On the Generic Regularity of Constant Mean Curvature Hypersurfaces

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“Probably the only thing one can really learn, the only technique to learn, is the capacity to be able to change.”

Philip Guston

Declaration

I, Kobe Marshall-Stevens, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Kobe Marshall-Stevens

Abstract

This thesis concerns the regularity properties of constant mean curvature hypersurfaces. These hypersurfaces arise naturally as boundaries to isoperimetric regions (regions with least boundary area for a fixed enclosed volume) and more generally as critical points of an area-type functional. Historically, constant (and other prescribed) mean curvature hypersurfaces have served as effective tools in developing our understanding of the interaction between the curvature and topology of Riemannian manifolds.

Constant mean curvature hypersurfaces may be produced by various minimisation and min-max procedures. Sharp regularity theory guarantees that, in both cases, the hypersurface produced will be smoothly immersed away from a closed singular set of codimension seven. In particular, when the ambient manifold is of dimension eight, one produces a constant mean curvature hypersurface which is smoothly immersed away from finitely many isolated singular points.

The presence of a singular set in high dimensional hypersurfaces of constant mean curvature means that, in general, they may fail to be an effective tool for geometric and topological application. One method to deal with the presence of a singular set is to show that generically one can remove it, resulting in an entirely smooth constant mean curvature hypersurface suitable for effective application. This generic regularity approach requires a finer understanding of the singularities that arise as well as the development of perturbation procedures to remove them.

This thesis utilises and develops techniques in the calculus of variations, elliptic partial differential equations and geometric measure theory in order to produce the first generic regularity results for the general class of constant mean curvature hypersurfaces when the ambient manifold is of dimension eight. The first part of this thesis establishes relevant background while the second part obtains generic regularity results under the assumption of positive Ricci curvature.

Impact Statement

Many natural phenomena are well approximated by solutions to geometric variational problems. In particular, constant mean curvature hypersurfaces provide mathematical realisations of soap films and bubbles. Historically, smooth constant (and other prescribed) mean curvature hypersurfaces have also served as effective tools for answering fundamental questions arising in both mathematical physics as well as the interaction between geometry and topology. Consequently, there is a large ongoing program, with contributions from researchers worldwide, dedicated to studying the regularity properties of constant mean curvature and more general variational hypersurfaces.

The scope of this thesis is to provide the very first generic regularity results for the general class of constant mean curvature hypersurfaces. These so called generic regularity results have in recent years seen significant development for the class of minimal hypersurfaces, a special instance of constant mean curvature hypersurfaces. In general, generic regularity results show that one may perturb away potential singular behaviour in order produce an entirely smooth variational hypersurface; the present work addresses the case of isolated singular points arising in constant mean curvature hypersurfaces. The resulting smooth constant mean curvature hypersurfaces produced by our results are thus suitable for effective application to problems arising in both mathematical physics as well as the interaction between geometry and topology.

One method used in this thesis to refine our understanding of the regularity properties of constant mean curvature hypersurfaces is to directly exploit their deep connection to phase transition phenomena, specifically arising from the Allen–Cahn energy. Over the past decade, there has been significant progress in our collective understanding of this connection. However, its profound potential in establishing geometrical results is only recently beginning to be realised.

Acknowledgements

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Finally, I'll give a little shout out to my dad and mum, for bringing me into this world and so on.

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Chapter 1

Background

Solutions to geometric variational problems, for example those arising via minimisation of an energy functional, provide mathematical realisations of various natural phenomena; as a pair of leading examples, one may consider soap films and bubbles. These problems have a rich history, with various formulations of geometric variational problems documented at least as far back as the ancient Greeks, who recorded that the resolution of the isoperimetric problem by Dido at Carthage took place around 800BC. The isoperimetric problem, which seeks the largest enclosed area by a boundary curve of fixed length, and generalisations thereof, remains an active area of mathematical research. Through deepening our understanding of these problems, significant developments in the interaction between geometry and topology have followed.

The area functional serves as perhaps the simplest geometric functional, and as such provides a good first approximation to various natural phenomena. One is thus interested in studying the behaviour of critical points of the area functional (and other area-type functionals) in a mathematical framework. Such a framework, provided by the fields of calculus of variations, geometric measure theory and partial differential equations, has seen a marked development in the past century. A major early success for these fields was the resolution of the Plateau problem, which seeks a surface of least area spanning a given curve, in [Dou31, Rad30] resolving a question of Lagrange that had remained open since 1760.

Historically, smooth hypersurfaces arising as critical points of area-type functionals have served as effective tools in their application to solve problems arising in low-dimensional geometry and topology. We highlight here the resolution of the positive mass conjecture in general relativity in [SY79a, SY79b], the proof of the finite extinction time of the Ricci flow in [Per08, CM05], the resolution of the Willmore conjecture in [MN14], and finally recent work concerning the topology of manifolds of positive scalar curvature in [CL24]. When hypersurfaces arising as critical points of area-type functions fail to be smooth however, they also in general fail to be effective tools in their application.

This thesis concerns the study of high-dimensional solutions to some geometric variational problems, in particular focusing on the singular behaviour of the hypersurfaces that arise as critical points to area-type functionals. In the lowest dimension in which singular behaviour occurs for these objects we establish, in various settings, that the presence of singularities in these hypersurfaces is not generic in a topological sense; more precisely, generically one can find a smooth hypersurface arising as a critical point.

In this chapter, we provide background on subjects relevant to the main body of the thesis. We begin with a description of the main result of the thesis, along with a brief summary of the proof strategy. We then discuss both the existence and corresponding regularity theory for the area functional, introducing various general techniques and notions used throughout the thesis. Finally, we conclude with a brief survey of historical developments in the field, accompanied by a summary of previous work on generic regularity.

1.1 Results and organisation

In this thesis we establish the first generic regularity results for the general class of constant mean curvature hypersurfaces. The new contributions of the thesis are contained in Chapter 2; the results of which were developed jointly with Costante Bellettini and appear in [BM23].

1.1.1 Main theorem and proof strategy

The main theorem obtained in the thesis is the following:

Theorem 1.1. *Given any $\lambda \in \mathbb{R}$, in a generic compact 8-dimensional Riemannian manifold with positive Ricci curvature, there exists a closed embedded smooth hypersurface with constant mean curvature λ.*

The term generic here is understood in the topological sense and refers to an open and dense subset of the metrics of positive Ricci curvature. A precise statement of Theorem 1.1 (and statements for other ancillary results of the thesis) may be found in Section 2.1 of Chapter 2. We now briefly discuss the proof of Theorem 1.1, and for further detail direct the reader to the expanded discussion of the proof strategy contained in Subsection 2.1.3 of Chapter 2.

In a compact Riemannian manifold, the work of [BW20a] established, for each $\lambda \in \mathbb{R}$, the existence of a hypersurface of constant mean curvature λ which is smoothly immersed away from a closed set of codimension 7. In particular, in ambient dimension 8, the constant mean curvature hypersurfaces produced by the Allen–Cahn min-max procedure of [BW20a] are smoothly immersed away from finitely many isolated points. This existence theory for constant mean curvature hypersurfaces relies both on the sharp regularity theory of [BW20b, BW20c] as well as a min-max procedure for a modified Allen–Cahn energy. A detailed discussion of this construction and the relevant preliminaries are contained in Subsection 2.1.2 of Chapter 2.

By directly exploiting this min-max construction, in [BW24] it was shown that under an assumption of positive Ricci curvature for the ambient manifold, the Allen–Cahn min-max procedure of [BW20a] above produces constant mean curvature hypersurfaces which are smoothly embedded (as opposed to smoothly immersed) away from a closed set of codimension 7; in particular, in ambient dimension 8 these hypersurfaces are smoothly embedded away from finitely many isolated points.

As the starting point for the proof of Theorem 1.1 we therefore know that for each $\lambda \in \mathbb{R}$, in a compact 8-dimensional Riemannian manifold with positive Ricci curvature, the Allen–Cahn min-max procedure of [BW20a] produces a hypersurface of constant mean curvature λ which is smoothly embedded away from finitely many isolated points. We work directly with such a hypersurface in order to establish Theorem 1.1.

Starting with the hypersurface of constant mean curvature λ produced by the Allen–Cahn min-max procedure of [BW20a] as a candidate, Theorem 1.1 ultimately follows by application of a local "cut-and-paste" surgery procedure that removes the finitely many isolated points at which the hypersurface fails to be embedded; yielding a closed embedded smooth constant mean curvature hypersurface after conformal metric perturbation. The surgery procedure we develop exploits local foliations by constant mean curvature hypersurfaces around such points as established in [HS85] and [Les23]; for further details on this procedure we refer the reader to Step 3 of Subsection 2.1.3 and Proposition 2.1 in Chapter 2. As a further application of this surgery procedure, we also partly answer an open question of Lawson; see Subsection 2.2.2 in Chapter 2.

However, in order to apply the aforementioned surgery procedure to each of the finitely many isolated points at which the constant mean curvature hypersurface fails to be embedded, we in fact need to know that the hypersurface is a local minimiser of the natural area-type functional around these points. Indeed, it is precisely this local minimisation condition that guarantees the existence of the local foliations of [HS85] and [Les23] with which our "cut-and-paste" surgery procedure is constructed.

We therefore turn our attention to showing that the candidate constant mean curvature hypersurface locally minimises the natural area-type functional around each of the finitely many isolated points at which it fails to be embedded. This is achieved by first relating this local minimisation to Allen–Cahn energy properties of specific functions constructed from the underlying hypersurface.

By pasting in the normal direction to our hypersurface (suitably modified) solutions to the one-dimensional Allen–Cahn equation for each scale $\varepsilon > 0$, we construct approximating functions with Allen–Cahn energy approaching the value of the natural area-type functional evaluated on our hypersurface as $\varepsilon \to 0$; see Subsection 2.3.3 of Chapter 2 for this construction. We then show that minimisation of the natural area-type functional in a small ball follows if the Allen–Cahn energy of these approximating function remains close, as $\varepsilon \to 0$, to that of the minimiser of the Allen–Cahn energy in this ball (with boundary conditions imposed by these approximating functions); this is discussed in further detail in Step 1 of Subsection 2.1.3 and established in Section 2.4 of Chapter 2.

Remark 1.1. *The manner in which we deduce local area minimisation (in the case that $\lambda = 0$) may be compared to the way in which optimal regularity is deduced for minimal hypersurfaces produced by the Almgren–Pitts min-max procedure. In [Pit81, Chapter 3] it is shown that the Almgren–Pitts min-max procedure produces varifolds satisfying an almost minimising property, allowing for stability in annular regions to be deduced (from which their regularity ultimately follows). The way in which we deduce local area minimisation is analogous to this notion in the sense that we require our approximating functions to remain close in Allen–Cahn energy (as opposed to close in area to local area minimisers in the definition of almost minimising) to that of Allen–Cahn energy minimisers in a ball.*

However, our approach differs in that for deducing local area minimisation we need not ask for the existence of a path of functions between *our approximating functions and the local minimiser with controlled increase in Allen–Cahn energy (as opposed to the isotopy with controlled area required in the definition of almost minimising). Paths between our approximating functions and local Allen–Cahn energy minimisers are in fact constructed as part of the proof of Theorem 1.1, but upper Allen–Cahn energy bounds along the paths produced are independent of the difference in energies between the functions and depend only on geometric properties of our underlying hypersurface; we refer the reader to Step 2 of Subsection 2.1.3 and Subsection 2.5.5 of Chapter 2 for further discussion on this point.*

Finally, we note that the conclusion of area minimisation (as opposed to the weaker notion of almost minimisation) for minimal hypersurfaces arising from the Allen–Cahn min-max procedure in compact 8*-dimensional Riemannian manifolds with positive Ricci curvature makes explicit use of the curvature assumption in its proof. It is, as yet, unclear whether the local area minimisation conclusion holds without this curvature assumption.*

Having related local minimisation of the natural area-type functional to the Allen–Cahn energy behaviour of specific functions approximating the underlying geometry, we now establish this local minimisation for the candidate constant mean curvature hypersurfaces by directly exploiting their construction via the Allen–Cahn min-max. Specifically, to derive a contradiction under the assumption that the local minimisation failed (i.e. assuming that the Allen–Cahn energy of our approximating functions differed by a fixed amount to that of the local Allen–Cahn energy minimisers for all $\varepsilon > 0$ sufficiently small), we exhibit a path of functions admissible in the min-max construction but with Allen–Cahn energy along this path bounded above by a value strictly less than the min-max value (which we show to be the value of the natural area-type functional evaluated on our hypersurface). This contradiction implies that constant mean curvature hypersurfaces produced by the Allen–Cahn min-max procedure in 8-dimensional compact manifolds with positive Ricci curvature in fact locally minimise their natural area-type functional. For a heuristic description of this path and of the reliance of the positive Ricci curvature assumption for controlling the Allen–Cahn energy, we refer the reader to a detailed discussion in Step 3 of Subsection 2.1.3 of Chapter 2; this path is explicitly constructed in Section 2.5 of Chapter 2.

To summarise the above, we will deduce Theorem 1.1 in three steps. First, we show that local minimisation of the natural area-type functional is related to the $\varepsilon \to 0$ behaviour of the Allen–Cahn energy for approximating functions constructed from the candidate constant mean curvature hypersurfaces. Second, we exploit the relation above and the Allen–Cahn min-max construction of these hypersurfaces in order to deduce local minimisation by exhibiting an admissible min-max path; this yields a contradiction under the assumption that local minimisation fails. Third, we employ a local "cut-and-paste" surgery procedure along with a conformal change of metric to perturb away the finitely isolated points at which the constant mean curvature hypersurface fails to be embedded; this final step concludes the proof of Theorem 1.1. A more detailed discussion of these three steps, alongside the necessary definitions and preliminaries, is the focus of Section 2.1 of Chapter 2.

Finally, for the reader interested in the results obtained in Chapter 2 solely for minimal (i.e. the case $\lambda = 0$) hypersurfaces we also provide, in Appendix 2.A, simplifications of various arguments which serve as a more direct route to establishing our results in this case. In particular, the simpler alternative upper energy bound calculations included in this appendix may be used directly in place of those computed in Subsections 2.5.3 and 2.5.4 in order to establish Theorem 1.1 in the minimal case.

1.2 A brief history of existence

For a given geometric variational problem, one is concerned with both the existence and corresponding regularity of its solutions. A major issue for existence theory is that often the desired solution space, for example the class of smooth hypersurfaces, lacks any natural compactness property. To surmount this issue, one typically enlarges the class of admissible solutions to a space with good compactness properties in order to produce a suitable candidate, a so-called *weak solution*; after which one may then turn their attention to its regularity.

This is in strong analogy with with the study of partial differential equations. For example, one can exploit the Rellich–Kondrachov compactness theorem in order to guarantee the existence of minimisers of the Dirichlet energy, subject to a boundary condition, in the Sobolev space $W^{1,2}$ (enlarging the class of smooth functions). One then proceeds to apply the Weyl lemma to the variational formulation of the problem in order to deduce regularity for this weak solution, i.e. establishing that this function is in fact smooth in the interior.

We begin by considering the model problem of finding closed geodesics in surfaces, emphasising the need for an alternative approach to direct minimisation in order to find critical points of the area functional. To address more general problems, we then introduce various generalisations of the notion of a submanifold, providing us with suitable solution spaces possessing favourable compactness properties. Finally, we discuss the deep connection between critical points of the area functional and phase transition phenomena.

