly solving the last equation, we have $(\mathrm{Id} - K^*)v = 0 \iff u^*(v) = v$, so:

$$\begin{aligned} 0 = (h, v)_{L^2(\Omega)} &= \big(L_\mu^{-1}(f), v\big) = \frac{1}{\mu}\big(K(f), v\big)_{L^2(\Omega)} \\ &= \frac{1}{\mu}\big(f, K^*(v)\big)_{L^2(\Omega)} = \frac{1}{\mu}(f, v)_{L^2(\Omega)}. \end{aligned}$$

This is equivalently that $(f, v)_{L^2(\Omega)} = 0$ for all such $v \in H_0^1(\Omega)$ that weakly solves $\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$ □

VII.5 Eigenvalues of Elliptic Operators

Recall that in finite dimensions, if $\dim V < \infty$, and $A : V \to V$ being a linear map. Then $\lambda \in \mathbb{R}$ is an eigenvalue of A if $Av = \lambda v$ for some $v \in V \setminus \{0\}$, which is equivalently that $(A - \lambda \,\mathrm{Id})v = 0$ for $v \in V \setminus \{)\}$, so we equivalent have $\ker(A - \lambda \,\mathrm{Id}) \neq \{0\}$.

In finite dimensions, injectivity is equivalent to surjectivity by **rank-nullity** theorem.

Example VII.5.1. Volterra Operator.
Note the Volterra operator for $f \in L^2\big((0,1)\big)$ as:

$$(Kf)(x) = \int_0^x f(t)dt \in L^2\big((0,1)\big),$$

and we saw that $K : L^2\big((0,1)\big) \to L^2\big((0,1)\big)$ was compact. But consider $Kf = \lambda f$ for $\lambda \in \mathbb{R}$, so we know $\lambda f'(x) = f(x)$ for $f \in C^1\big((0,1)\big)$, which implies that $f(x) = Ae^{\frac{x}{\lambda}}$ for $A \in \mathbb{R}$.
Since $(Kf)(0)$, which implies that $\lambda f(0) = \lambda A$, and in either cases, $f \equiv 0$, and so no eigenfunctions can exist for K (so it has no eigenvalues).
Hence, $\ker(K - \lambda \,\mathrm{Id}) = \{0\}$ but K is not surjective, as $(Kf) \in C^0\big((0,1)\big)$ but not every $L^2\big((0,1)\big)$. ◊

Hence, we can see that **rank-nullity** no longer works for infinite dimensional space, so we need some new definition.

Definition VII.5.2. Spectrum.
If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear map on a real Hilbert space $\mathcal{H}$, then the (real) **spectrum**, $\sigma(A)$ of A, is:

$$\sigma(A) := \{\lambda \in \mathbb{R} : A - \lambda \,\mathrm{Id} \text{ is not invertible}\}.$$

The (real) **point spectrum** (or eigenvalues) $\sigma_p(A) \subset \sigma(A)$ is the set:

$$\sigma_p(A) := \{\lambda \in \mathbb{R} : \ker(A - \lambda \,\mathrm{Id}) \neq \{0\}\}.$$

With this new definition, we can have the following.

Theorem VII.5.3. Specturm of Compact Operators.
If $K : \mathcal{H} \to \mathcal{H}$ is compact, bounded, linear map on an infinite dimensional real Hilbert space $\mathcal{H}$, then:

(i) $0 \in \sigma(K)$, *i.e.*, K is not invertible.

(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$.

(iii) Either $\sigma(K)$ is a finite set or $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$ is countable and converges to 0.

This proof appears on the appendix of the textbook. We will use this theorem to our existence for elliptic PDEs.

Theorem VII.5.4. Existence for Elliptic PDEs, III.
If $\Omega \subset \mathbb{R}^n$ is open and bounded, L is elliptic with coefficients in $L^\infty(\Omega)$, then:

(i) There is an at most countable set, $\Sigma \subset \mathbb{R}$ such that for each $f \in L^2(\Omega)$, there is a unique solution $u \in H_0^1(\Omega)$ to the PDE:

$$\begin{cases} Lu = \lambda u + f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

if and only if $\lambda \notin \Sigma$.

(ii) If Σ is infinite, then $\Sigma = \{\lambda_k\}$, $\lambda_k \to \infty$, $\lambda_k \leq \lambda_{k-1}$ for $k \geq 1$.

Remark VII.5.5.

- Σ is the spectrum of L.
- From (i), there exists unique weak solutions to:

$$\begin{cases} Lu = \lambda u, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

is and only if $\lambda \notin \Sigma$, we call λ an eigenvalue of L and u as its eigenfunction.

Proof of Theorem VII.5.4. Let γ be the constant from the Garding inequality for L, and without loss of generality, let's assume $\gamma > 0$.
If $\lambda > -\gamma$, by the first existence theorem (Theorem VII.3.3), for elliptic PDE, we have a unique weak solution to $\begin{cases} Lu = \lambda u + f, & \text{on } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$ if and only if the only weak solution to $\begin{cases} Lu = \lambda u, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$ is zero.
Now by adding γu, we have $u = 0$ being the only solution to:

$$\begin{cases} L_\gamma(u) = Lu + \gamma u = (\lambda + \gamma)u, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

which is equivalently $u = (\lambda + \gamma)L_\gamma^{-1}(u)$.
By setting $K = \gamma L_\gamma^{-1}$, we have that:

$$K : L^2(\Omega) \to L^2(\Omega)$$

is bounded and compact, so now we can replace the above equivalence to $u = \frac{\lambda+\gamma}{\gamma} \cdot K(u)$ only has solution $u = 0$.
Thus, this implies that $\frac{\gamma}{\lambda+\gamma}$ is not an eigenvalues of K.
Hence, in summary:

$$\begin{cases} Lu = \lambda u + f, & \text{on } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \text{had unique weak solution} \iff \frac{\gamma}{\lambda + \gamma} \text{ not being an eigenvalue of } K.$$