1.2.1 A case study in closed geodesics

Recall that a closed geodesic in a Riemannian manifold is a smooth, closed curve of locally shortest length between any two points contained on it. Equivalently, closed geodesics, $\gamma : S^1 \to \Sigma$, in a Riemmanian surface (Σ^2, g) are characterised as critical points to the length functional

$$L(\gamma) = \int_{S^1} \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt.$$

Here, a critical point of the length functional is a closed curve, γ, as above such that we have

$$\left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) = 0,$$

for all smooth variations, γ_s, with $\gamma_0 = \gamma$ (precisely, $\gamma_s(t) = H(t,s)$ for some smooth map $H : S^1 \times (-\varepsilon, \varepsilon) \to \Sigma$ with $\varepsilon > 0$); for a proof of this equivalence see [Lee18, Corollary 6.7].

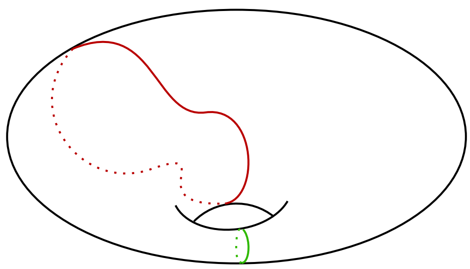

Figure 1.1: A torus showing two closed curves in the same homotopy class. The curve in green indicates a length minimising closed geodesic.

A natural question one may ask is whether there exist closed geodesics in any compact Riemannian surface. To answer this, one may first try to minimise the length functional among all curves in a given homotopy class and, upon taking a sub-sequence converging to the infimum of the lengths of these curves, find a length minimising geodesic in this homotopy class. This method was successfully carried out in [Had98] and in particular, when the homotopy class is non-trivial, produces a non-trivial geodesic (i.e. a curve of non-zero length); this procedure is depicted in Figure 1.1.

The issue one encounters with this approach is that if the homotopy class is trivial, namely when our surface is a topological sphere, one will simply minimise the length to zero and produce a point; see Figure 1.2.

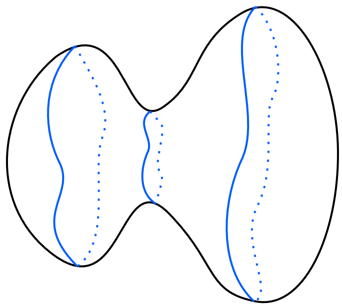

Figure 1.2: A sphere showing a sweepout by closed curves. Trying to minimise length in the homotopy class of the sphere would result in a trivial geodesic (a point).

As a concrete example, in the round sphere all closed geodesics are given by great circles (intersections of the sphere, centered at the origin, with planes in $\mathbb{R}^3$), which are all unstable critical points of the length (pushing the curves in their normal direction strictly decreases their length). In order to locate these unstable non-trivial geodesics we thus require a general procedure to locate critical points that may no longer be global minimisers.

In [Bir17] the existence of a closed geodesic on any topological sphere was established by introducing a technique now well known as a *min-max*. Loosely speaking, in this construction one considers sweep-outs (by which we mean a set of pairwise disjoint closed curves, $\Gamma \subset S^2$, whose union is S^2) of the ambient Riemannian surface, (S^2, g), by closed curves and keeps track of the lengths of the curves along the sweep-out; see Figure 1.3.

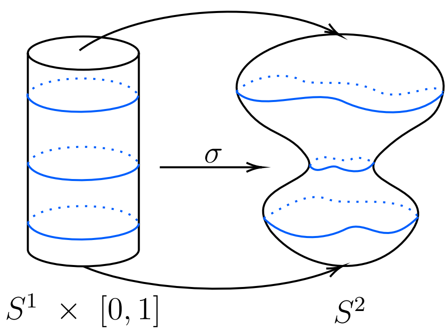

Figure 1.3: A sweep-out of S^2 from a continuous map σ.

More precisely, one considers the set, $\mathcal{S}$, of continuous maps

$$\sigma : S^1 \times [0,1] \to S^2,$$

that map $S^1 \times \{0\}$ and $S^1 \times \{1\}$ to points, that for each $t \in [0,1]$ the map $\sigma(\cdot, t) \in W^{1,2}$, and are continuous in the second variable as a map from $[0,1]$ to $W^{1,2}$. For a given $\tilde{\sigma} \in \mathcal{S}$ we let $\mathcal{S}_\sigma$ denote the set of all maps $\sigma \in \mathcal{S}$ that are homotopic to σ through maps in $\mathcal{S}$. Given $\tilde{\sigma} \in \mathcal{S}$ one then considers the following min-max quantity or *width*

$$W_{\tilde{\sigma}} = \inf_{\sigma \in \mathcal{S}_{\tilde{\sigma}}} \max_{t \in [0,1]} E(\sigma(\cdot, t));$$

here in the above, for each closed curve $\gamma : S^1 \to S^2$, we define

$$E(\gamma) = \int_{S^1} g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\, dt,$$

so that $L(\gamma)^2 \leq 2\pi E(\gamma)$ with equality if and only if γ is of constant speed.

In fact, $W_{\tilde{\sigma}} = 0$ if and only if $\tilde{\sigma}$ is homotopic to a constant map, as we now show. Whenever $\tilde{\sigma} \in \mathcal{S}$ is homotopic to a constant map we immediately note that $W_{\tilde{\sigma}} = 0$. For the reverse direction we note that if $W_{\tilde{\sigma}} = 0$ we can find some minimising sequence, $\sigma_k \in \mathcal{S}_{\tilde{\sigma}}$, such that

$$\lim_{k \to \infty} \max_{t \in [0,1]} E(\sigma_k(\cdot, t)) = 0.$$

This implies that there is a $\delta > 0$ such that for sufficiently large k and each $t \in [0,1]$ the images $\sigma_k(S^1, t)$ are contained within a convex geodesic ball of radius δ centred at points $p_k(t)$ (where these centres may be chosen to vary continuously in t). We may then produce a homotopy from $\sigma_k(\cdot, t)$ to the constant map $p_k(t)$ by considering the shortest geodesics from points on $\sigma_k(\cdot, t)$ to $p_k(t)$. We then have that σ_k is homotopic to p_k, with the latter null homotopic as $p_k([0,1])$ is contractible to a point.

By exploiting the fact that the widths are positive on non-trivial homotopy classes, Birkhoff was able to show that each non-trivial homotopy class of maps in $\mathcal{S}$ realises a non-trivial closed geodesic in S^2. This requires the additional development of a curve shortening process (which we omit for brevity) with full details of the entire min-max construction found in [CM11, Chapter 5]. This therefore, coupled with the minimisation procedure above, establishes the existence of a closed geodesic in any compact Riemannian surface.

The above min-max technique is a specific instance of a more general principle for a wide range of functionals satisfying an appropriate compactness condition, allowing one to establish the existence of non-trivial critical points; generally these will be unstable or saddle type critical points rather than minima. The idea is similar to the above whereby one considers continuous paths, now in the domain of the functional, between two critical points of low energy (the trivial length minimisers for the length above). If the Palais-Smale condition (which guarantees the extraction of weakly converging sub-sequences) is satisfied, one can consider the maximum value of the functional along each of these paths, extract a sub-sequence which converges in an appropriate weak topology to the infimum of these values, and produce a non-trivial critical point. The general result here is aptly named a "mountain pass" theorem; as depicted in Figure 1.4; for details see [Eva10, Section 8.5].

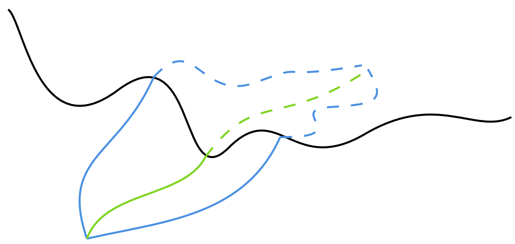

Figure 1.4: A depiction of the mountain pass theorem. Along the black curve are the maximum values of the functional along the paths between the critical points of low energy (the endpoints of the blue and green curves). The green curve depicts an "optimal" path with maximum value of the functional along the path realising the min-max value.

We conclude this subsection by emphasising the importance of utilising mountain pass min-max arguments in order to produce critical points of the area functional is essential in general. Ambient Riemannian manifolds with positive Ricci curvature (which we will focus on in Chapter 2) contain no area-minimising (and more generally contain no stable) minimal hypersurfaces. Similarly, for producing hypersurfaces with constant mean curvature of a specific value (a problem we will consider in Chapter 2), a minimisation procedure will not suffice (regardless of curvature assumption) as such a procedure provides no control on the mean curvature of the resulting hypersurface (e.g. consider the isoperimetric problem on the flat torus). Thus, in order to produce critical points of the area in either of the above settings (i.e. under an assumption positive Ricci curvature or specifying a constant for the mean curvature) a method for producing unstable critical points of the area is essential.

1.2.2 Weak notions of a submanifold

A prototypical existence problem related to the area functional is that of the Plateau problem. Namely, given a boundary in $\mathbb{R}^n$, find a surface of least area with that boundary. In order to answer such a question, one must first concern themselves as to which space of surfaces to consider.

Perhaps the simplest example is provided by considering the unit circle inside $\mathbb{R}^3$, where an obvious minimising candidate is the flat unit disk. One may then suppose that a reasonable space of surfaces to consider are those smooth surfaces with common boundary given by the unit circle. Within this space of surfaces one can then take a sequence whose area approaches the infimum, hoping to extract the smooth disk in the limit; such a procedure fails dramatically. Consider the following figures taken from [Mor16]:

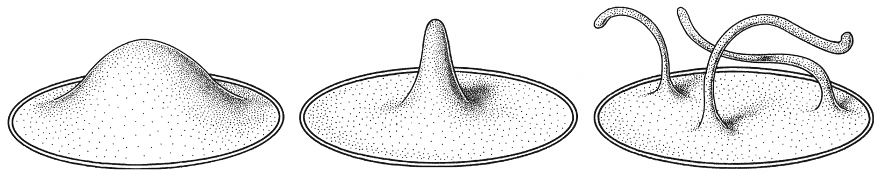

Figure 1.5: A minimising sequence of smooth surfaces.

The "tentacles" depicted in Figure 1.5 can in fact be arranged so that the sequence includes the entirety of $\mathbb{R}^3$ in its closure. Here we have exhibited a sequence of smooth surfaces with area approaching that of the minimiser (the area of disk) but not converging to a smooth surface. Thus, the lack of compactness exemplified by this "filigree" issue above must be accounted for in any reasonable formulation of the Plateau problem, and more generally for the study of general critical points of the area functional.

Another issue to consider, again within the class of smooth surfaces, is that of so-called "mass drop". Consider the simple example of two vertically stacked straight unit line segments, separated by a distance of width ε; for each positive ε the total length along the sequence is 2. Upon sending $\varepsilon \to 0$ however, the resulting limit will be a single line; which as a set may be considered now to have length 1.

In order to account for this loss of length in the limit we need to introduce a notion of integer multiplicity; namely, we will need to allow our space of surfaces (still to be determined) to carry a (potentially non-unitary) integer multiplicity function. In the example of the two line segments above this would correspond to equipping the limiting unit line segment with multiplicity 2 (i.e. counting its length twice), thus accounting for all of the length along the sequence. The introduction of the notion of integer multiplicity will later serve to ensure continuity of the area functional when taking limits.

To summarise the above two examples, any space of surfaces considered for formulating problems regarding critical points of the area functional must accommodate singular behaviour (regions where these surfaces fail to be smooth manifolds), higher integer multiplicity (to ensure continuity of the area along sequences of these surfaces), and posses a notion of compactness (in order to extract limits). We now proceed to examine what it means for a smooth submanifold to be a critical point of the area functional, from which we will be able to appropriately weaken our notion of a surface.

Definition 1.1. *We say that a smooth k-dimensional submanifold, Σ, (possibly with boundary) of a Riemannian manifold, (N, g), is stationary or a critical point for the area functional if*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{H}^k(\Sigma_t) = 0,$$

for all smooth variations, Σ_t, of Σ with compact support and fixed boundary.

Here, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure (which agrees with the k-dimensional area/Lebesgue measure on smooth submanifolds) and by a smooth variation we mean precisely that $\Sigma_t = F(\Sigma, t)$ for some smooth map $F : \Sigma \times (-\varepsilon, \varepsilon) \to N$ with $F(x, 0) = x$ for all $x \in \Sigma$, F equal to the identity outside of a compact subset of Σ, and $F(x, t) = x$ for all $x \in \partial\Sigma$ and $t \in (-\varepsilon, \varepsilon)$.

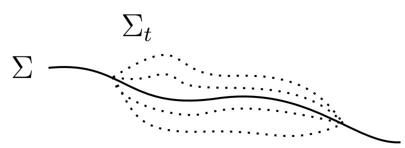

Figure 1.6: A smooth variation of a submanifold.

In fact, one can compute that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{H}^k(\Sigma_t) = -\int_\Sigma g(F_t, H)\, d\mathcal{H}^k = \int_\Sigma \mathrm{div}_\Sigma F_t\, d\mathcal{H}^k,$$

where F_t is the vector field given by the derivative of F with respect to the variable t, H denotes the mean curvature vector of Σ, and div_Σ

denotes the divergence operator acting on vector fields at points of Σ (for full details of this computation one may consult [CM11, Section 1.3]).

From this we deduce that Σ being stationary has both a geometric and variational formulation. Geometrically, through the vanishing of the mean curvature, for which we say that Σ is a *minimal submanifold*; notice in particular that 1-dimensional minimal submanifolds are in fact geodesics as in Subsection 1.2.1. In its variational formulation, the stationarity condition is equivalent to the identity

$$\int_{\Sigma} \mathrm{div}_{\Sigma} X \, d\mathcal{H}^k = 0 \tag{1.1}$$

for all compactly supported smooth vector fields, X, that vanish on $\partial\Sigma$.

We note that while the vanishing of the mean curvature vector makes sense pointwise whenever Σ is a C^2 submanifold, the variational formulation above makes sense for much weaker regularity assumptions on the submanifold Σ. In particular, the identity (1.1) makes sense when Σ is merely *k-rectifiable.*

Definition 1.2. *We say that $\Sigma \subset N$ is k-rectifiable if Σ has locally finite $\mathcal{H}^k$ measure and a k-dimensional tangent plane at $\mathcal{H}^k$ almost every point. Equivalently, Σ is contained, up to a set of zero $\mathcal{H}^k$ measure, in a countable union of C^1 submanifolds.*

One can therefore, at least in principle, establish the existence of a k-dimensional minimal submanifold by first producing a k-rectifiable set, Σ, that is stationary for the area functional, and then deduce the regularity of Σ directly from the above variational formulation, (1.1) (this second step will be discussed further in Subsection 1.3).

This class of k-rectifiable sets is broad enough to capture the singular behaviour alluded to above and, when these sets are equipped with a locally integrable integer multiplicity function, form a space of surfaces with favourable compactness properties.

Definition 1.3. *An integer rectifiable k-varifold (as a contraction of the term variational manifold), or simply k-varifold, is a pair $V = (M, \theta)$, where $M \subset N$ is k-rectifiable and $\theta : M \to \mathbb{Z}_{\geq 0}$ (the integer multiplicity function) is locally $\mathcal{H}^k$ integrable.*

By setting $\theta = 0$ on $N \setminus M$ we define the *weight measure*, $||V||$, of a varifold, $V = (M, \theta)$, by setting $||V|| = \mathcal{H}^k \llcorner \theta$ (so that $||V||(A) = \int_{A \cap M} \theta \, d\mathcal{H}^k$).