By spectral theorem for compact operators (iii), we see that eigenvalues of K are at most countable, so we have proven (i) by setting:

$$\Sigma = \left\{ \lambda \in \mathbb{R} : \frac{\gamma}{\lambda + \gamma} \text{ is an eigenvalue of } K \right\}.$$

For (ii), we have $\sigma_p(K) \setminus \{0\} = \frac{\gamma}{\lambda_k+\gamma} \to 0$ as $k \to \infty$, which implies that $\lambda_k \nearrow \infty$ as $k \to \infty$. □

Theorem VII.5.6. Boundedness of L^{-1}.
With the same assumption from the last theorem, if $\lambda \notin \Sigma$, there is a constant $c > 0$ depending on λ, Ω, and L, such that $\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ if f is in $L^2(\Omega)$ and $u \in H_0^1(\Omega)$ is the unique weak solution to:

$$\begin{cases} Lu = \lambda u + f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, $C \to \infty$ as λ converges to eigenvalues of L.

Proof. If no such $C > 0$ exists for a given $\lambda \notin \Sigma$, we can find $\{f_k\} \subset L^2(\Omega)$ and $\{u_k\} \subset H_0^1(\Omega)$ with:

$$\begin{cases} Lu_k = \lambda u_k + f_k, & \text{on } \Omega, \\ u_k = 0, & \text{on } \partial\Omega \end{cases}$$

being weakly solved, but $\|u_k\|_{L^2(\Omega)} > K\|f_k\|_{L^2(\Omega)}$. We may assume, since L is linear, that:

$$\|u_k\|_{L^2(\Omega)} = 1 \implies f_k \to 0 \text{ in } L^2(\Omega).$$

As for $\beta > 0$ and $\gamma \geq 0$, we have:

$$\begin{aligned}\beta\|u_k\|^2_{H^1_0(\Omega)} &\leq B_{-\lambda}(u_k, u_k) + \gamma\|u_k\|^2_{L^2(\Omega)} \\ &= (f_k, u_k)_{L^2(\Omega)} + \gamma \leq \|f_k\|_{L^2(\Omega)} + \gamma,\end{aligned}$$

since by Cauchy-Schwartz, $|(f_k, u_k)_{L^2(\Omega)}| \leq \|f_k\|_{L^2(\Omega)} \cdot \|u_k\|_{L^2(\Omega)}$, in which $\|u_k\|_{L^2(\Omega)} = 1$, hence:

$$\|u\|_{H^1_0(\Omega)} \leq \frac{\|f_k\|_{L^2(\Omega)} + \gamma}{\beta} < A, \qquad \text{where } A > 0.$$

Hence, we have $\{u_k\}$ being bounded in $H + 0^1(\Omega)$, so by weak compactness, we have $u_k \rightharpoonup u$ in $H^1_0(\Omega)$, so $u_k \to u$ in $L^2(\Omega)$, which implies that $B_{-\lambda}(u, u) = 0$ if and only if u weakly solves:

$$\begin{cases} Lu = \lambda u, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

However, since $\lambda \notin \Sigma$, so $u = 0$, so $\|u\|_{L^2(\Omega)} = 0$, but we had $\|u_k\|_{L^2(\Omega)} = 1$, so $\|u\|_{L^2(\Omega)} = 1$, which is a contradiction. $\square$

VIII Regularity Theory

VIII.1 Intuitions on Regularity

For many PDEs, we want to show that if $Lu = f$ weakly, then u has *better* regularity than f.

We will see that $f \in L^2(\Omega)$ implies that $u \in H^2(\Omega)$, so that $Lu = f$ makes sense almost everywhere (this is not just weakly).

More generally, $f \in H^k(\Omega)$ implies that $u \in H^{k+2}(\Omega)$.

This says that, if L is elliptic, controlling *some of* u's second derivatives, in fact, allows us to control *all* of them. (This is exactly an aspect of ellipticity: Control of $Lu \rightsquigarrow$ control of D^2u if L is elliptic).

Remark VIII.1.1. The *gain* of two derivatives applies to some but not all function spaces:

- **Schaudes theory**: $f \in C^{0,\alpha}$ implies that $u \in C^{2,\alpha}$.
- **Calderon-Zygmind theory**: $f \in W^{k,p}$ implies that $u \in W^{k+2,p}$ for $p \in (1, \infty)$.

⌟

Example VIII.1.2. $f \in C^0$ does not imply $u \in C^2$, since we have:

$$u(x,y) = xy \log\left(-\log\left(\frac{x^2+y^2}{e}\right)\right)$$

is a weak solution to $-\Delta u = f$ for some $f \in C^0$, but u is not C^2 near $(0,0)$. ♢

VIII.2 Interior Regularity

We first need to work "away" from the boundary and show that if $Lu = f$ weakly for $f \in L^2(\Omega)$, implying that $u \in H^2_{\text{loc}}(\Omega)$.

As motivation, if $u \in C_c^\infty(B)$ for $B = B_1(0) \subset \mathbb{R}^n$, then:

$$\|\Delta u\|_{L^2(\Omega)} = \int_B |\Delta u|^2 = \int_B (D_iD_iu)(D_jD_ju) = \int_B (D_iD_ju)(D_iD_ju) = \int_B |D^2u|^2 = \|D^2u\|_{L^2(B)}.$$

Hence, control of Δu in L^2 implies the control of D^2u in L^2.

Ellipticity guarantees Lu in L^2 controls D^2u in L^2 for general L and Ω.

Theorem VIII.2.1. Interior Estimate for Compact Support.
If $\Omega \subset \mathbb{R}^n$ is open and $Lu = -D_j(a^{i,j}D_iu)$ is elliptic with $A^{i,j} \in W^{1,\infty}(\Omega)$ for $i,j = 1, \cdots, n$, then whenever $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ for some $f \in L^2(\Omega)$ with $\{u \neq 0\} \Subset \Omega$, we have

$u \in H^2(\Omega)$ and:

$$\|D^2 u\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}\big),$$

for a constant $C > 0$ depending on n and $a^{i,j}$.

Proof. We assume without loss of generality that $\Omega = \mathbb{R}^n$. Now, $u \in H^1(\Omega)$ weakly solves $Lu = f$, if:

$$\int a^{i,j} D_i u D_j v = \int f v \qquad \text{for all } v \in H^1_0.$$

If $v \in H^1_0$ is compactly supported, so is $D^{-h}_u v$ for $k = 1, \cdots, n$ and $h \in \mathbb{R}$.
By the product rule and integration by parts for difference quotients:

$$\begin{aligned}\int f D^{-h}_k v = \int a^{ij} D_i u D_j (D^{-h}_k v) &= \int a^{i,j} D_i u D^{-h}_k (D_j v) \\ = -\int D^h_k (A^{i,j} D_i u) D_j v &= -\int D^h_k (a^{i,j}) D_i u D_j v - \int a^{ij}(x + he_k)(D_i u) D_j v.\end{aligned}$$

Recall that:

- $\|D^{-h}_k v\|_{L^2} \leq \|Dv\|_{L^2}$.
- As $a^{i,j} W^{1,\infty}(\Omega)$, $\|D^h_k(a^{i,j})\|_{L^\infty} \leq \|Da^{i,j}\|_{L^\infty} \leq C$.

Therefore, from above, we have:

$$\begin{aligned}\left|\int a^{i,j}(x + he_k) D^h_k (D_i u) D_j v\right| &\leq \left|\int f D^{-h}_k v\right| + \left|\int D^h_k(a^{i,j}) D_i u D_j v\right| \\ &\leq C\|f\|_{L^2}\|Dv\|_{L^2} + \|Du\|_{L^2}\|Dv\|_{L^2}.\end{aligned}$$

As L is elliptic, there is $\lambda > 0$ such that:

$$\lambda \|Du\|_{L^2} = \lambda \int |Dv|^2 \leq \int a^{i,j}(x + he_k) D_i v D_j v \qquad \text{for all } v \in H^1_0 \text{ (or } C^\infty_c\text{)}.$$

We choose $v = D^h_k u$ later on.
Hence, we have:

$$\lambda \|Dv\|^2_{L^2} \leq \int a^{i,j}(x + he_k) D_i v D_j v \leq C\big(\|f\|_{L^2} + \|Du\|_{L^2}\big)\|Dv\|_{L^2}.$$

Hence, by dividing $\|Dv\|_{L^2}$ (if $= 0$, we are done), we have:

$$\|D^h_k(Du)\|_{L^2} \leq C\big(\|f\|_{L^2} + \|Du\|_{L^2}\big),$$

where C changes line by line only increasing.
Result for difference quotients implies that:

$$\|D^2 u\|_{L^2} \leq C\big(\|f\|_{L^2} + \|Dv\|_{L^2}\big).$$

□

Theorem VIII.2.2. Interior Estimate for Elliptic PDE.
If $\Omega \subset \mathbb{R}^n$ is open, L is elliptic with $a^{i,j} \in W^{1,\infty}(\Omega)$, $b^i, c \in L^\infty(\Omega)$ for $i, j = 1, \cdots, n$, then if $u \in H^1(\Omega)$ weakly solves $Lu = f$ for $f \in L^2(\Omega)$, we have $u \in H^2_{\text{loc}}(\Omega)$ and for each $\tilde{\Omega} \Subset \Omega$, we have:

$$\|u\|_{H^2(\tilde{\Omega})} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)$$

for a constant $C > 0$ depending on Ω, $\tilde{\Omega}$, and L.

Remark VIII.2.3. C may degenerate (*i.e.*, blow up) as $\tilde{\Omega}$ gets larger, so we will not be able to say that $u \in H^2(\Omega)$ in general, if $\partial\Omega$ is regular enough, we can, however. ⌟

Proof of Theorem VIII.2.2. For $\tilde{\Omega} \Subset \Omega$, let $\varphi \in C_c^\infty(\Omega)$ be such that $\varphi \equiv 1$ on $\tilde{\Omega}$ and $0 \leq \varphi \leq 1$. We will see what PDE the product $\varphi \cdot u \in H_0^1(\Omega)$ solves weakly.
Since $Lu = f$ weakly, if $v \in H_0^1(\Omega)$, then $\varphi v \in H_0^1(\Omega)$ as well. Then we have:

$$\int_\Omega f(\varphi v) = \int_\Omega a^{i,j} D_i u D_j(\varphi v) + b^i D_i u(\varphi v) + Cu(\varphi v),$$

which is equivalently as:

$$\int_\Omega (f - b^i D_i u - cu)\varphi v - (a^{i,j} D_i u D_j \varphi) v = \int_\Omega (a^{i,j} D_i u D_j v)\varphi.$$

Note that $\varphi \cdot D_i u = D_i(\varphi u) - u D_i \varphi$, and we equivalently have:

$$\int_\Omega (f - b^i D_i u - cu)\varphi v - (a^{i,j} D_i u D_j \varphi) v - D_j(a^{i,j} u D_i \varphi) v = \int_\Omega a^{i,j} D_i(\varphi u) D_j v.$$