More generally, by letting $G(n,k)$ denote the set of k-dimensional subspaces of $\mathbb{R}^n$, one can define a *general k-varifold* on an open set $U \subset \mathbb{R}^n$ to be any Radon measure on $U \times G(n,k)$. Given an integer rectifiable k-varifold, $V = (M,\theta)$, there is a corresponding general k-varifold, $\widetilde{V}$, defined for $A \subset U \times G(n,k)$ by

$$\widetilde{V}(A) = ||V|| \left(\pi(TM \cap A)\right),$$

where π is the projection from $U \times G(n,k)$ onto the first factor and $TM = \{(x, T_xM) \mid M \text{ has a } k\text{-dimensional tangent plane at } x\}$ (such a tangent plane is defined for $\mathcal{H}^k$ almost every $x \in M$ by definition). This notion can be readily extended to Riemannian manifolds and is of use in establishing compactness of the class of integer rectifiable k-varifolds; we refer the reader to [Sim84, Chapter 8] for further details on the theory of general varifolds.

Definition 1.4. *We define the first variation, δV, of a k-varifold, $V = (M,\theta)$, on compactly supported smooth vector fields, X, by setting*

$$\delta V(X) = \frac{d}{dt}\bigg|_{t=0} ||V_t||(\mathbb{R}^{n+1}),$$

where here $V_t = (\varphi_t)_\# V = (\varphi_t(M), \theta \circ \varphi_t^{-1})$ is the varifold given by the pushforward of V by the flow, φ_t, induced by the vector field X.

By a near identical calculation to the smooth case above (see for example [Sim84, Section 39]) we then observe that for each compactly supported smooth vector field, X, it holds that

$$\delta V(X) = \int_M \theta \, \mathrm{div}_M X \, d\mathcal{H}^k = \int_M \mathrm{div}_M X \, d||V||.$$

In analogy with the smooth case above, if for all compactly supported smooth vector fields, X, we have that

$$\delta V(X) = \int_M \mathrm{div}_M X \, d||V|| = -\int_M g(X,H) \, d||V||,$$

then we say that V has generalised mean curvature H (which is locally integrable with respect to the weight measure $||V||$).

Definition 1.5. *We say that V is a stationary k-varifold if it has vanishing generalised mean curvature, thus satisfying (1.1) for the weighted area functional (i.e. by integration with respect to $||V|| = \mathcal{H}^k \llcorner \theta$).*

In particular, this generalises (1.1) for smooth k-dimensional submanifolds, Σ, as above by considering the varifold $V = (\Sigma, 1)$. Another important class of codimension 1 varifolds (i.e. n-varifolds in $\mathbb{R}^{n+1}$) we will consider throughout this thesis are provided by those with generalised mean curvature equal to a constant (scalar) multiple of the unit normal, thus subsuming the class of stationary codimension 1 integral varifolds; these hypersurfaces of *constant mean curvature* will be the main focus of Chapter 2. Constant mean curvature hypersurfaces arise naturally as solutions to the isoperimetric problem and more generally as critical points of area-type functionals; see the introductions of Chapter 2 for details. In particular, the constant for the mean curvature may be seen loosely as a Lagrange multiplier for the area functional, finding critical points of the area subject to an enclosed volume constraint.

As an aside, in [Men11] it has been shown that a stationary (and more generally for those with locally bounded first variation) k-varifold in fact has a C^2 structure for its underlying k-rectifiable set (in that it is contained, up to a set of zero $\mathcal{H}^k$ measure, in a countable union of C^2 submanifolds).

A fundamental compactness result, established in [Alm65] and [All72] (see also [Sim84, Theorem 42.7]), for the class of integer rectifiable varifolds is the following:

Theorem 1.2. *(Allard–Almgren compactness theorem) Sequences of k-varifolds with locally bounded weight measure and first variation converge (in the sense of Radon measures) sub-sequentially to k-varifolds of locally bounded weight measure and first variation.*

The above result shows that the class of integer rectifiable varifolds possesses favourable compactness properties and is suitable for studying critical points of the area (and other area-type functionals). In fact, it was also shown in [Alm65] that, by carrying out a mountain pass style min-max (see Subsection 1.2.1 above) in the space of k-cycles, every compact $(n+1)$-dimensional Riemannian manifold contains a stationary k-varifold for each $k \in \{0, \ldots, n\}$. For further detail on the theory of varifolds we refer the reader to [Sim84].

We shall also make use of the alternative weak notions of submanifolds provided by *integral currents* and *Caccioppoli sets*, each of which has an underlying varifold associated to it. We now briefly summarise each of these notions, providing references for the interested reader.

Integral currents, originally introduced in [FF60] to study generalisations of the Plateau problem, may be thought of as integer rectifiable varifolds that posses both a notion of continuously varying orientation (associated to each of the almost everywhere defined tangent planes to the underlying rectifiable set) along with a boundary (making them well suited to formulations of the Plateau problem). We now define integer rectifiable currents in Euclidean space, a notion which can be readily extended to Riemannian manifolds.

Definition 1.6. *An integer rectifiable k-current, $T = (M, \theta, \xi)$, on an open set $U \subset \mathbb{R}^n$, or simply k-current, is a continuous linear functional on the space of compactly supported k-forms, $\Omega^k_c(U)$, such that for each $\omega \in \Omega^k_c(U)$ we have*

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^k(x).$$

Here, M is a k-rectifiable subset of U, the multiplicity, $\theta : M \to \mathbb{Z}_{\geq 0}$, is locally $\mathcal{H}^k$ integrable, and the orientation, ξ, is an $\mathcal{H}^k$ measurable function such that at $\mathcal{H}^k$ almost every point of $x \in M$ we have that $\xi(x) = \tau_1 \wedge \cdots \wedge \tau_k$ (where $\tau_1, \ldots, \tau_k$ form an orthonormal basis for the k-dimensional tangent plane at $x \in M$). We define the boundary, ∂T, of a k-current T to be the $(k-1)$-current formed by setting

$$\partial T(\omega) = T(d\omega)$$

(where d is the exterior derivative on forms) for each $\omega \in \Omega^{k-1}_c(U)$.

To each k-current $T = (M, \theta, \xi)$ we associate an underlying k-varifold by considering $V = (M, \theta)$ (dropping the orientation). As an important example, for an open set $U \subset \mathbb{R}^n$ and a k-dimensional oriented (by some k-vector ξ) smooth submanifold $M \subset U$ with locally finite $\mathcal{H}^k$ measure, there is an associated k-current, which we hereafter denote by $[M]$, defined on $\omega \in \Omega^k_c(U)$ by setting

$$[M](\omega) = \int_M \langle \omega(x), \xi(x) \rangle d\mathcal{H}^k(x).$$

We refer the reader to [Fed96], [Sim84] and [Mor16] for further detail on the theory of integral currents.

A special class of codimension 1 integral currents are the Caccioppoli sets (also often referred to as sets of finite perimeter), originally introduced in [Cac27]; see Subsection 2.1.1 of Chapter 2 for a precise definition. Caccioppoli sets are well suited for studying problems involving interactions between sets and their boundary (c.f. the isoperimetric problem introduced above). These are sets whose indicator function is of bounded variation (i.e. has a weak derivative given by a Radon measure), and to whose boundary we associate a codimension 1 integer rectifiable varifold with multiplicity function identically equal 1 (often referred to as a multiplicity 1 varifold). We refer to [Mag12] for a comprehensive introduction to the theory of Caccioppoli sets.

1.2.3 Phase transitions

We now discuss more recent developments in the study of phase transition phenomena that have established deep connections with the area functional, notably providing an alternative approach to the existence theory for critical points of the area functional in codimension 1. As we will describe, this existence theory exploits a mountain-pass style min-max argument in the Sobolev space $W^{1,2}$ (just as for geodesics in Subsection 1.2.1) along with an approximation scheme, with parameter ε, that recovers critical points of the area functional as $\varepsilon \to 0$.

To this end, on a compact Riemannian manifold, (N^{n+1}, g), of dimension $n+1 \geq 3$, for each $\varepsilon \in (0,1]$, we denote the *Allen–Cahn energy* of a function $u \in W^{1,2}(N)$ by

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2\sigma}\int_N \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}.$$

Here, W is a double-well potential with unique global minima at ± 1 and $\sigma = \int_{-1}^{1}\sqrt{W(t)/2}\,dt$. A typical choice for the potential is

$$W(t) = \frac{1}{4}(1-t^2)^2$$

(appropriately modified to ensure quadratic growth outside of $[-2,2]$ so that this functional is well defined on $W^{1,2}(N)$).

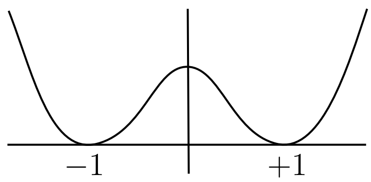

Figure 1.7: The double-well potential, $W(t) = \frac{1}{4}(1-t^2)^2$.

As alluded to above, the $\varepsilon \to 0$ behaviour of the Allen–Cahn energy defined above turns out to be closely related to critical points of the area functional. Loosely speaking, one can think of the Allen–Cahn energy as a "mollification" of the area functional. Let us now make this heuristic more precise.

Suppose a sequence of functions, $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset W^{1,2}(N)$, have uniformly bounded Allen–Cahn energies, i.e. such that $\sup_{\varepsilon\in(0,1)} \mathcal{E}_\varepsilon(u_\varepsilon) \leq C$ for some constant, $C > 0$, independent of ε. In order for the Allen–Cahn energies to remain uniformly bounded as $\varepsilon \to 0$, we must have that $W(u_\varepsilon) \to 0$ as $\varepsilon \to 0$ at almost every point. From the structure of W we then have that $u_\varepsilon \to \pm 1$ (up to a sub-sequence we will not relabel) as $\varepsilon \to 0$ at almost every point of N. The pointwise limit of the functions u_ε as $\varepsilon \to 0$ is thus a function, u_0, with $u_0 = \pm 1$ at almost every point (in fact u_0 will be of locally bounded variation, see [MM77]).

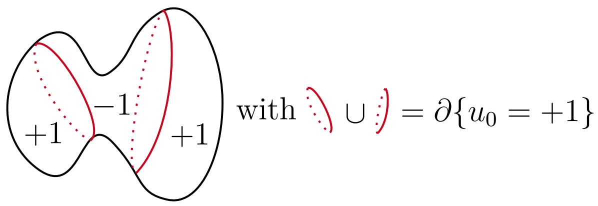

Figure 1.8: A depiction of how the function u_0 splits N.

In fact, if the u_ε are further assumed to be critical points of the Allen–Cahn energy, then the boundary of the region $\{u_0 = +1\}$ will be a minimal hypersurface (precisely, a stationary n-varifold); we now sketch this in the case where the u_ε are local minimisers of the Allen–Cahn energy.

We assume that the u_ε minimise the Allen–Cahn energy in some ball, B, and compute that by the Cauchy–Schwartz inequality

$$\mathcal{E}_\varepsilon(u_\varepsilon, B) = \frac{1}{2\sigma}\int_B \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} \geq \frac{1}{2\sigma}\int_B \sqrt{2W(u_\varepsilon)}|\nabla u|,$$

and by the co-area formula (see [Fed96] or [Sim84])

$$\frac{1}{2\sigma}\int_B \sqrt{2W(u_\varepsilon)}|\nabla u| = \frac{1}{2\sigma}\int_{\mathbb{R}} \sqrt{2W(s)}\mathcal{H}^n(\{u_\varepsilon = s\})\, ds.$$

The quantity $\mathcal{E}_\varepsilon(u_\varepsilon, B)$ is thus minimised on B when almost every level set, $\{u_\varepsilon = s\}$, is area-minimising in B and the above inequality is in fact an equality (i.e. when $|\nabla u_\varepsilon| = \frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}$). This brief sketch indicates that level sets of local minimisers to the Allen–Cahn energy accumulate on minimal surfaces as $\varepsilon \to 0$ and suggests a deeper connection to the area functional. We refer the interested reader to [Sav10, Section 2] and [Cho19, Section 3] (and the references therein) for more complete details of the above sketch.

The seminal work of [HT00], and its extension to Riemannian manifolds in [Gua18, Appendix B], shows that for general critical points of the Allen–Cahn energy this analogy also holds; in that for sequences of critical points with uniformly bounded energy one obtains a limiting minimal hypersurface. More precisely, the level sets of such critical points accumulate, as $\varepsilon \to 0$, around the support of the weight measure of a stationary n-varifold.

Critical points of $\mathcal{E}_\varepsilon$ arise as solutions, u, to the following semi-linear elliptic equation:

$$\Delta u = \frac{W'(u)}{\varepsilon^2};$$

(such solutions will be smooth by elliptic regularity, e.g. see [GT01]). The unconstrained global minima, uniquely given by the constants ± 1, are the only functions with zero Allen–Cahn energy. One can show however, see [Gua18, Lemma 4.2], that for a fixed ε the $W^{1,2}(N)$ functions with zero average have a uniformly positive lower bound on their Allen–Cahn energy; serving as an ideal barrier for a mountain pass, see Figure 1.9. One could then hope to show the existence of a non-constant critical point of $\mathcal{E}_\varepsilon$ through a mountain pass style min-max argument by using paths between ± 1.

This was carried out in [Gua18], upon showing that the Allen–Cahn energy functional $\mathcal{E}_\varepsilon$ satisfies the Palais–Smale compactness property, by considering the min-max quantity

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} \mathcal{E}_\varepsilon(\gamma(t)),$$

where Γ denotes the set of continuous paths in $W^{1,2}(N)$ between the global minima ± 1, i.e.

$$\Gamma = \{\gamma \in C^0([-1,1]; W^{1,2}(N)) \mid \gamma(-1) = -1, \gamma(+1) = +1\}.$$

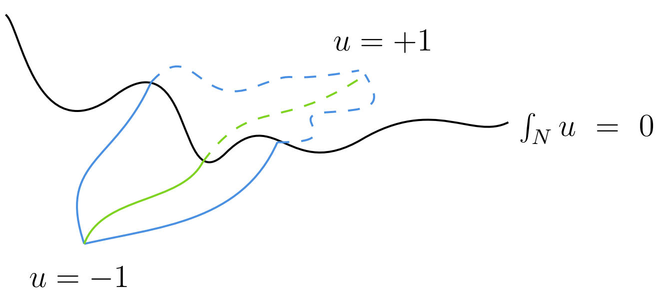

Figure 1.9: The mountain pass for $\mathcal{E}_\varepsilon$.

One thus obtains a smooth critical point, u_ε, of the Allen–Cahn energy which attains the min-max value (i.e. such that $\mathcal{E}_\varepsilon(u_\varepsilon) = c_\varepsilon$). We remark here for later reference that as a consequence of this construction, the Morse index of these solutions is bounded above by 1 (this is a general principle of min-max constructions, where one expects the number of parameters used in the construction to upper bound the Morse index of the solutions obtained).

It is then shown, in [Gua18, Sections 6 and 7] by means of an isoperimetric type inequality and construction of an explicit sweepout from the level sets of the distance function for the lower and upper bounds respectively, that

$$0 < \liminf_{\varepsilon \to 0} c_\varepsilon \leq \limsup_{\varepsilon \to 0} c_\varepsilon < \infty.$$

The theory of [HT00], in particular its extension to Riemannian manifolds in [Gua18, Appendix B], then guarantees the existence of a non-trivial stationary n-varifold by sending $\varepsilon \to 0$ (along a sub-sequence).

Though the existence theory through the Allen–Cahn energy described above only produces stationary codimension 1 varifolds, as opposed to those of any codimension as produced in [Alm65], this approach (and those using alternative approximation schemes, e.g. recent work of [PS20] for producing stationary codimension 2 varifolds and [GL24] for Plateau solutions) is significantly less technically involved and, as such, has led to incredibly fruitful results in recent years; we refer to Subsection 1.4.1 for further references in this direction. One work of particular note for this thesis is the recently developed existence theory provided by [BW20a] for constant (and more general prescribed) mean curvature hypersurfaces. This latter approach exploits min-max techniques for a modified Allen–Cahn energy functional and is the main focus of our attention in Chapter 2; further details of the explicit construction of these constant mean curvature hypersurfaces may be found in Subsection 2.1.2.