Hence, this implies that φu weakly solves the PDE, with:

$$-D_j(a^{i,j} D_i(\varphi u)) = \hat{f} = \varphi(f - b^i D D_i u - cu) - a^{i,j} D_i u D_j \varphi - D_j(a^{i,j} u D_i \varphi) \in L^2(\Omega),$$

and φu has compact support in Ω. Hence, by the previous theorem (Theorem VIII.2.1), $\varphi u \in H^2(\Omega)$, and:

$$\|D^2(\varphi u)\|_{L^2(\Omega)} \leq C\big(\|\hat{f}\|_{L^2(\Omega)} + \|D(\varphi u)\|_{L^2(\Omega)}\big) \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\big).$$

As $\|D(\varphi u)\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)}$. Since $\varphi u = u$ on $\tilde{\Omega}$, this says that:

$$\|u\|_{H^2(\tilde{\Omega})} \leq \|D^2(\varphi u)\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\big).$$

Finally, if we can prove the following:

Lemma VIII.2.4. For $\tilde{\tilde{\Omega}} \Subset \Omega$:

$$\|u\|_{H^1(\tilde{\tilde{\Omega}})} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big).$$

Now, we are just left to show that we can squeeze out a little bit more, as we can replace Ω with $\tilde{\tilde{\Omega}}$ so $\tilde{\Omega} \Subset \tilde{\tilde{\Omega}} \Subset \Omega$ for the above inequality. □

Proof of Lemma VIII.2.4. Let $\varphi \in C_c^\infty(\Omega)$ with $\varphi \equiv 1$ on $\tilde{\tilde{\Omega}}$, $0 \le \varphi \le 1$ and consider $\varphi^2 \cdot u \in H_0^1(\Omega)$ in the weak formulation of $Lu = f$ to obtain:

$$\int_\Omega f(\varphi^2 u) = \int_\Omega a^{i,j} D_i u D_j(\varphi^2 u) + (b^i D_i u)(\varphi^2 u) + cu^2\varphi^2.$$

Note that $D_j(\varphi^2 u) = \varphi^2 D_j u + 2u\varphi D_j \varphi$. Rearranging and Cauchy-Schwartz gives us that:

$$\int_\Omega a^{i,j} a^{i,j} \varphi^2 D_i u D_j u \le C\big(\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big).$$

Now, ellipticity of L gives $\lambda > 0$ with:

$$\lambda\|Du\|^2_{L^2(\tilde{\Omega})} \le \int_{\tilde{\tilde{\Omega}}} a^{i,j} D_i u D_j v \le \int_\Omega a^{i,j} \varphi^2 D_i u D_j v.$$

Hence, we have that:

$$\lambda\|Du\|^2_{L^2(\tilde{\tilde{\Omega}})} \le C\big(\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big).$$

Here, we choose $\epsilon \cdot C < \frac{\lambda}{2}$ say, we have that:

$$\|Du\|^2_{L^2(\tilde{\tilde{\Omega}})} \le C\big(\|f\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big).$$

□

VIII.3 Higher Interior Regularity

The idea is that if $f \in H^1(\Omega)$ and $u \in H^2_{\text{loc}}(\Omega)$ is a weak solution to $Lu = f$ (for appropriate coefficients), then we can differentiate the PDE.

Example VIII.3.1. If $f \in H^1(\Omega)$ and $u \in H^2_{\text{loc}}(\Omega)$ weakly solves $-\Delta u = f$, set $v = D_i\varphi$ for $\varphi \in C_c^\infty)\Omega$ to get:

$$-\int_\Omega (D_i f)\varphi = \int_\Omega f(D_i\varphi) = \int_\Omega fv = \int_\Omega Du \cdot Dv = \int_\Omega Du(D_i\nabla\varphi) = -\int_\Omega \nabla_i(Du)\cdot D\varphi.$$

We can write the right hand side as $\nabla(D_i u)D\varphi$, so we have $D_i u \in H^1_{\text{loc}}(\Omega)$ weakly solves:

$$-\Delta(D_i u) = D_i f \in L^2(\Omega).$$

Then, the interior regularity implies that $D_i u \in H^2_{\text{loc}}(\Omega)$, which implies that $u \in H^3_{\text{loc}}(\Omega)$. Iterating this, we get that $f \in H^k(\Omega)$ implies that $u \in H^{k+2}_{\text{loc}}(\Omega)$.
Thus, for k being large, we have the Sobolev embedding for $u \in C^2_{\text{loc}}(\Omega)$ implying that it is the classical solution to $-\Delta u = f$ on $\tilde{\Omega} \Subset \Omega$. ♢

Theorem VIII.3.2. Higher Interior Regularity.
If $\Omega \subset \mathbb{R}^n$ is open, L is elliptic with $a^{i,j} \in W^{k+1,\infty}(\Omega)$, $b^i, c \in W^{k,\infty}(\Omega)$ for $i, j = 1, \cdots, n$, then if

$u \in H^k(\Omega)$ is a weak solution to $Lu = f$ for some $f \in H^k(\Omega)$, we have $u \in H^{k+2}_{\text{loc}}(\Omega)$ and for $\tilde{\Omega} \Subset \Omega$:

$$\|u\|_{H^{k+2}(\tilde{\Omega})} \leq C\big(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\big)$$

for $C > 0$ dependent on $K, \Omega, \tilde{\Omega}$, and L.