1.3 Optimal and generic regularity

In order to provide a suitable space of surfaces in which to establish the existence of critical points to the area functional, in Subsection 1.2.2 we enlarged the class of smooth submanifolds to that of the integer rectifiable varifolds. This was done in analogy with enlarging the class of smooth functions to the Sobolev space $W^{1,2}$ in order to find minimisers of the Dirichlet energy. There, regularity (smoothness of the solutions) is then deduced directly from the variational formulation of the problem via the Weyl lemma.

Due to the nonlinear nature of the area functional, one does not generally expect its critical points to lie in the class of smooth submanifolds. Nevertheless, extensive developments in the regularity theory in the past century have established an optimal regularity theory for these objects in various settings. Analogously to the Dirichlet energy, this regularity is often deduced by direct exploitation of the variational formulation of stationarity; given by (1.1).

Note that, although we will primarily restrict our attention to the case of stationary codimension 1 varifolds in this section, much of the regularity theory discussed here also applies (with appropriate modifications that we will indicate in Chapter 2) to constant mean curvature hypersurfaces.

We begin this section by discussing the regularity theory of stationary varifolds, with a specific focus on the optimal regularity conclusions in codimension 1. Subsequently, we explore various perturbation procedures aimed at removing singularities that arise in such varifolds; leading us to the main focus of the thesis, the notion of *generic regularity*. The interested reader may consult the beautiful survey provided in [Wic14b] for a more comprehensive survey of the results discussed here.

1.3.1 The singular set

When discussing the regularity of a stationary k-varifold, $V = (M, \theta)$, we are concerned with the geometric structure and properties of the k-rectifiable set M. Note that (e.g. see [Sim84, Theorem 38.3]) as

$$\mathcal{H}^k((M \setminus \mathrm{Spt}||V||) \cup (\mathrm{Spt}||V|| \setminus M)) = 0,$$

this is equivalent to studying the support of the weight measure, $||V||$. While we are interested in the general case of Riemannian manifolds, regularity questions are of a local nature and, as such, extend to this level of generality with only minor technical modifications to the results in the Euclidean case (i.e. in ambient $\mathbb{R}^{n+1}$), which we now restrict to.

Definition 1.7. *The regular set,* $\mathrm{Reg}(V)$*, of a stationary k-varifold V is the set of points in* $\mathrm{Spt}||V||$ *near which* $\mathrm{Spt}||V||$ *is locally a smoothly embedded k-dimensional submanifold; consequently, we define the singular set,* $\mathrm{Sing}(V)$*, of V to be* $\mathrm{Spt}||V|| \setminus \mathrm{Reg}(V)$*. In this way* $\mathrm{Reg}(V)$ *is relatively open, and* $\mathrm{Sing}(V)$ *is relatively closed, inside of* $\mathrm{Spt}||V||$.

Before discussing the general theory, we describe three instructive examples of singular stationary varifolds for the reader to keep in mind throughout this subsection. These examples show that, even in some of the simplest settings, singular behaviour in these objects is unavoidable. They also serve to indicate the striking difference in behaviour we will later observe between stationary varifolds of codimension 1 and those of higher codimension.

Firstly, one reason to expect the presence of singularities is through topological obstructions to smoothness. For example, if one formulates the Plateau problem for a boundary with odd Euler characteristic, no solution can be a smooth submanifold. Concretely, we embed $\mathbb{C}P^2$ into a sphere of some dimension m, and take it as the boundary to solve the Plateau problem in $\mathbb{R}^{m+1}$. There has to be a singularity in any Plateau solution since $\mathbb{C}P^2$ cannot bound a smooth compact oriented manifold, and thus any solution must contain an interior singular point (as a technical aside, since the sphere is convex there are no boundary singularities by [All75]). However, we remark that for Plateau solutions of codimension 1, there is no topological obstruction to the existence of a smooth hypersurface (with boundary) arising as a solution.

Secondly, one can consider the multiplicity 1 varifold given by the locally area-minimising 2-dimensional complex analytic variety

$$\{(z, w) \mid z^2 = w^3\} \subset \mathbb{C}^2 \cong \mathbb{R}^4,$$

which has a branch point singularity at the origin (i.e. where the variety is not smooth locally but infinitesimally looks like a plane of high multiplicity). As we shall see however, this phenomena of branch point singularities can be ruled out, under appropriate assumptions, in codimension 1.

Finally, we consider the simple example of a multiplicity 1 stationary varifold given by a pair of transversely intersecting hyperplanes in Euclidean space. Such a pair of crossed planes is a smooth hypersurface away from their intersection but, in ambient $\mathbb{R}^{n+1}$, exhibits a singular set of dimension $n-1$; as we shall discuss shortly, this is in fact the best general dimension bound on $\mathrm{Sing}(V)$ one can expect for multiplicity 1 stationary varifolds.

1.3.2 Tangent cones

In order to probe the structure of the singular set of V, we first exploit the variational formulation of stationarity given by (1.1) for the weighted area. Recall, (1.1) was shown to be equivalent to the fact that for each compactly supported smooth vector field, X, we have

$$\delta V(X) = \int_M \mathrm{div}_M X \, d||V|| = 0.$$

Upon an appropriate choice of radial test vector field (e.g. see [Sim84, Section 17]), one obtains the *monotonicity formula* which states that, for each $x \in \mathbb{R}^{n+1}$, the mass ratio function

$$\rho \mapsto \frac{||V||(B_\rho(x))}{\rho^k},$$

is increasing in ρ, where $B_\rho(x)$ is the open ball, in $\mathbb{R}^{n+1}$, of radius ρ centred at x. An important consequence of this, when combined with the Allard–Almgren compactness theorem discussed in Subsection 1.2.2, is the existence of *tangent cones*; the blowup models at singular points of stationary varifolds.

Definition 1.8. *Given a point $p \in \mathrm{Spt}||V||$ and a sequence, $\{\lambda_j\}_{j\geq 1}$ of positive numbers, $\lambda_j \to 0$ as $j \to \infty$, there exists a sub=sequence, $\{\lambda_{j_l}\}_{l\geq 1}$, and a non-trivial stationary k-varifold, C_p with*

$$(\eta_{p,\lambda_{j_l}})_\# V \to C_p.$$

Here in the above, for each $x, y \in \mathbb{R}^{n+1}$ and $\lambda > 0$ we define $\eta_{y,\lambda}(x) = \frac{x-y}{\lambda}$, the pushforward is defined as in Subsection 1.2.2, and the convergence is in the sense of Radon measures. Furthermore, C_p is a cone, in the sense that for each $\lambda > 0$ we have $(\eta_{0,\lambda})_\# C_p = C_p$ and as such we call C_p a tangent cone to V at the point p.

One may hope that, as a blowup model, tangent cones reflect the local geometric behaviour of the varifold. It follows from the above that if $p \in \mathrm{Spt}||V||$ is a point where a k-dimensional tangent plane exists (which holds at at $\mathcal{H}^k$ almost every point of M by definition) then the unique tangent cone to V at p is this tangent plane equipped with integer multiplicity given by $\theta(p)$; in particular, this holds whenever $p \in \mathrm{Reg}(V)$.

For singular points of the support however, the behaviour of tangent cones in relation to the structure of the support is, as yet, less clear; we highlight here the recent constructions in [Szé21] of stationary varifolds with isolated singularities but cylindrical tangent cones of the form $C \times \mathbb{R}$, for some singular cone C. Another severe issue at singular points is whether or not a given tangent cone at a point $p \in \mathrm{Sing}(V)$ depends on the choice of sub-sequence, i.e. whether tangent cones are unique.

This remains a largely open question, with a complete resolution only for the case where $k = 1$ in [AA76]. Another important instance, particularly relevant to the thesis, is from the seminal work of [Sim83] (see also the recent work of [EM24]), where uniqueness is known for *regular tangent cones* (which we define as tangent cones with multiplicity 1 and an isolated singularity, i.e. $\mathrm{Sing}(C_p) = \{0\}$); this will play a key role in the work in Chapter 2. Although there are no known examples of non-unique tangent cones to stationary integer rectifiable varifolds, we mention here the constructions in [Kol15] of non-unique tangent cones for stationary rectifiable varifolds with *non-integer* multiplicity.

While the presence of higher integer multiplicity allows for the development of a suitable existence theory for stationary varifolds, it creates severe difficulties in establishing their regularity. In the most general case of a stationary k-varifold, $V = (M, \theta)$, the best known regularity conclusions are that $\mathrm{Reg}(V)$ is dense in $\mathrm{Spt}||V||$, as shown in the landmark work of [All72]. In particular, it is not even known whether or not $\mathcal{H}^k(\mathrm{Sing}(V)) = 0$; which could fail if for example $\mathrm{Sing}(V)$ arose as a positive measure Cantor set (relevant here are the recent constructions of stationary varifolds with prescribed singular sets in [Sim23]).

The density of the regular set in the support in fact follows as a corollary of an "ε-regularity theorem" in [All72], which we state as follows:

Theorem 1.3. *(Allard's theorem) If at a point in* $\mathrm{Spt}||V||$ *the multiplicity function, θ, is sufficiently close to* 1*, then in a neighbourhood of this point* $\mathrm{Spt}||V||$ *is a smoothly embedded k-dimensional submanifold. With reference to the definition of tangent cone above, this says that if a multiplicity* 1 *tangent plane arises as a tangent cone at a point of the support of a stationary varifold, then locally the varifold is in fact a smooth submanifold.*

Although it is not known to what extent the support of a stationary varifold is modelled by its tangent cones, one is able to transfer information from the level of the tangent cone to the singular set of the varifold by studying the dimensions along which they are translation invariant. We summarise this in the following theorem, established in [Fed70] and [Alm00] (see also [Sim84, Appendix A]):

Theorem 1.4. *(Almgren–Federer stratification theorem) Let the spine, $S(C)$, of a given stationary cone, C, denote the linear subspace along which* $\mathrm{Spt}||C||$ *is translation invariant, we define for each $j \in \{0, \ldots, k\}$*

the jth strata as the set

$$\mathcal{S}_j = \{p \in \mathrm{Sing}(V) \mid \textit{every tangent cone, } C_p, \textit{ has } \dim(S(C_p)) \leq j\};$$

so that $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{k-1} \subset \mathcal{S}_k = \mathrm{Sing}(V)$. *We then have that*

$$\dim_{\mathcal{H}}(\mathcal{S}_j) \leq j.$$

It has more recently been shown in [NV20] that each strata, $\mathcal{S}_j$, as defined above is in fact j-rectifiable.

By controlling the maximal spine dimension of tangent cones, one can thus hope to establish dimension bounds on the singular set of a stationary varifold. In particular, if one can rule out the possibility of a high multiplicity k-plane from arising as a tangent cone at a singular point, one immediately deduces that the dimension of the singular set is at most $k-1$ by noticing that if a multiplicity 1 tangent plane arises at a point, such a point must be in the regular set by Allard's theorem as stated above. As a consequence, if a stationary k-varifold, $V = (M, \theta)$, is of multiplicity one, then we can in fact conclude from Allard's theorem that the dimension of the singular set is at most $k-1$; this is sharp in view of the example of transversely intersecting planes given in Subsection 1.3.1. Whether or not this dimension bound remains true for higher multiplicity stationary varifolds remains a largely open problem.

1.3.3 Stability and Simons classification

It is perhaps not so surprising that relying solely on such a weak condition as stationarity is unable to yield the strongest possible regularity conclusions. This is again analogous to the case of partial differential equations, where often one needs to exploit second order behaviour (such as stability arising from a minimisation property) of solutions in order to deduce further regularity. Therefore, in order to yield improved dimension bounds on the singular set, we will now restrict our attention to subclasses of stationary varifolds satisfying appropriate second order conditions.

Definition 1.9. *We define the second variation of a stationary k-varifold, $V = (M, \theta)$, for a compactly supported smooth vector field, X, by setting*

$$\delta^2 V(X) = \frac{d^2}{dt^2}\bigg|_{t=0} ||V_t||(\mathbb{R}^{n+1}),$$

where, the V_t are defined as in Subsection 1.2.2. We say that V is stable if the second variation is non-negative, i.e. $\delta^2 V(X) \geq 0$.

Stationary varifolds that correspond to locally area-minimising currents, a condition which immediately implies non-negativity of their second variation, form an important subclass of stable varifolds (e.g. for solving the Plateau problem). In the monolithic work of [Alm00] it was shown that the dimension of the interior singular set of these varifolds is at most $k-2$ (we refer to Subsection 1.4.1 for further discussion of results in high codimension); this dimension bound is sharp in view of the example $\{(z,w) \mid z^2 = w^3\} \subset \mathbb{C}^2$, as discussed in Subsection 1.3.1.

An important consequence of stability in codimension 1 (i.e. when V is a stationary n-varifold), which we now restrict to for the remainder of the subsection, is the *stability inequality.* This states, see [Sim84, Section 9], that if $\mathrm{Reg}(V)$ is orientable with continuous unit normal, ν, then we may consider $X = \xi\nu$ (extended to $\mathbb{R}^{n+1}$) for some compactly supported smooth function, ξ, on $\mathrm{Reg}(V)$ and deduce from the non-negativity of the second variation that

$$\int_{\mathrm{Reg}(V)} |A|^2\xi^2 \, d\mathcal{H}^n \leq \int_{\mathrm{Reg}(V)} |\nabla\xi|^2 \, d\mathcal{H}^n,$$

where here $|A|$ denotes the length of the second fundamental form, and ∇ represents the gradient on $\mathrm{Reg}(V)$.

We can now state the following powerful result for stable minimal hypercones as established in [Sim68].

Theorem 1.5. *(Simons' classification theorem) If* $n \in \{2,3,4,5,6\}$ *and* C *is a stable conical* n*-varifold with* $\mathrm{Sing}(C) \subset \{0\}$*, then* C *is a hyperplane.*

We remark that this result is sharp. It fails for $n = 1$ by virtue of transversely intersecting lines, and for $n = 7$ by the existence of the so called *Simons cone*, given by

$$C^{3,3} = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |x| = |y|\} \subset \mathbb{R}^8.$$

This was first observed to be stationary and stable in [Sim68]; which follows from a simple calculation involving the first variation, see [Sim84, Appendix B]. In fact, the Simons cone was also shown to be locally area-minimising in [BDG69] and thus serves as the prototypical counterexample to the smoothness of high-dimensional area-minimising (and more generally stable) minimal hypersurfaces.

As a consequence of the Almgren–Federer stratification theorem mentioned above, the dimension of the singular set of a stationary n-varifold in $\mathbb{R}^{n+1}$ is at most $n-l$, where $l \in \{0, \dots n\}$ is the smallest integer such that, at some singular point, there is some cylindrical tangent cone of the form $C \times \mathbb{R}^{k-l}$, with C a stationary conical l-varifold (in $\mathbb{R}^{l+1}$) with $\text{Sing}(C) \subset \{0\}$ (if $l = 0$ here we mean that the tangent cone is a, necessarily higher multiplicity, hyperplane). Thus, if one can rule out both high multiplicity n-planes and cones of the form $C \times \mathbb{R}^{n+1}$, where C is a stationary conical 1-varifold (in $\mathbb{R}^2$), arising as tangent cones at a singular point, then one in fact concludes from Simons classification theorem above that the singular set is of dimension at most $n-7$. This dimension bound is optimal in view of the example of the Simons cone introduced above. As such, we say that a stationary n-varifold with singular set of dimension at most $n-7$ has *optimal regularity.*

1.3.4 Codimension 1 theory

What is perhaps surprising, and that we now turn our attention to, is that the optimal regularity conclusion in codimension 1 in fact holds for a wide range of stationary varifolds under rather weak stability assumptions. This includes stationary varifolds corresponding to locally area-minimising currents in codimension 1 and, as we will define, stable codimension 1 varifolds which admit no singularities with a specific geometric structure (which in fact subsumes the class of local area-minimisers). We thus restrict our attention to the codimension 1 case for this subsection (without further comment) and will consider stationary n-varifolds, $V = (M, \theta)$, in $\mathbb{R}^{n+1}$.

We first discuss why local area-minimisers in codimension 1 have optimal regularity, i.e. we want to show that if V corresponds to a locally area-minimising current, T, then $\dim(\text{Sing}(V)) \leq n-7$. This is equivalent, by the Almgren–Federer stratification theorem, to showing that $\text{Sing}(V) = \mathcal{S}_{n-7}$ (where the strata, $\mathcal{S}_j$, were defined in Subsection 1.3.2). For each $p \in \text{Sing}(V) \setminus \mathcal{S}_{n-7}$, we can find (through iteratively taking tangent cones if necessary) a tangent cone of the form $C \times \mathbb{R}^{n-l}$, where here $l \in \{0, 1, 2, 3, 4, 5, 6\}$, and C is an area-minimising l-dimensional cone (and hence a stable l-varifold in $\mathbb{R}^{l+1}$) with $\text{Sing}(C) \subset \{0\}$. Simons classification theorem above thus tells us that if $l \in \{2, 3, 4, 5, 6\}$ then C as above is a hyperplane, for which we then know we are in the case $l = 0$ (and our tangent cone must a high multiplicity plane by Allard's theorem); thus we have that $l \in \{0, 1\}$.

If $l = 1$, then C is an area-minimising 1-dimensional cone in $\mathbb{R}^2$, which one can easily check (by means of a simple comparison argument) is a straight line, and thus $l = 0$ is the only possibility. If $l = 0$ this means that we are at a singular point that admits a high multiplicity planar tangent cone, which we now preclude. As a consequence of the decomposition theorem for codimension 1 integral currents, see [Sim84, Corollary 27.8], we may write T as a sum of Caccioppoli sets, i.e. in a ball, B, we have $T \llcorner B = \sum_{i=1}^{\infty} \partial[E_i]$ for open sets $E_{j+1} \subset E_i \subset B$, where here we recall that $[A]$ is the $n+1$-current associated to an open set A. Under the assumption that there is a high multiplicity planar tangent at a singular point, $p \in \mathrm{Sing}(V)$, the problem thus reduces to studying the multiplicity 1 boundaries, $\partial[E_i]$, with planar tangent cone; which Allard's theorem tells us are locally smooth. We have a sum of smooth hypersurfaces which touch at the point p, but do not cross (as for each j we have $E_{j+1} \subset E_j$), from which the maximum principle (for minimal graphs) tells us that these hypersurfaces coincide. We therefore conclude that in fact $p \in \mathrm{Reg}(T)$ and so the case $l = 0$ cannot occur, therefore $\mathrm{Sing}(V) = \mathcal{S}_{n-7}$ and so $\dim(\mathrm{Sing}(V)) \leq n - 7$ as desired.

As an aside, we remark that without the application of Simons classification theorem in the above, one is able to establish that the dimension of the singular set of an area-minimiser is at most $n - 3$, i.e. that $l \geq 3$. To see this, one repeats the arguments to rule out $l \in \{0, 1\}$ as above and to rule out the case $l = 2$, uses the fact that a 2-dimensional minimal cone in $\mathbb{R}^3$ yields a smooth closed geodesic, a great circle, when intersected with the round 2-sphere (else $\mathrm{Sing}(C) \subset \{0\}$ fails). We conclude that the tangent cone is a thus a hyperplane (i.e. $l = 0$), which we had already ruled out.

The area-minimisation assumption was only utilised in the above arguments in order to rule out tangent cones arising as high multiplicity hyperplanes and those with codimension 1 spine. With this established, we concluded optimal regularity by applying Simons classification theorem, which only required the weaker assumption of stability. One can thus posit whether this optimal regularity conclusion may be obtained for stable codimension 1 varifolds, provided one can preclude the two types of tangent cones mentioned above from arising.

The first key result was obtained in [SS81] where, under an a priori smallness assumption on the size of the singular set, they conclude optimal regularity for stable codimension 1 varifolds. This smallness assumption, precisely assuming $\mathcal{H}^{n-2}(\mathrm{Sing}(V)) = 0$ (more generally one can assume the singular set has vanishing 2-capacity), allows for the stability inequality, which originally held only on the regular part, to be extended across the singular set. By exploiting this smallness assumption on the singular set, they exploit stability in order to rule out tangent cones with codimension 1 spine. Furthermore, they are able to show that if such a stable codimension 1 varifold is sufficiently close to a high multiplicity hyperplane, then it decomposes as a sum of smooth graphs (one may think of this theorem as saying that the stability assumption rules out the formation of small “necks”), such a result is known as a *sheeting theorem*. This result in particular rules out tangent cones at singular points arising as high multiplicity hyperplanes, from which one thus may apply Simons classification and conclude optimal regularity.

A natural open question that remained after the work of [SS81] was whether the smallness assumption on the size of the singular set could be relaxed to $\mathcal{H}^{n-1}(\mathrm{Sing}(V)) = 0$; which is sharp in view of a pair of transversely intersecting planes as discussed in Subsection 1.3.1. This was entirely resolved in the monumental work of [Wic14a] which, by providing sharp results for the class of stable codimension 1 varifolds, subsumes both the results for codimension 1 area-minimisers as well as those of [SS81]. The results in [Wic14a] have also found effective application in the codimension 1 existence theory through the Allen–Cahn min-max approach, which we will discuss in the next subsection.

We conclude this subsection by briefly discussing [Wic14a], starting with a key definition.

Definition 1.10. *A point* $p \in \mathrm{Sing}(V)$ *is said to be a classical singularity if there exists a* $\rho > 0$ *and* $\alpha \in (0,1)$ *such that* $\mathrm{Spt}||V|| \cap B_\rho(p)$ *is the finite union of three or more embedded* n*-dimensional* $C^{1,\alpha}$ *submanifolds with boundary in* $B_\rho(y)$*, meeting only along a common* $(n-1)$*-dimensional boundary,* Γ*, containing* p *with at least one pair meeting transversly; see Figure 1.10.*

One can in fact upgrade this $C^{1,\alpha}$ regularity to analyticity, for analytic ambient spaces, by the work of [Kru14].

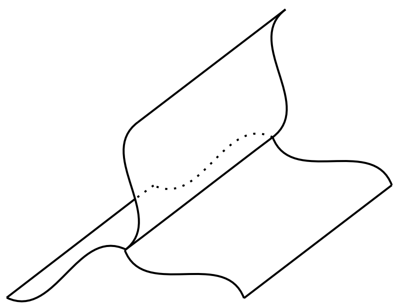

Figure 1.10: A classical singularity formed by three submanifolds meeting along a common boundary, Γ.

The main result of relevance for us from [Wic14a] is the following:

Theorem 1.6. *(Wickramasekera's theorem) If V is a stationary codimension 1 varifold with* $\mathrm{Reg}(V)$ *stable and* $\mathrm{Sing}(V)$ *containing no classical singularities, then V is of optimal regularity.*

We remark that since any tangent cone at a classical singularity is necessarily unique and has a codimension 1 spine, the above comparison argument for currents implies that any stable codimension 1 varifold corresponding to a locally area-minimising current contains no classical singularities; we then apply Wickramasekera's theorem to conclude optimal regularity. Furthermore, if a stable codimension 1 varifold has a classical singularity then, by definition, it must have singularities along an $(n-1)$-dimensional submanifold. Thus, under the assumption that $\mathcal{H}^{n-1}(\mathrm{Sing}(V)) = 0$, V cannot contain any classical singularities; by applying Wickramasekera's theorem we again conclude optimal regularity. The above arguments therefore show that Wickramasekera's theorem implies both the optimal regularity results for codimension 1 area-minimisers, as well as those of [SS81] (and in doing so answers the above question regarding the smallness assumption on the singular set).

The proof of Wickramasekera's theorem requires the use of two separate theorems; a sheeting theorem and the so called *minimum distance theorem*. This sheeting theorem is similar in spirit to that of [SS81] (which follows as a special case), but for stable codimension 1 varifolds assumed now to be close in area as well as distance to a higher multiplicity hyperplane. The minimum distance theorem, which builds upon techniques initially developed in [Sim93], shows that if a stable codimension 1 varifold contains no classical singularities, then no tangent cone to V can have a codimension 1 spine. With both of these theorems at hand, one readily establishes the optimal regularity conclusion by an application of Simons classification theorem in the manner described above; ruling out tangent cones at singular points arising as high multiplicity hyperplanes and those with codimension 1 spine by applying the sheeting and minimum distance theorems respectively.

An explanation of the proof of the sheeting and minimum distance theorems (which involves proving these theorems by induction simultaneously) is unfortunately beyond the scope of our exposition and, as such, we defer to the detailed exposition of the proof provided in [Wic14b].

1.3.5 Applications to existence

We now detail how the regularity theory we have discussed above is applied to stationary codimension 1 varifolds produced by the various existence methods discussed in Section 1.2. For this subsection we again consider (without further comment) stationary n-varifolds, $V = (M, \theta)$, in $\mathbb{R}^{n+1}$.

Following the min-max construction of stationary codimension 1 varifolds in [Alm65], in [Pit81] it was established that such varifolds are in fact entirely smooth hypersurfaces whenever $n \leq 5$. The arguments there relied on establishing an "almost-minimising" property (which in particular rules out tangent cones arising as high multiplicity hyperplanes and those with codimension 1 spine) for the stationary varifolds produced by the min-max, and crucially exploited curvature estimates for stable immersed minimal hypersurfaces, valid for $n \leq 5$, that had been proven in [SSY75] (see recent work of [Bel23a] for the $n = 6$ case). As a consequence of the work of [SS81], these curvature estimates were extended to the embedded case for $n = 6$ and moreover showed that the stationary codimension 1 varifolds produced in [Alm65] were in fact of optimal regularity. We now summarise this celebrated result, which is often referred to in the literature as the *Almgren–Pitts–Schoen–Simon existence theory* or *Almgren–Pitts min-max procedure*:

Theorem 1.7. *Let (N^{n+1}, g) be a compact Riemannian manifold. For $n \leq 6$ then there exists an entirely smooth minimal hypersurface, if $n = 7$ then there exists a minimal hypersurface smooth away from finitely many isolated singularities, and if $n \geq 8$ there exists a minimal hypersurface which is smooth away from a closed singular set of dimension at most $n - 7$.*

In particular, the existence of smooth minimal hypersurfaces for $n \leq 6$ has paved the way for significant developments in low-dimensional geometry and topology.

For the existence theory provided by the Allen–Cahn min-max approach of [Gua18], the regularity theory provided in [SS81] does not suffice to establish the optimal regularity of the stationary codimension 1 varifolds produced. This is due to the fact that there is no obvious "almost-minimising" property that one can establish for these varifolds and so a priori, one has no way to rule out tangent cones at singular points arising as high multiplicity hyperplanes. Thus, in order to establish the desired optimal regularity, one must appeal to the full strength of Wickramasekera's theorem. Although one needs to appeal to stronger regularity theory here, one major advantage of the Allen–Cahn min-max approach stems from the fact that the existence and the regularity components of the theory are entirely distinct. This is not the case for the Almgren–Pitts min-max procedure described above however, which requires additional technical work in order to exploit both the explicit min-max construction as well as the regularity theory simultaneously.

The work of [Ton05] showed that if a sequence of critical points of the Allen–Cahn energy with uniformly bounded energy is additionally assumed to be stable (i.e. with non-negative second variation), the stationary codimension 1 varifold around which they accumulate (as guaranteed by [HT00]) is also stable. By precluding classical singularities from arising in these varifolds, in [TW12] it was shown that Wickramasekera's theorem may be applied in order to guarantee that stable critical points of the Allen–Cahn energy with uniformly bounded energy accumulate around stationary codimension 1 varifolds with optimal regularity.

As an aside, it is interesting to note that the work of [Ton05] shows that the limit varifold is in fact stable on the entirety of its support, and not only on the regular set as required in the assumptions of Wickramasekera's theorem, suggesting an alternate method to establish optimal regularity for such varifolds by exploiting this stability information across the singular set. Such a method would have the advantage of avoiding needing the full strength of the Wickramasekera's theorem, and perhaps lead to a simplified proof of the existence and regularity theory in codimension 1.

An elegant observation made in [Gua18, Section 3] was that if the Morse index of a critical point of the Allen–Cahn energy (the dimension of the subspace along which the second variation is negative) is at most 1, then for any ambient ball the critical point is stable either in the ball or its complement (but not both). As remarked in Subsection 1.2.3, the min-max critical points of the Allen–Cahn energy constructed in [Gua18] have Morse index bounded above by 1, and so using this observation along with the conclusions of [TW12] we deduce that the non-trivial stationary codimension 1 varifolds produced by sending $\varepsilon \to 0$ have optimal regularity (a slick alternative argument for deducing optimal regularity in this setting was provided in [Hie18, Section 4.2]). In particular, in low-dimensions ($n \leq 6$) the Allen–Cahn min-max produces entirely smooth minimal hypersurfaces, recovering the conclusions of the Almgren–Pitts–Schoen–Simon existence theory in all dimensions.

In the recent work of [BW20b, BW20c], sharp regularity theorems were established for constant (and more generally prescribed) mean curvature hypersurfaces, generalising the results of [Wic14a] in the stationary case. These regularity theorems in particular show that under a suitably modified notion of stability, and again subject to the absence of classical singularities, codimension 1 varifolds with constant generalised mean curvature (i.e. constant mean curvature hypersurfaces) are smoothly immersed (in fact quasi-embedded, see Subsection 2.1.2 of Chapter 2 for a precise definition) away from a singular set of dimension at most $n-7$, i.e. of optimal regularity. In particular, this generalised the optimal regularity conclusions obtained in [GMT83] for isoperimetric regions.

We conclude this subsection by remarking that in the lowest singular dimension (i.e. in ambient dimension 8), the optimal regularity conclusions (both for the case of minimal hypersurfaces and constant mean curvature hypersurfaces) guarantee that the support will be a smooth hypersurface away from a singular set consisting of finitely many isolated points (e.g. see the end of the proof in [Wic14a, Section 17]). This case will be the main focus of Chapter 2 where we will show that, in various settings, one can perturb away all isolated singularities and produce an entirely smooth hypersurface.

1.3.6 Perturbing away the singular set

As mentioned at the beginning of the Chapter, smooth hypersurfaces arising as critical points of area-type functionals have proven to be incredibly effective through their applications in low-dimensional geometry and topology. In the presence of a singular set however, such hypersurfaces in general fail to be effective tools for applications. As such, one may posit whether the singularities that appear in such hypersurfaces are generic in a topological sense, and whether they can be perturbed away. We now discuss some results in this direction that are of particular relevance to the thesis, a more complete summary of which may be found in Subsection 1.4.2.

As mentioned above, the Simons cone, $C^{3,3}$, introduced in Subsection 1.3.3 was first shown to be area-minimising in [BDG69]; establishing the sharp regularity conclusions possible for stable codimension 1 varifolds. In the proof of this fact, it was shown that the Simons cone is foliated by entirely smooth area-minimising hypersurfaces. Namely, perturbations of the Simons cone, which is singular with an isolated singularity at the origin, are entirely smooth.

This was generalised in the fundamental work of [HS85], where it was shown that locally area-minimising currents containing only isolated singularities with regular tangent cone (i.e. multiplicity 1 with an isolated singularity) were also locally foliated by entirely smooth area-minimisers. As a consequence, this showed that subject to a small perturbation of the boundary, area-minimisers in ambient dimension 8 were in fact smooth. In recent work of [Les23], an analogous foliation result was established for constant mean curvature hypersurfaces that locally minimise the appropriate area-type functional around each isolated singularity.