Proof. We induct on $K \geq 0$, with the base case $k = 0$ precisely being the previous theorem (Theorem VIII.2.2). Now, we assume that:

$$\|u\|_{H^{k+2}(\tilde{\Omega})} \leq C\big(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\big)$$

holds for some $C > 0$ dependent on $K, \Omega, \tilde{\Omega}$, and L on $k \geq 0$, and we assume that $a^{i,j} \in W^{k+2,\infty}(\Omega)$, $b^i, c \in W^{k+1,\infty}(\Omega)$, $f \in H^{k+1}(\Omega)$ and $u \in H^1(\Omega)$ weakly solves $Lu = f$. The inductive assumption tells us $u \in H^{k+2}_{\text{loc}}(\Omega)$. Fix $\tilde{\Omega} \Subset \hat{\Omega} \Subset \Omega$, with multi-indices $\alpha \in \mathbb{N}^n$ that $|\alpha| = k+1$, and $\varphi \in C_c^\infty(\Omega)$. Let $v = (-1)^{|\alpha|} D^\alpha \varphi$ in the weak formulation, so we have:

$$B(u, v) = (f, v)_{L^2(\Omega)}.$$

By **integration by parts**, we obtain that:

$$B(D^\alpha u, \varphi) = (\hat{f}, \varphi)_{L^2(\Omega)},$$

where $\hat{f} \in L^2(\Omega)$ is given by:

$$\hat{f} = D^\alpha f - \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} \binom{|\alpha|}{|\beta|} \big[-D_j\big(D^{\alpha-\beta}(a^{i,j}) D^\beta(D_i u) + D^{\alpha-\beta}(b^i) D^\beta(D_i u)\big)\big],$$

where $\beta \leq \alpha$ implies that $\beta_i \leq \alpha_i$ for all i. Hence, we obtain that $D^\alpha u$ is a weak solution to $L(D^\alpha u) = \hat{f} \in L^2(\tilde{\Omega})$ on $\hat{\Omega}$.

Hence, by interior estimate, we have $D^\alpha u \in H^2_{\text{loc}}(\hat{\Omega})$ and:

$$\|D^\alpha u\|_{H^2(\tilde{\Omega})} \leq C\big(\|\hat{f}\|_{L^2(\hat{\Omega})} + \|D^\alpha u\|_{L^2(\hat{\Omega})}\big).$$

Now, we have:

$$\|\hat{f}\|_{L^2(\hat{\Omega})} \leq C\big(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{H^{k+1}(\Omega)}\big).$$

Now, we have:

$$\begin{aligned} \|\hat{f}\|_{L^2(\hat{\Omega})} &\leq C\big(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{H^{k+2}(\hat{\Omega})}\big) \leq C\big(\|f\|_{H^{k-1}(\Omega)} + \|u\|_{L^2(\Omega)}\big), \\ \|D^\alpha u\|_{L^2(\hat{\Omega})} &\leq \|u\|_{H^{k+1}(\hat{\Omega})}. \end{aligned}$$

Now, combining all three, we see that as α is arbitrary:

$$\|u\|_{H^{k+2}(\tilde{\Omega})} \leq C\big(\|f\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}\big),$$

which implies that $u \in H^{k+3}_{\text{loc}}(\Omega)$ (as $\tilde{\Omega}$ was arbitrary). □

Theorem VIII.3.3. Smoothness on Interior.
If $\Omega \subset \mathbb{R}^n$ is open, L is elliptic with $a^{i,j}, b^i, c \in C^\infty(\Omega)$ for $i, j = 1, \cdots, b$, then if $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ for $f \in C^\infty(\Omega)$, we have $u \in C^\infty(\Omega)$.

Proof. By Higher Interior Regularity (Theorem VIII.3.2), we have $u \in H^k_{\text{loc}}(\Omega)$ for every $k \geq 0$, hence by Sobolev embedding, we have $u \in C^\ell(\Omega)$ for each $\ell \geq 0$ ($k - \lfloor \frac{n}{2} \rfloor > \ell$) and thus $u \in C^\infty(\Omega) = \bigcap_{k\geq 0} C^\ell(\Omega)$. □

Remark VIII.3.4.

- $u \in H^2_{\text{loc}}(\Omega)$ implies that $Lu = f$ makes sense almost everywhere, but now we have $u \in C^2(\Omega)$ if k is large enough ($k - \lfloor \frac{n}{2} \rfloor > \ell$), so $Lu = f$ makes sense everywhere, *i.e.*, u is a classical solution to the PDE.
- The technique of "improving" regularity from the PDE itself is called **bootstrapping**.
- We have not said anything about the regularity of u near $\partial\Omega$.

┘

VIII.4 Boundary Regularity

We aim to extend the interior estimates for weak solutions *up to* the boundary, assuming the boundary is sufficiently regular and the weak solution satisfies some boundary conditions, *e.g.* $u \in H^1_0$.

The question now becomes: Why do we need assumptions on $\partial\Omega$ to get regularity?

Example VIII.4.1. Consider for $\beta \in (0, 2\pi)$, we have:

$$\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < \beta\},$$

Here, we can visualize some sectors:

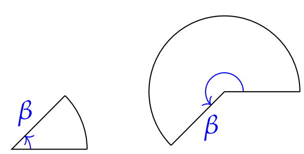

Figure VIII.1. Some sectors with angle β.

then we have $\Omega \subset \mathbb{R}^2$, and $\partial\Omega$ is Lipschitz but not C^1.
In polar coordinates, we let:

$$u(r, \theta) = r^{\pi/\beta} \sin\left(\frac{\pi\theta}{\beta}\right) \in C^\infty(\Omega).$$

Then, if $\theta = 0$, we have $u = 0$, and writing:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Hence, we have $\Delta u = 0$ strongly in Ω and one can check that $u \in H^1(\Omega)$ with:

$$\int_\Omega |Du|^2 = \frac{\pi}{2}.$$

Now, if $\beta > \pi$, we have $u \notin H^2(\Omega)$ since:

$$\int_\Omega |D^2 u|^2 \geq \int_0^\beta \int_0^1 \left|\frac{\partial^2 u}{\partial r^2}\right|^2 r dr d\theta \geq C \int_0^1 r^{2\pi/\beta - 3} dr.$$

As $\frac{2\pi}{\beta} - 3 < -1$ if and only if $\beta > \pi$, which implies that:

$$\int_0^1 r^{2\pi/\beta - 3} dr = +\infty,$$

hence $u \notin H^2(\Omega)$.
If we let $r > 0$ (not < 1), we have $u \in H_0^1(\Omega)$. $\Diamond$

Now, we have an analogue of the interior estimate.