In [Sma93], using the foliation of [HS85], the *generic regularity* of area-minimisers in each non-trivial homology class in dimension 8 was established. By generic regularity here we mean that, for an open and dense subset of metrics on an 8-dimensional compact Riemannian manifold, one can find a smooth area-minimising hypersurface in each non-trivial homology class.

In [CLS22] a generic regularity result was established in 8-dimensional compact Riemannian manifolds with positive Ricci curvature. In this setting it was shown, through the Almgren–Pitts min-max procedure and a local metric surgery exploiting the foliation of [HS85], that for a generic metric there existed a smooth minimal hypersurface. In Chapter 2 we develop a similar local metric surgery procedure for isolated singularities of constant mean curvature hypersurfaces that exploits the above mentioned foliation of [Les23]. We then use this to establish, through the Allen–Cahn min-max procedure of [BW20a], analogous generic regularity results for constant mean curvature hypersurfaces in 8-dimensional compact Riemannian manifolds with positive Ricci curvature.

By developing specific global metric perturbations, as opposed to the local ones exploited in the above mentioned perturbation results, in [LW21] the generic regularity of minimal hypersurfaces in 8-dimensional compact Riemannian manifolds (with no curvature assumptions) was established; guaranteeing the existence of a smooth minimal hypersurface. This was further extended in [LW22] to show that full generic regularity held, namely that for an open and dense subset of the Riemannian metrics on an 8-dimensional compact Riemannian manifold every minimal hypersurface (with no classical singularities) is in fact smooth.

The results of [HS85] and [Sma93] were generalised in more recent remarkable work of [CMS23a, CMS23b] (building on developments in the mean curvature flow, see Subsection 1.4.2), where generic regularity results were established up to ambient dimension 10. More precisely, they established that, subject to a perturbation of the boundary or of the ambient metric, area-minimising hypersurfaces are generically smooth.

We also mention here the interesting recent work of [ST20], showing that (under appropriate decay rate conditions) singular points of stationary, multiplicity 1, codimension 1 varifolds admitting high multiplicity hyperplane tangent cones are in an appropriate sense dynamically unstable with respect to the mean curvature flow; suggesting that the stability of stationary varifolds with respect to the mean curvature flow can rule out certain types of singularities.

1.4 Historical primer

Having provided relevant background in the previous two sections, we now offer a very brief survey summarising developments in the existence and regularity theory for the area functional as discussed in Subsections 1.2 and 1.3. Following this, we summarise some generic regularity results for a range of geometric variational problems.

1.4.1 A brief survey

The Plateau problem for surfaces was originally investigated by Lagrange in 1760, and was first resolved independently in [Dou31, Rad30]; the surfaces produced by these results were in fact shown to be smooth in [Oss70]. We also mention here the alternative approach taken to the Plateau problem in the novel work of [Rei60], producing solutions of varying topological type.

The pioneering work of [DG61] showed that locally area-minimising Caccioppoli sets in fact have smooth n-dimensional boundary away from a closed interior singular set of zero n-dimensional Hausdorff measure. This regularity conclusion was successively strengthened and, through the work of [FF60] (which also proved more general existence), [Sim68] and [Fed70], led to the conclusion that the interior singular set of the locally area-minimising hypersurface is in fact empty if $n \leq 6$, discrete if $n = 7$, and has Hausdorff dimension at most $n - 7$ if $n \geq 7$; which we refer to as optimally regular. The work of [BDG69] showed that this regularity conclusion was sharp for local area-minimisers by establishing that the singular Simons cone, $C^{3,3}$, introduced in [Sim68] was in fact locally area-minimising.

For regularity at the boundary, the work of [HS79] completed the resolution of the Plateau problem in codimension 1 by showing that solutions are entirely smooth in a neighbourhood of any $C^{1,\alpha}$-regular boundary; see also [All75] and [Bou15] concerning boundary regularity properties of stationary varifolds.

In the higher codimension case, the monolithic work of [Alm00] showed that the interior singular set of a k-dimensional area-minimiser was of Hausdorff dimension at most $k-2$; which is sharp in view of examples of the locally area-minimising singular complex analytic variety, $\{(z,w) \mid z^2 = w^3\} \subset \mathbb{C}^2$, as discussed in Subsection 1.3.1. This was revisited in a series of papers, [DS11, DS14, DS16a, DS16b], providing a more streamlined proof of the result. We also mention here the recent works [KW23] and [DMS24] concerning finer properties of the singular set of high codimension area-minimisers and refer the reader to the references therein for a summary of modern theory in this direction.

The regularity conclusions for codimension 1 area-minimisers were shown to hold for stable minimal hypersurfaces in [SS81] and led, in combination with the works [Alm65], [SSY75] and [Pit81], to the existence of optimally regular minimal hypersurfaces in any compact Riemannian manifold. In particular, the Almgren–Pitts min-max procedure was utilised in [MN14] in order to establish the validity of the Willmore conjecture and has found a number of other fascinating results in low-dimensional geometry, for which we refer to the survey [MN20] and the references therein. This ultimately culminated in establishing the existence of infinitely many minimal hypersurfaces in low dimensions in [Son23]; see also [Li23] which established the existence of infinitely many optimally regular minimal hypersurfaces in a generic Riemannian manifold.

The Allen–Cahn min-max approach introduced in [Gua18], building on work of [HT00], [Ton05], [TW12] and [Wic14a], provided an alternative route to establish the existence of optimally regular minimal hypersurfaces in any compact Riemannian manifold of dimension at least 3; this min-max procedure was also carried out on surfaces in [Man21]. In recent years there has been a large amount of interest in developing this Allen–Cahn min-max approach to better understand minimal hypersurfaces; we highlight here the works [Hie18], [Gas19], [GG18], [GG19], [CM20] and [CM23].

In [BW20b] and [BW20c], the sharp regularity theory for hypersurfaces of prescribed mean curvature was developed. These were exploited in [BW20a], alongside a min-max procedure for a modified Allen–Cahn energy functional, in order to produce optimally regular hypersurfaces with mean curvature prescribed by any non-negative Lipschitz function; in particular, when the prescribing function is taken to be a constant, this min-max produces constant mean curvature hypersurfaces.

1.4.2 Previous work on generic regularity

As this thesis is specifically concerned with the generic regularity of constant mean curvature hypersurfaces in ambient dimension 8, and is related to previous work concerning minimal hypersurfaces, we first briefly summarise those works on generic regularity for minimal hypersurfaces in ambient dimension 8 relevant to the thesis:

- The existence of the Simons cone, introduced in [Sim68], showed that stable minimal hypersurfaces may admit isolated singularities in dimension 8.
- In [HS85] it was shown that every area-minimising cone with an isolated singularity is foliated (to both sides) by entirely smooth area-minimising hypersurfaces.
- The generic regularity of area-minimisers in each non-zero homology class was established in [Sma93], using the above foliation result of [HS85].
- In [CLS22], using the Almgren–Pitts min-max procedure and the foliation of [HS85] (in particular the one-sided extension in [Liu19]), the existence of smooth minimal hypersurfaces was established in manifolds equipped with a generic metric of positive Ricci curvature.
- In [LW21] it was shown that every 8-dimensional closed manifold equipped with a generic metric (with no curvature assumption) admits a smooth minimal hypersurface. See also [LW22], where it is shown that for a generic metric, every embedded locally stable minimal hypersurface is smooth in dimension 8.

We also provide a non-exhaustive summary of work on generic regularity for other geometric variational problems:

- In [CMS23a] and [CMS23b] generic regularity of area-minimising minimal hypersurfaces is established, in various settings, up to ambient dimension 10.
- In [Whi85] and [Whi19] it is shown that for a generic ambient metric, every 2-dimensional surface (integral current or flat chain mod 2) without boundary that minimises area in its homology class has support equal to a smoothly embedded minimal surface.
- In [Moo06] and [Moo17] it is shown that for a generic ambient metric, parameterised 2-dimensional minimal surfaces are free of branch points.
- In [CCMS20] and [CCMS22] an analogy was established between mean curvature flow with generic initial data and the generic regularity of area-minimising hypersurfaces.
- In [FRS20] the generic regularity of free boundaries for the obstacle problem is established up to ambient dimension 4.

Chapter 2

Generic regularity via min-max

2.1 Introduction

The Allen–Cahn min-max procedure in [BW20a], with constant prescribing function, shows that in a compact 8-dimensional Riemannian manifold there exists a quasi-embedded hypersurface of constant mean curvature, with a singular set consisting of finitely many points (see Subsection 2.1.2 for a precise description). One may thus conjecture the existence of a smoothly embedded constant mean curvature hypersurface in all 8-dimensional Riemannian manifolds under some assumption on the metric; for example a genericity assumption. As a first step, in this chapter we resolve this for manifolds with positive Ricci curvature:

Theorem 2.1. *Let N be a smooth compact 8-dimensional manifold and $\lambda \in \mathbb{R}$. There is an open and dense subset, $\mathcal{G}$, of the smooth metrics with positive Ricci curvature such that for each $g \in \mathcal{G}$, there exists a closed embedded smooth hypersurface of constant mean curvature λ in (N, g).*

We will actually prove more general results valid in higher dimensions, showing the generic existence of a closed embedded hypersurface of constant mean curvature, with singular set of codimension 7, containing no isolated singularities with regular tangent cone. Indeed, Theorem 2.1 is a consequence of the following:

Theorem 2.2. *Let N be a smooth compact manifold of dimension $n+1 \geq 3$ and $\lambda \in \mathbb{R}$. There is a dense subset, $\mathcal{G}$, of the smooth metrics with positive Ricci curvature such that for each $g \in \mathcal{G}$, there exists a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set of Hausdorff dimension at most $n-7$, containing no isolated singularities with regular tangent cone in (N, g).*

Remark 2.1. *Let* $\mathrm{Met}^{k,\alpha}_{\mathrm{Ric}_g>0}(N)$*, for each* $k \geq 1$ *and* $\alpha \in (0,1)$*, denote the open subset of Riemannian metrics of regularity* $C^{k,\alpha}$ *on* N *with positive Ricci curvature. In the proof of Theorem 2.2 we in fact establish that there exists a dense set,* $\mathcal{G}_k \subset \mathrm{Met}^{k,\alpha}_{\mathrm{Ric}_g>0}(N)$*, such that for each* $g \in \mathcal{G}_k$*, the same existence conclusion of Theorem 2.2 holds.*

The focus of the present chapter concerns the generic regularity of constant mean curvature hypersurfaces in ambient dimensions 8 or higher and is related to previous work concerning minimal hypersurfaces. We now briefly recall those works on generic regularity for minimal hypersurfaces in ambient dimension 8 that are relevant to this chapter:

- The existence of the Simons cone, introduced in [Sim68], showed that stable minimal hypersurfaces may admit isolated singularities in dimension 8.
- The generic regularity of area-minimising hypersurfaces in each non-trivial homology class was established in [Sma93], using the fundamental foliation result of [HS85].
- In [CLS22], using the Almgren–Pitts min-max procedure and the foliation of [HS85] (in particular the one-sided extension in [Liu19]), the existence of smooth minimal hypersurfaces was established in manifolds equipped with a generic metric of positive Ricci curvature.

Both [Sma93] and [CLS22] exploit local foliations by area-minimising hypersurfaces, provided by [HS85], allowing for a surgery procedure to be established in order to perturb away an isolated singularity with regular tangent cone. Recently an analogous foliation was established in [Les23] for constant mean curvature hypersurfaces that locally minimise a prescribed mean curvature functional to at least one side. Such a foliation, to one side of the hypersurface, provides a natural means to perturb away an isolated singularity with regular tangent cone via a surgery procedure; we develop such a procedure in Section 2.2.

Remark 2.2. *The surgery procedure developed in Section 2.2 of this chapter may be immediately applied to isolated singularities with regular tangent cone that arise in boundaries of isoperimetric regions. In particular, this surgery procedure allows for all such singularities to be perturbed away in dimension 8 and results in a smooth hypersurface of constant mean curvature; for details and further discussion see Subsection 2.2.2.*

In order to guarantee the existence of such a local foliation, one needs to establish that the tangent cones to isolated singularities of a candidate hypersurface are area-minimising; we establish this for all hypersurfaces of constant mean curvature arising from the Allen–Cahn min-max procedure in [BW20a] (with constant prescribing function) in manifolds with positive Ricci curvature.

The Allen–Cahn min-max procedure in [BW20a] produces in the first instance a quasi-embedded hypersurface of constant mean curvature with a possibly non-empty singular set of codimension at least 7. The results of [BW24] then establish that in manifolds with positive Ricci curvature the above hypersurface of constant mean curvature in fact attains the min-max value (in a manner made precise in Subsection 2.1.2) and is embedded, with the above dimension bound on the singular set. We thus start to work with this hypersurface as a candidate to perturb away isolated singularities with regular tangent cone via our surgery procedure.

For a compact Riemannian manifold (N, g) and $\lambda \in \mathbb{R}$ we define the $\mathcal{F}_\lambda$ functional on a Caccioppoli set, $F \subset N$, by

$$\mathcal{F}_\lambda(F) = \mathrm{Per}_g(F) - \lambda \mathrm{Vol}_g(F).$$

Recall that (e.g. as shown in [BW20b, Proposition B.1]) smooth constant mean curvature hypersurfaces are locally $\mathcal{F}_\lambda$-minimising. The main technical result of this chapter shows that this $\mathcal{F}_\lambda$-minimisation also holds in sufficiently small balls around isolated singularities for constant mean curvature hypersurfaces produced by the Allen–Cahn min-max in manifolds with positive Ricci curvature:

Theorem 2.3. *Let (N, g) be a smooth compact Riemannian manifold of dimension $n + 1 \geq 3$, with positive Ricci curvature, and $\lambda \in \mathbb{R}$. The one-parameter Allen–Cahn min-max procedure in [BW20a], with constant prescribing function λ, produces a closed embedded hypersurface of constant mean curvature λ which is smooth away from a closed singular set of Hausdorff dimension at most $n - 7$, locally $\mathcal{F}_\lambda$-minimising in balls around each isolated singularity. Precisely, this constant mean curvature hypersurface arises as the boundary of a Caccioppoli set, $E \subset N$, and for each isolated singularity, $p \in \overline{\partial^* E} \setminus \partial^* E$, there exists an explicit $r > 0$ such that*

$$\mathcal{F}_\lambda(E) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(G) \mid G \setminus B_r(p) = E \setminus B_r(p)\},$$

where $\mathcal{C}(N)$ is the set of Caccioppoli sets in N. Consequently, the hypersurface has area-minimising tangent cones at each isolated singularity.

Remark 2.3. *In the case that $\lambda = 0$, Theorem 2.3 shows that minimal hypersurfaces produced by the Allen–Cahn min-max procedure in [Gua18] in manifolds with positive Ricci curvature are in fact locally area-minimising (to both sides), as opposed to just one-sided homotopy minimising as obtained via the Almgren–Pitts min-max procedure in the results of [CLS22]. These minimising properties immediately pass to tangent cones. Stable regular minimal cones that do not minimise area are known to exist, for example the Simons cone*

$$C^{1,5} = \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^6 \mid 5|x|^2 = |y|^2\right\},$$

is stable and one-sided area-minimising, but is not area-minimising to the other side (see [Law91]). Such a cone is not explicitly ruled out in [CLS22] from arising as a tangent cone to a min-max minimal hypersurface at an isolated singularity. Theorem 2.3 precludes such tangent cones. We also note that in the Allen–Cahn framework, obtaining an absolute area-minimisation property, as opposed to a homotopic minimisation property, appears to be natural; indeed the relevant space where the min-max is carried out is $W^{1,2}(N)$, which is contractible.

2.1.1 Chapter notation

We now collect various notation and definitions that will be used throughout the chapter:

- Unless otherwise stated, throughout this chapter we let (N^{n+1}, g) be a compact (with empty boundary) Riemannian manifold of dimension $n+1 \geq 3$ with positive Ricci curvature, $\mathrm{Ric}_g > 0$. We will always implicitly assume that N is connected.

- We let $M \subset N$ be denote a non-empty, smooth, two-sided, separating, embedded hypersurface of constant mean curvature $\lambda \in \mathbb{R}$, with closed *singular set*, $\mathrm{Sing}(M) = \overline{M} \setminus M$, of Hausdorff dimension at most $n - 7$ (we adopt this slight abuse of notation throughout, where precisely we are considering the multiplicity one varifold, $V = (M, 1)$, with $\mathrm{Reg}(V) = M$ and thus set $\mathrm{Sing}(M) = \mathrm{Sing}(V) = \overline{M} \setminus M$ as defined in Subsection 1.3.1). As M is a separating hypersurface, we may write the complement of its closure, $N \setminus \overline{M}$, as two disjoint open sets, E and $N \setminus E$, with common boundary $\overline{M}$.

- We say that $p \in \mathrm{Sing}(M)$ is an *isolated singularity* of M if there exists some $R_p > 0$ such that

$$\text{Sing}(M) \cap \overline{B_{R_p}}(p) = \{p\},$$

i.e. such that $M \cap \overline{B_{R_p}}(p)$ is smooth. Moreover, we say that a multiplicity one tangent cone, C_p, to M at an isolated singularity, $p \in \text{Sing}(M)$, is a *regular tangent cone* if $\text{Sing}(C_p) = \{0\}$, where 0 here denotes the origin in $\mathbb{R}^{n+1}$. We note that the tangent cone to M at an isolated singularity with regular tangent cone is necessarily unique by the work of [Sim83].

- A measurable set $F \subset N$ is a *Caccioppoli set* (often called a set of finite perimeter) if

$$\text{Per}_g(F) = \sup\left\{\int_E \text{div}_g\varphi \;\middle|\; \varphi \in \Gamma(TN), ||\varphi||_\infty \leq 1\right\} < \infty,$$

where div_g is the divergence with respect to the metric g, $\Gamma(TN)$ is the set of vector fields of regularity C^1 on N and $|| \cdot ||_\infty$ denotes the supremum norm. We denote by $\mathcal{C}(N)$ the set of Caccioppoli sets in N. Our main reference for Caccioppoli sets will be [Mag12].

- By De Giorgi's Structure Theorem we have that the distributional derivative, $D_g\chi_F$ (which is a Radon measure), of the the indicator function, χ_F, of a Caccioppoli set F is given by

$$D_g\chi_F = -\nu_F\mathcal{H}^n \llcorner \partial^* F,$$

where $\partial^* F$ is the reduced boundary of F (an n-rectifiable set along which there is an inward pointing unit normal for F, for a precise definition see e.g. [Mag12, Section 15]), $\mathcal{H}^n = \mathcal{H}^n_g$ (we will omit the subscript g when the metric choice is clear from context) is the n-dimensional Hausdorff measure and ν_F is the unit normal to $\partial^* F$ pointing into F defined $\mathcal{H}^n$-a.e. (almost everywhere). Note then that $\text{Per}_g(F) = \mathcal{H}^n(\partial^* F)$.

- We define the following prescribed mean curvature functional on measurable subsets of N: for a measurable set $F \subset N$ we let

$$\mathcal{F}_\lambda(F) = \text{Per}_g(F) - \lambda\text{Vol}_g(F) + \frac{\lambda}{2}\text{Vol}_g(N),$$

where here Vol_g denotes the $n+1$-dimensional Hausdorff measure $\mathcal{H}^{n+1} = \mathcal{H}^{n+1}_g$. We remark that this definition differs from that of Section 2.1 by the addition of the constant $\frac{\lambda}{2}\text{Vol}_g(N)$. Note however this addition of a constant does not affect the set of critical points of the functional $\mathcal{F}_\lambda$ and is made purely for convenience of notation in forthcoming computations.

- With the above two definitions in mind, we denote throughout this chapter ν to be the unit normal to M pointing into E and write $M = \partial^* E$; by viewing $M = \partial^* E$ as the reduced boundary of the Caccioppoli set E we have that E is a critical point for $\mathcal{F}_\lambda$ (as M is assumed to have constant mean curvature).

- We will frequently utilise the notions of integer rectifiable currents and varifolds throughout this chapter; the main reference for the notation and definitions used throughout is [Sim84].

- Let dist_N denote the Riemannian distance (implicitly with respect to the metric g) on N and define the distance function to $\overline{M}$, $d_{\overline{M}}$, on N by setting for each $x \in N$,

$$d_{\overline{M}}(x) = \mathrm{dist}_N(x, \overline{M}).$$

We then have that $d_{\overline{M}}$ is Lipschitz on N (with Lipschitz constant equal to 1) and, as N is complete, for each $x \in N$ there exists a geodesic realising the value $d_{\overline{M}}(x)$. Furthermore, we let

$$d(N) = \sup_{x,y \in N} d_N(x, y)$$

denote the diameter of N, which is finite as N is compact.

- We fix an $R_l > 0$ (dependent on the metric g) such that for every $R \in (0, R_l)$ and each point $p \in N$ we have that the ball $B_R(p) \subset N$ of radius R centred at a point $p \in N$ is 2-bi-Lipschitz diffeomorphic, via a geodeisic normal coordinate chart, to the Euclidean ball, $B_R^{\mathbb{R}^{n+1}}(0) \subset \mathbb{R}^{n+1}$ of radius R centred at the origin in $\mathbb{R}^{n+1}$.

- For $\varepsilon \in (0, 1)$ we denote the *Allen–Cahn energy* of a function $u \in W^{1,2}(N)$ by

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2\sigma} \int_N e_\varepsilon(u) = \frac{1}{2\sigma} \int_N \frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon},$$

where W is a C^2 double-well potential with non-degenerate minima at ± 1, $c_W \leq W''(t) \leq C_W$ for constants $c_W, C_W > 0$ for all $t \in \mathbb{R} \setminus [-2, 2]$ and $\sigma = \int_{-1}^{1} \sqrt{W(t)/2}\, dt$. A standard choice of double-well potential is

$$W(u) = \frac{(1 - u^2)^2}{4}.$$

Note here that we are considering the energy density $e_\varepsilon(u)$ as the measure

$$e_\varepsilon(u) = \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right) d\mathcal{H}^{n+1}_g.$$

- We will consider the following functional, which we shall frequently refer to throughout this chapter simply as the *energy*, defined for functions $u \in W^{1,2}(N)$ by

$$\mathcal{F}_{\varepsilon,\lambda}(u) = \mathcal{E}_\varepsilon(u) - \frac{\lambda}{2}\int_N u.$$

With reference to the definition of the functional $\mathcal{F}_\lambda$ above, we remark that the addition of the constant $\frac{\lambda}{2}\mathrm{Vol}_g(N)$ in the definition was made to ensure that (in a manner made precise in Subsection 2.3.3) the energy, $\mathcal{F}_{\varepsilon,\lambda}$, provides a suitable approximation of the functional $\mathcal{F}_\lambda$ as $\varepsilon \to 0$.

2.1.2 Allen–Cahn min-max preliminaries

The Allen–Cahn min-max procedure in [BW20a] produces a hypersurface with mean curvature prescribed by an arbitrary non-negative Lipschitz function, and provides sharp dimension bounds on the singular set.

We recall this procedure in the case relevant to this chapter, in which the metric is assumed to have positive Ricci curvature, and the prescribing function is a non-negative constant λ; for producing a candidate hypersurface of constant mean curvature $\lambda < 0$ one can simply consider $-\lambda$ in the results below (this amounts to a change in the choice of unit normal to M); thus, without loss of generality, in this chapter we assume that $\lambda \geq 0$. The constant mean curvature hypersurfaces produced by this procedure will, after establishing Theorem 2.3, be the candidates for our surgery procedure established in Section 2.2.

For $\varepsilon \in (0,1)$ there exist two constant functions, a_ε and b_ε, on N, arising as stable critical points of $\mathcal{F}_{\varepsilon,\lambda}$ with the properties that

$$\begin{cases} -1 < a_\varepsilon < -1 + c\varepsilon \\ +1 < b_\varepsilon < +1 + c\varepsilon \end{cases},$$

and thus

$$\begin{cases} a_\varepsilon \to -1 \text{ as } \varepsilon \to 0 \\ b_\varepsilon \to +1 \text{ as } \varepsilon \to 0 \end{cases}.$$

Here in the above the constant $c > 0$ depends only on the choices of W

and λ. These functions are constructed in [BW20a, Section 5] by means of the negative gradient flow, through constant functions, of $\mathcal{F}_{\varepsilon,\lambda}$ starting at ± 1. In particular, there are continuous paths of functions in $W^{1,2}(N)$ connecting a_ε to -1 and b_ε to $+1$, provided by the negative gradient flow of $\mathcal{F}_{\varepsilon,\lambda}$; in particular the energy along these paths is bounded from above by $\mathcal{F}_{\varepsilon,\lambda}(-1)$ and $\mathcal{F}_{\varepsilon,\lambda}(+1)$ respectively.

For each $\varepsilon \in (0,1)$ a min-max critical point, $u_\varepsilon \in W^{1,2}(N)$, of $\mathcal{F}_{\varepsilon,\lambda}$ may be constructed, with $\sup_N |u_\varepsilon|$ uniformly bounded and $\mathcal{E}_\varepsilon(u_\varepsilon)$ uniformly bounded from above and below by positive constants (independently of ε). This is done by applying a mountain pass lemma, for paths between the two stable critical points a_ε and b_ε, based on the fact that the energy, $\mathcal{F}_{\varepsilon,\lambda}$, satisfies a Palais-Smale condition. The Morse index of the critical points, u_ε, will then all be equal 1, as noted in [BW20a, Remark 6.7], by virtue of the fact that if $\mathrm{Ric}_g > 0$ then a_ε and b_ε are the only stable critical points of the energy functional $\mathcal{F}_{\varepsilon,\lambda}$.

By general principles, the uniform bounds on $\sup_N |u_\varepsilon|$ and $\mathcal{E}_\varepsilon(u_\varepsilon)$ above imply that there exist a sequence $\varepsilon_j \to 0$, a non-zero Radon measure, μ, on N and a function, $u_\infty \in BV(N)$ with $u_\infty = \pm 1$ for a.e. $x \in N$, such that for the min-max critical points, $\{u_{\varepsilon_j}\}_{j=1}^\infty$, we have, as $\varepsilon_j \to 0$, that:

$$\begin{cases} \frac{1}{2\sigma} e_{\varepsilon_j}(u_{\varepsilon_j}) \to \mu \text{ weakly as measures} \\ u_{\varepsilon_j} \to u_\infty \text{ strongly in } L^1(N) \end{cases}.$$

Defining $E = \{u_\infty = 1\}$, we note that E is a Caccioppoli set with its reduced boundary $\partial^* E \subset \mathrm{Spt}\mu$; moreover, as $\mathrm{Ric}_g > 0$ we have that $E \neq \emptyset$ by the arguments in [BW20a, Remark 6.7].

We now introduce a key definition:

Definition 2.1. *[BW20a, Definition 8] A quasi-embedded hypersurface is a smooth immersion such that in an open neighbourhood of each non-embedded point, the image of the immersion is the union of two embedded $C^{2,\alpha}$ disks intersecting tangentially, with each disk lying on one side of the other. Equivalently, near each non-embedded point of the image, the immersion is a union of two graphs over a common tangent plane, with one graph lying above the other.*

Prototypical examples of quasi-embedded constant mean curvature hypersurfaces are provided by two spheres touching at a point and two cylinders touching along a line.

In [BW20a], relying on the combined works of [HT00] and [RT07], it is then established that the measure μ above is in fact the weight measure of an integer rectifiable n-varifold V with the following properties:

- $V = V_0 + V_\lambda$.
- V_0 is a (possibly zero) stationary n-varifold on N with $\mathrm{Sing}(V_0)$ empty if $2 \leq n \leq 6$, $\mathrm{Sing}(V_0)$ discrete if $n = 7$ and $\mathrm{Sing}(V_0)$ of Hausdorff dimension $\leq n - 7$ when $n \geq 8$.
- $V_\lambda = |\partial^* E| \neq 0$ (by [BW20a, Remark 6.7] when $\mathrm{Ric}_g > 0$ as noted above), is the multiplicity one n-varifold associated with the reduced boundary $\partial^* E$. Moreover, $\mathrm{Spt}(V_\lambda)$ is a quasi-embedded hypersurface of constant mean curvature λ with respect to the unit normal pointing into E, away from a closed set which is empty if $2 \leq n \leq 6$, discrete if $n = 7$ and of Hausdorff dimension $\leq n - 7$ when $n \geq 8$.

For a more detailed description of the definitions and results above we refer the reader to [BW20a, Sections 3 and 4].

As in [BW24, Theorem 2], whenever we assume $\lambda \neq 0$ the path we exhibit, for all $\varepsilon > 0$ sufficiently small, in Subsection 2.5.4 with the upper energy bounds provided by Lemma 2.5 (these bounds are depicted by the dashed lines in Figure 2.1) between $+1$ and -1, along with short paths of constant functions connecting $+1$ to b_ε and -1 to a_ε, proves that $V_0 = 0$; i.e. that the min-max procedure produces no minimal piece in manifolds with positive Ricci curvature when $\lambda \neq 0$. Using this we then note that as we have $\frac{1}{2\sigma} e_{\varepsilon_j}(u_{\varepsilon_j}) \to \mu$ as $\varepsilon_j \to 0$ and $E = \{u_\infty = 1\}$ we have

$$\mathcal{F}_{\varepsilon_j,\lambda}(u_{\varepsilon_j}) \to \mathcal{F}_\lambda(E) \text{ as } \varepsilon_j \to 0, \tag{2.1}$$

i.e. by which we have that the constant mean curvature hypersurface attains the min-max value in an appropriate sense. In the proof of Theorem 2.3, under the contradiction assumption that our candidate hypersurface produced by the above procedure does not satisfy a local minimisation property, we will exploit (2.1) by constructing continuous paths of functions in $W^{1,2}(N)$ for all $\varepsilon > 0$ sufficiently small, admissible in the min-max construction above, with energy along the paths bounded above by a value strictly below $\mathcal{F}_\lambda(E)$ (independently of ε); thus violating the min-max characterisation of E.

In fact, by [BW24, Theorem 4], for $\lambda \neq 0$ we have that $\partial^* E$ is fully embedded (rather than quasi-embedded). Moreover, $\partial^* E$ is connected, has index 1 and is separating in the sense that $N \setminus \partial E$ may be written as the union of two disjoint open sets whose common boundary is ∂E. The same properties hold for E in the case that $\lambda = 0$ by combining the results of [Gua18, Theorem A] with [Bel23b, Theorem 1.8].

To summarise, we know that for a compact Riemannian manifold of dimension $n + 1 \geq 3$ with positive Ricci curvature, the properties of M as stated in Subsection 2.1.1 hold for any constant mean curvature hypersurface produced by the Allen–Cahn min-max procedure of [BW20a] when the prescribing function is taken to be the constant λ.

2.1.3 Proof strategy

The results of this chapter can be divided up into the following three distinct steps that combine to prove Theorems 2.1, 2.2 and 2.3:

1. **Functions to geometry:** We relate the local geometric behaviour of constant mean curvature hypersurfaces produced by the Allen–Cahn min-max procedure to the $\varepsilon \to 0$ energy properties of specific $W^{1,2}(N)$ functions.