Theorem VIII.4.2. Boundary Estimate.
If $\Omega \subset \mathbb{R}^n$ is open, $\partial\Omega$ is C^2, L is elliptic with $a^{i,j} \in W^{1,\infty}(\Omega)$, $b^i, c \in L^\infty(\Omega)$ for $i, j = 1, \cdots, n$, then if $u \in H_0^1(\Omega)$ is a weak solution to:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

for $f \in L^2(\Omega)$, we have that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and:

$$\|u\|_{H^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)$$

for a constant $C > 0$ depending on Ω, n, and L.

Remark VIII.4.3. If such a weak solution to the differential equation is unique, then one can show that:

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

(*since if $Lu, L\tilde{u} = f$, then we have $L(u - \tilde{u}) = 0$ and so $u = \tilde{u}$ almost everywhere.*)

- We **require** $\partial\Omega$ is C^2 and $u \in H_0^1(\Omega)$.

$\lrcorner$

Proof of sketch. Consider that we have $\partial\Omega$ to be C^2, we consider the straightening:

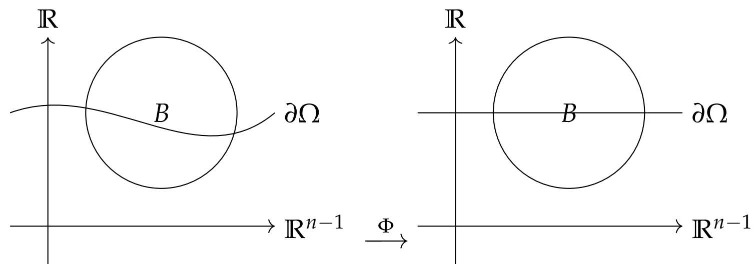

Figure VIII.2. Straightening $\partial\Omega$ and use interior ideas/estimates.

□

Theorem VIII.4.4. Higher Boundary Regularity.
If $\Omega \subset \mathbb{R}^n$ is open, $\partial\Omega$ is C^{k+2}, L is elliptic with $a^{i,j} \in W^{k+1,\infty}(\Omega)$, $b^i, c \in W^{k,\infty}(\Omega)$ for $i, j = 1, \cdots, n$, $k \in \mathbb{N}$, then if $u \in H_0^1(\Omega)$ is a weak solution to:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

for $f \in H^k(\Omega)$, we have $u \in H^{k+2}(\Omega) \cap H_0^1(\Omega)$ and $\|u\|_{H^{k+2}(\Omega)} \leq C\big(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\big)$ for a constant $c > 0$ depending on k, Ω, L, and n.

By repeatedly applying higher boundary regularity, we have the following.

Theorem VIII.4.5. Smoothness up to the Boundary.
If $\Omega \subset \mathbb{R}^n$ is open, $\partial\Omega$ is C^∞, L is elliptic with $a^{i,j}, b^i, c \in C^\infty(\overline{\Omega})$ for $i, j = 1, \cdots, n$, then if $u \in H_0^1(\Omega)$ is a weak solution to:

$$\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $f \in C^\infty(\overline{\Omega})$, we have $u \in C^\infty(\overline{\Omega})$.

Remark VIII.4.6. Now, we can entirely solve the Poisson equation on smooth domains in the classical sense. (*I.e.*, the elliptic PDEs can ne connected to the Laplacian operator.) ⌟

The proofs can be found from Evans' section 6.3.2.

Part 4

Second Order Nonlinear PDEs

IX Extending from Second Order Linear PDEs

IX.1 Existence for a Nonlinear PDE

The aim is to think of $-\Delta u = u^p$ for $p \in [1, 2^*]$, which is a *quasi-linear* PDE, and we want to think about the existence of weak solutions using **direct method** and use regularity to show that weak solutions lead to strong solutions.

Consider $\Omega \subset \mathbb{R}^n$ being open and bounded, with $\partial\Omega$ being C^3 and for $n \geq 3$, we seek a non-negative function $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ with $u > 0$ in Ω, and solving:

$$\begin{cases} -\Delta u = u^p, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

for $p \in \left(1, \frac{n+2}{n-2}\right)$.

Remark IX.1.1. The above example is semi-linear as it is linear in its highest (2nd) order derivatives. The method we will use to find weak solutions applies more generally to PDEs of the form:

$$\begin{cases} Lu = f(u), & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Provided that L is elliptic, and f has controlled growth.

- $\left(\frac{n+2}{n-2}\right)$ has an upper bound for p will be used since:

$$p + 1 = \frac{n+2}{n-2} + 1 = \frac{2n}{2n-2} = 2^*,$$

  i.e., $p+1$ is critical exponent for the Sobolev embedding to be compact.

- To solve the equation for $p + 1 = 2^*$, one needs more refined methods, and this value of p is related to the **Yamate problem**.

⌟

Definition IX.1.2. Yamate Problem.
Suppose (M, g) is a compact Riemannian manifold with dimension $n \geq 3$, is there a metric $\tilde{g} = e^{2u} g$, such that $\tilde{g}$ has constant scalar curvature? ⌟

Existence: Let's drop the assumption $\partial\Omega$ is C^3 for now. We will now find $u \geq 0$ and $u \not\equiv 0$ over Ω. Hence, the weak solution must solve that:

$$\int_\Omega Du \cdot D\varphi = \int_\Omega u^p \varphi \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

This makes sense since $u^p \in L^1(\Omega)$ as:

$$p < \frac{n+2}{n-2} < \frac{2n}{n-2} = 2^* \implies u \in L^p(\Omega),$$

since $u \in H_0^1(\Omega)$ and the Poincaré inequality.

Let's try to use the **direct method** where we saw critical points of:

$$E(u) = \int_\Omega |Du|^2 \text{ gives weak solutions to } \Delta u = 0 \text{ on } \Omega.$$

We modify this and for $q \in \left(2, \frac{2n}{n-2}\right)$, let:

$$E_q(u) = \frac{E(u)}{\|u\|_{L^q(\Omega)}} \text{ for } u \in H_0^1(\Omega) \setminus \{0\},$$

again, by using that $H_0^1 \hookrightarrow L^q$.
Note that:

$$E_q(u) = \frac{E(u)}{\|u\|_{L^q(\Omega)}} = \frac{\|Du\|^2_{L^2(\Omega)}}{\|u\|_{L^q(\Omega)}} \geq \frac{1}{c_p} > 0,$$

and by the Poincaré inequality (as $q < 2^*$ and Ω is bounded), we set:

$$\mathcal{E}_q = \inf\left\{ E_q(u) \mid u \in H_0^1(\Omega) \setminus \{0\} \in \right\} > 0.$$

We let $(u_i) \subset H_0^1(\Omega) \setminus \{0\}$ be such that $E_q(u_i) \to \mathcal{E}_q$.

As $E_q(\lambda u) = E_q(u)$ for $\lambda > 0$, $u \in H_0^1(\Omega) \setminus \{0\}$, so we may suppose (without loss of generality) that $\|u_i\|_{L^q(\Omega)} = 1$ for each $i \geq 1$, hence we have $E_q(u_i) = E(u_i)$.

So (u_i) is bounded in $H_0^1(\Omega)$ since for large $i \geq 1$, we have:

$$\|u_i\|^2_{H_0^1(\Omega)} \leq C\|Du\|^2_{L^2(\Omega)} = CE(u_i) = CE_q(u_i) < C(\mathcal{E}_q + 1).$$

Hence, by weak compactness in $H_0^1(\Omega)$, there is a weakly compact subsequence $u_{\iota(i)} \rightharpoonup u \in H_0^1(\Omega)$, and thus by Rellich-Kandrachov, this implies that $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$, so $u_{\iota(i)} \to u \in L^q(\Omega)$, and in particular (since $\|u_{\iota(i)}\|_{L^q(\Omega)} = 1$), we have $\|u\|_{L^q(\Omega)}$.
We also have $E(u) \leq \liminf_{i\to\infty} E(u_{\iota(i)})$. Since the norms are lower-semi-continuous under weak convergence, we have:

$$E_q(u) = E(u) \leq \liminf_{i\to\infty} E(u_{\iota(i)}) = \mathcal{E}_q,$$

and since $u \in H_0^1(\Omega) \setminus \{0\}$, we have $E_q(u) = \mathcal{E}_q$.
To ensure that $u \geq 0$ almost everywhere, note that $|u| \in H_0^1(\Omega) \setminus \{0\}$ with $|Du| = \big|D|u|\big|$, so $E_q(|u|) = E_q(u) = \mathcal{E}_q$, so by replacing that $u = |u|$, we ensure that $u \geq 0$ almost everywhere.

Let's see what PDE u solves, if $\varphi \in C_c^\infty(\Omega)$, and $u + t\varphi \in H_0^1(\Omega) \setminus \{0\}$ (for small t) so that:

$$E_q(u + t\varphi) \geq E_q(u),$$

and hence $\frac{d}{dt}E_q(u+t\varphi)\big|_{t=0} = 0$, which is:

$$\begin{aligned}
\frac{d}{dt}E_q(u+t\varphi) &= \frac{d}{dt}\left[E(u+t\varphi)\|u+t\varphi\|^{-2}_{L^q(\Omega)}\right] \\
&= 2\int_\Omega Du\cdot D\varphi + E(u)\left(-2\|u\|^{-3}_{L^q(\Omega)}\frac{d}{dt}\|u+t\varphi\|_{L^q(\Omega)}\right) \\
&= 2\int_\Omega Du\cdot D\varphi - 2\mathcal{E}_q\frac{d}{dt}\left(\int_\Omega |u+t\varphi|^q\right)^{\frac{1}{q}} \\
&= 2\int_\Omega Du\cdot D\varphi - 2\mathcal{E}_q\cdot\frac{1}{q}\left(\int_\Omega |u|^q\right)^{\frac{1}{q}-1}\left(\int_\Omega q|u|^{q-1}\varphi\right).
\end{aligned}$$

Hence, we now have:

$$\int_\Omega Du\cdot D\varphi = \frac{\mathcal{E}_q}{q}\int_\Omega qu^{q-1}\varphi$$

for all $\varphi\in C_c^\infty(\Omega)$. Hence, we have shown that u weakly solves:

$$\begin{cases} -\Delta u = \mathcal{E}_q u^{q-1}, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Given $p\in\left(1,\frac{n+2}{n-2}\right)$, we set $q = p+1 < 2^*$ so that u above weakly solves:

$$\begin{cases} -\Delta u = \mathcal{E}_q u^{p}, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Consider $\mathcal{E}_q^{\frac{1}{p-1}}$, then it weakly satisfies that:

$$-\Delta\big(\mathcal{E}_q^{\frac{1}{p-1}}u\big) = \mathcal{E}_q^{\frac{1}{p-1}}(-\Delta u) = \mathcal{E}_q^{\frac{1}{p-1}}\mathcal{E}_q u^p = \mathcal{E}_q^{\frac{p}{p-1}}u^p = (\mathcal{E}_q^{\frac{1}{p-1}}u)^p,$$

so $u = \mathcal{E}_q^{\frac{1}{p-1}}u$ is ≥ 0 almost everywhere, and weakly solves:

$$\begin{cases} -\Delta u = u^{p}, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $u\in H_0^1(\Omega)\setminus\{0\}$ and ≥ 0 almost everywhere.

Now, if $u^p\in L^2(\Omega)$, we have $u\in H^2_{\text{loc}}(\Omega)$ (by our interior regularity, "freezing" $f = v^p$).
Also, if $v^p\in H^k(\Omega)$ so we have $v\in H^{k+2}_{\text{loc}}(\Omega)$.

IX.2 Regularity for a Nonlinear PDE

Under the assumption that $\partial\Omega$ being C^3, we want to then show that:

$$u\in C^\infty(\Omega)\cap C^{2,\alpha}(\overline{\Omega}).$$

Recall we saw that if $-\Delta u = f\in L^2(\Omega)$ is solved weakly by $v\in H_0^1(\Omega)$, then in fact we have $v\in H_0^2(\Omega)$.

This principle holds if $f \in L^q(\Omega)$ for $q \in (1,\infty)$ by Calderon-Zygmind estimates which then ensure that $w \in W_0^{2,q}(\Omega)$.

In our situation that $f = u^p$, we now use these estimates to argue that $u \in C^{0,\alpha}(\overline{\Omega})$.

Suppose we had $u \in W_0^{1,s}(\Omega)$ for some $s \geq 2$ (we already have $s = 2$). If $s > n$, we can apply Morrey inequality that $u \in C^{0,\alpha}(\overline{\Omega})$. If $s \in [2,n]$, however, we consider the following cases:

- First, if $s \in [2,n)$, the GNS inequality gives $u \in L^{\frac{ns}{n-s}}(\Omega)$ so we have $u^p \in L^{\frac{ns}{p(n-2)}}(\Omega)$.
  As $s \geq 2$, $p \in \left(1, \frac{n+2}{n-2}\right)$ and $n > 2$, so we have:
  $$\frac{ns}{p(n-s)} \geq \frac{2n}{p(n-2)} > \frac{2n}{n+2} > 1.$$
  So by Calderon-Zygmind, we have $u \in W_0^{2,\frac{ns}{p(n-s)}}(\Omega)$.
- Second, if $s \in \left[2, \frac{np}{1+p}\right)$, we have:
  $$\frac{ns}{p(n-s)} < \frac{n\left(\frac{np}{1+p}\right)}{p\left(n - \frac{np}{1+p}\right)} = \frac{n^2}{n} = n.$$
  Now, by applying GNS inequality to $D_i u$ for $i = 1, \cdots, n$, we have:
  $$D_i u \in W^{1,\frac{ns}{p(n-s)}}(\Omega)$$
  with:
  $$\sigma = \frac{n\left(\frac{ns}{p(n-s)}\right)}{n - \left(\frac{np}{p(n-s)}\right)} = \frac{ns}{(n-s)p - s} = \underbrace{\left(\frac{ns}{np - s(p+n)} \geq \frac{n}{np - 2(p+1)}\right)}_{\gamma} s = \gamma s.$$
  Now, since $p \in \left(1, \frac{n+2}{n-2}\right)$, so we have:
  $$\gamma = \frac{n}{np - 2(p+1)} = \frac{n}{p(n-2) - 2} > \frac{n}{n+2-2} = 1.$$
  Thus, we have:
  $$u \in W_0^{1,s}(\Omega) \qquad \text{for } s \in \left[2, \frac{np}{1+p}\right),$$
  so we have $u \in W_0^{1,\gamma s}(\Omega)$ for some $\gamma > 1$.
- Starting at $s = 2$, we iteratively apply the previous step, we get $u \in W_0^{1,\gamma^k s}(\Omega)$ for all $k \geq 1$ such that $\gamma^k s < \frac{np}{n-p}$, so one application give $u \in W_0^{1,t}(\Omega)$ for some $t > \frac{np}{1+p}$, so we have $u \in W_0^{1,\frac{np}{1+p}}(\Omega)$.
  As $\frac{np}{1+p} < n$, by applying GNS inequality again, we have $u \in L^{\frac{n\left(\frac{np}{1+p}\right)}{n-\left(\frac{np}{1+p}\right)}}(\Omega) = L^{np}(\Omega)$, so we have $u^p \in L^n(\Omega)$.

One further application of the Calderon-Zygmind gives $u \in W_0^{2,n}(\Omega)$, and by Sobolev embedding, as $2 > \frac{n}{p}$, we have that $u \in C^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$.

We now use Schauder estimates which says that $-\Delta v = f \in C^{0,\alpha}(\overline{\Omega})$ implies $v \in C^{2,\alpha}(\overline{\Omega})$.

As we have $-\Delta u = u^p \in C^{0,\alpha}(\overline{\Omega})$ as $x \mapsto x^p$ locally Lipschitz as $p > 1$ and so Schauder estimates give $u \in C^{2,\alpha}(\overline{\Omega})$. Hence, u solves (7).

Iteratively applying the Scauder estimates to $\tilde{\Omega} \Subset \Omega$ with $\partial\tilde{\Omega}$ being smooth (differentiating the PDE: $-\Delta u = u^p$ into $-\Delta D_i^k u = D_i^k(u^p) \in C^{0,\alpha}(\overline{\Omega})$, so $u \in C^{k+2,\alpha}(\overline{\Omega})$) to get $u \in C^\infty(\overline{\tilde{\Omega}})$, *i.e.*, $u \in C^\infty(\Omega)$.
Thus by Calderon-Zygmind and Schauder, we have $u \in C^{2,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$.