2. **Paths of functions:** By exhibiting an admissible min-max path, we establish that the energy properties for the functions from Step 1 hold. We use this to establish that the constant mean curvature hypersurfaces generated through the Allen-Cahn min-max procedure in positive Ricci curvature are locally $\mathcal{F}_\lambda$-minimising around their isolated singularities.

3. **Surgery procedure:** We show how to perturb constant mean curvature hypersurfaces that are locally $\mathcal{F}_\lambda$-minimising around isolated singularities with regular tangent cone, resulting in a smooth hypersurface of constant mean curvature.

We now sketch these three steps in more detail in order to clearly outline our arguments and later technical work for the reader:

1. **Functions to geometry:** We aim to show that, when the ambient metric is assumed to have positive Ricci curvature, the hypersurfaces of constant mean curvature λ produced by the Allen–Cahn min-max procedure in [BW20a] are in fact locally $\mathcal{F}_\lambda$-minimising. In order to do this we will first relate the local $\mathcal{F}_\lambda$-minimisation we desire to the $\varepsilon \to 0$ energy behaviour of specific $W^{1,2}(N)$ functions defined from such a hypersurface.

Rather than work directly with the min-max critical points of $\mathcal{F}_{\varepsilon,\lambda}$ produced in [BW20a] we instead introduce, in Subsection 2.3.3, a function, $v_\varepsilon \in W^{1,2}(N)$, which we call the *one-dimensional profile*. This function is constructed by placing a truncated version of the one-dimensional solution to the Allen–Cahn equation, $\overline{\mathbb{H}}^\varepsilon$, (explicitly constructed in Subsection 2.3.3) in the normal direction to an underlying hypersurface, M, of constant mean curvature λ as defined in Subsection 2.1.1. Precisely, we define

$$v_\varepsilon = \overline{\mathbb{H}}^\varepsilon \circ d^{\pm}_{\overline{M}},$$

where here $d^{\pm}_{\overline{M}}$ is the Lipschitz *signed distance function* to $\overline{M}$, taking positive values in E and negative values in $N \setminus E$. The function v_ε is then shown to act as an approximation of the hypersurface M in the sense that, analogously to (2.1), we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \to \mathcal{F}_\lambda(E) \text{ as } \varepsilon \to 0. \tag{2.2}$$

In Subsection 2.4.2, for an isolated singularity, $p \in \mathrm{Sing}(M)$, $\varepsilon > 0$ sufficiently small and radius $\rho > 0$, we minimise $\mathcal{F}_{\varepsilon,\lambda}$ over a class of functions $\mathcal{A}_{\varepsilon,\rho}(p)$. The set $\mathcal{A}_{\varepsilon,\rho}(p)$ is, roughly speaking, all $W^{1,2}(N)$ functions agreeing with v_ε outside of the ball of radius ρ centred at p. The minimiser of this problem is thus a function, $g_\varepsilon \in W^{1,2}(N)$, that agrees with v_ε outside of the ball $B_\rho(p)$ and is such that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_{\varepsilon,\rho}(p)} \mathcal{F}_{\varepsilon,\lambda}(u). \tag{2.3}$$

Note that the notation used for g_ε suppresses the dependence on $p \in \mathrm{Sing}(M)$ and $\rho > 0$ used in the construction; in each instance that the functions g_ε are utilised the choice of isolated singularity and radius in question will be made explicit. We then produce, in Subsection 2.4.3, a sequence of "recovery functions", admissible in the minimisation problem that produced g_ε above, for any local $\mathcal{F}_\lambda$-minimiser. Precisely, in the vein of [KS89], for each local $\mathcal{F}_\lambda$-minimiser, $F \in \mathcal{C}(N)$, agreeing with E outside of $B_{\frac{\rho}{2}}(p)$, we show that there exists a sequence of functions, $f_\varepsilon \in \mathcal{A}_{\varepsilon,\rho}(p)$, for all $\varepsilon > 0$ sufficiently small such that

$$\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F) \text{ as } \varepsilon \to 0. \tag{2.4}$$

As $f_\varepsilon \in \mathcal{A}_{\varepsilon,\rho}(p)$ we conclude that by (2.3) we have

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon).$$

In particular, by (2.2), (2.3) and (2.4), if we assume that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \text{ for some sequence } \tau_\varepsilon \to 0 \text{ as } \varepsilon \to 0, \tag{2.5}$$

then

$$\mathcal{F}_\lambda(E) \leq \mathcal{F}_\lambda(F),$$

so that E is $\mathcal{F}_\lambda$-minimising in $B_{\frac{\varrho}{2}}(p)$. In this manner we have related, via (2.5), the $\varepsilon \to 0$ energy behaviour of specific $W^{1,2}(N)$ functions, namely v_ε and g_ε, defined from a hypersurface as produced by the Allen–Cahn min-max procedure, to the geometric behaviour of the underlying hypersurface; precisely, if (2.5) holds then E is locally $\mathcal{F}_\lambda$-minimising. In order to prove Theorem 2.3 we will therefore turn our attention to establishing, as sketched in Step 2 below, that (2.5) holds for all constant mean curvature hypersurfaces produced by the Allen–Cahn min-max procedure in manifolds with positive Ricci curvature.

In order to produce the "recovery functions" as described above we first establish a local smoothing procedure for Caccioppoli sets that are smooth in an annular region. This is done by a "cut-and-paste" style argument using Sard's Theorem on the level sets of mollified indicator functions for the Caccioppoli set, full details of this procedure can be found in Subsection 2.4.1. We emphasise that this local smoothing procedure we exhibit for Caccioppoli sets is not, in and of itself, sufficient to establish Theorem 2.1. The reason for this is that the local smoothing we produce provides no control on the mean curvature near the isolated singularity, and this lack of control makes it difficult to perturb the metric in the same manner as in Step 3. However, the foliation provided by [Les23] ensures there always exists a sequence of smooth hypersurfaces of constant mean curvature λ converging to the singular hypersurface, allowing for the surgery to be carried out directly in the manner described in Step 3 below.

2. **Paths of functions:** Similarly to the strategy employed in previous works on hypersurfaces produced by the Allen–Cahn min-max procedure, for example in [Bel23b], [BW20a], and [BW24], establishing that (2.5) holds, in order to conclude the proof of Theorem 2.3, is achieved by exhibiting a suitable continuous path in $W^{1,2}(N)$.

Under the assumption that a hypersurface produced by the min-max procedure violates a desired property, one basic idea is to exploit its min-max characterisation as follows. If a path admissible in the min-max procedure may be produced, with energy along this path bounded above by a constant strictly less than the min-max value, then one contradicts the assumption that such a hypersurface arose from the min-max. Thus, the desired property must hold for all hypersurfaces produced by the min-max procedure.

We first emphasise that the paths we construct in $W^{1,2}(N)$ reflect the underlying geometry imposed by the assumption of positive Ricci curvature. We denote the super-level sets and level sets of the signed distance function, for each $s \in \mathbb{R}$, by

$$E(s) = \{x \in N \mid d^{\pm}_{\overline{M}}(x) > s\} \text{ and } \Gamma(s) = \{x \in N \mid d^{\pm}_{\overline{M}}(x) = s\},$$

respectively; so that $E(0) = E$ and $\Gamma(0) = \overline{M}$. Formally computing we have, for almost every $s \in \mathbb{R}$, that

$$\begin{cases} \frac{d}{ds}\mathcal{H}^n(\Gamma(s)) = -\int_{\Gamma(s)} H(x,s)\, d\mathcal{H}^n(x) \\ \frac{d}{ds}\mathrm{Vol}_g(E(s)) = \mathcal{H}^n(\Gamma(s)) \end{cases}, \tag{2.6}$$

where here $H(x,s)$ denotes the mean curvature of the level set $\Gamma(s)$ at a point $x \in N$. By denoting $m = \min_N \mathrm{Ric}_g > 0$, the assumption of positive Ricci curvature implies the following relation between the mean curvature of M and the level sets $\Gamma(s)$:

$$\begin{cases} H(x,s) \geq \lambda + ms \text{ for } s > 0 \\ H(x,0) = \lambda \\ H(x,s) \leq \lambda + ms \text{ for } s < 0 \end{cases}, \tag{2.7}$$

for further explanation see Subsection 2.3.2. Therefore, by (2.6) and (2.7), for each $t \in \mathbb{R} \setminus \{0\}$ we compute that

$$\begin{aligned} \mathcal{F}_\lambda(E(t)) - \mathcal{F}_\lambda(E) &= \int_0^t \frac{d}{ds}\mathcal{H}^n(\Gamma(s)) - \lambda \frac{d}{ds}\mathrm{Vol}_g(E(s))\, ds \\ &= \int_0^t \int_{\Gamma(s)} \lambda - H(x,s)\, d\mathcal{H}^n(x)\, ds < 0. \end{aligned}$$

From the assumption of positive Ricci curvature we thus conclude that for each $t \in \mathbb{R} \setminus \{0\}$ we have

$$\mathcal{F}_\lambda(E(t)) < \mathcal{F}_\lambda(E). \tag{2.8}$$

We then replicate this geometric behaviour of the super-level sets at the diffuse level by considering the continuous path of *sliding functions*, $v^t_\varepsilon \in W^{1,2}(N)$, produced by sliding the zero level set of v_ε from $\overline{M}$ to $\Gamma(t)$

for each $t \in \mathbb{R}$; that is we define

$$v_\varepsilon^t = \overline{\mathbb{H}}^\varepsilon \circ (d^\pm_{\overline{M}} - t),$$

which satisfy the following properties for each $t \in \mathbb{R}$ (whenever $\varepsilon > 0$ is sufficiently small with respect to the diameter, $d(N)$)

$$\begin{cases} \{v_\varepsilon^t = 0\} = \Gamma(t) \\ v_\varepsilon^0 = v_\varepsilon \text{ on } N \\ v_\varepsilon^{2d(N)} = -1 \text{ on } N \\ v_\varepsilon^{-2d(N)} = +1 \text{ on } N \end{cases}.$$

The geometric relation (2.8), induced by the positive Ricci curvature assumption, then translates to the level of functions and allows us to compute, in Subsection 2.5.4, that the v_ε^t have the following $\varepsilon \to 0$ energy property:

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) \leq \mathcal{F}_\lambda(v_\varepsilon) + E(\varepsilon) \text{ where } E(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

As the nodal sets of the functions v_ε^t are precisely the level sets of the signed distance function to $\overline{M}$, the functions v_ε^t may be directly interpreted as a path of functions analogous to a sweep-out of N by the level sets of the signed distance function to $\overline{M}$.

Thus, by concatenating the path of the sliding functions $v_\varepsilon^t \in W^{1,2}(N)$ with the energy reducing paths from -1 and $+1$ provided by negative gradient flow of the energy to a_ε and b_ε respectively, provides a "recovery path" for the value $\mathcal{F}_\lambda(E)$; this path connects a_ε to b_ε, passing through v_ε, with the maximum value of the energy along this path approximately $\mathcal{F}_\lambda(E)$ (by virtue of (2.2)); approximate upper energy bounds along this path are depicted by the thick dashed lines in Figure 2.1. In this manner, as mentioned in Subsection 2.1.2, such a path establishes that the Allen–Cahn min-max procedure in positive Ricci curvature produces no minimal piece. Furthermore, in combination with (2.2) it guarantees that (2.1) holds for the min-max critical points of the energy introduced in Subsection 2.1.2.

We now exhibit (for all $\varepsilon > 0$ sufficiently small) a continuous path in $W^{1,2}(N)$ between a_ε and b_ε which, under the assumption that E is not $\mathcal{F}_\lambda$-minimising a small ball around an isolated singularity, contradicts the min-max characterisation of E and proves Theorem 2.3. We emphasise that this path is constructed with energy bounded above by a value strictly below $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ (independently of ε); approximate upper energy bounds along the path constructed are depicted by the solid curve in Figure 2.1. We thus conclude, by the arguments in Step 1, that the Allen–Cahn min-max procedure in positive Ricci curvature produces a hypersurface which is locally $\mathcal{F}_\lambda$-minimising.

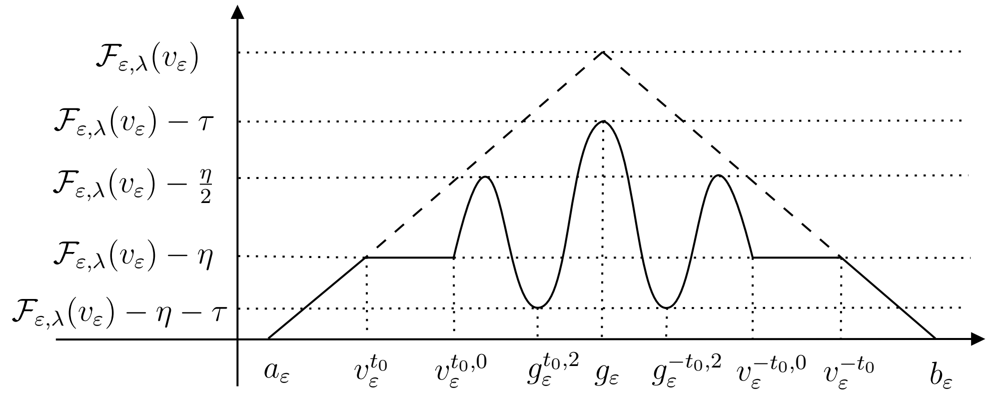

Figure 2.1: The solid curve depicts approximate (i.e. up to the addition of an error term that converges to zero as $\varepsilon \to 0$) upper energy bounds along the path taken from a_ε to b_ε constructed for the proof of Theorem 2.3 in Subsection 2.6.1. The horizontal axis identifies some specific functions in the path and the vertical axis depicts the approximate upper bound on the energy of the path between each identified function. The thick dashed lines depict an approximate upper energy bound along the path of functions in Lemma 2.5.

We will now sketch the various path constructions and motivate the upper energy bounds as a diffuse reflection of the underlying geometry. The explicit constructions of the portions of the path along with computations for their approximate upper energy bounds are carried out in full in Section 2.5.

Firstly, for a given isolated singularity, $p \in \mathrm{Sing}(M)$, we are able to continuously deform the super-level set $E(t_0)$ (for a fixed $t_0 > 0$ sufficiently small) locally around p so that the resulting deformation agrees with E inside of a fixed ball $B_{r_0}(p)$ (for some $r_0 > 0$ determined only by the area of M). This is done by exploiting (2.8) in such a way that the $\mathcal{F}_\lambda$-energy of the deformations remain a fixed amount below $\mathcal{F}_\lambda(E)$. At the diffuse level this is replicated by placing $\overline{\mathbb{H}}^\varepsilon$ in the normal direction to the deformations as in the construction of v_ε, producing a $W^{1,2}(N)$ continuous path of functions with controlled energy; as depicted in Figure 2.2.

Specifically, we produce a continuous path of *shifted functions*,

$$s \in [0,1] \to v_\varepsilon^{t_0,s} \in W^{1,2}(N),$$

which satisfy the following properties

$$\begin{cases} v_\varepsilon^{t_0,1} = v_\varepsilon^{t_0} \text{ on } N \\ v_\varepsilon^{t_0,0} = v_\varepsilon \text{ in } B_{r_0}(p) \end{cases}.$$

Furthermore, there exists a fixed $\eta > 0$ such that for each $s \in [0,1]$ we have the following upper energy bound

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t_0,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta,$$

as depicted in Figure 2.1. In this manner we have exhibited a continuous path in $W^{1,2}(N)$ from the sliding function $v_\varepsilon^{t_0}$ to a function $v_\varepsilon^{t_0,0}$, which is equal to v_ε in a fixed ball $B_{r_0}(p)$, and with the energy along this path a fixed amount below the min-max value.

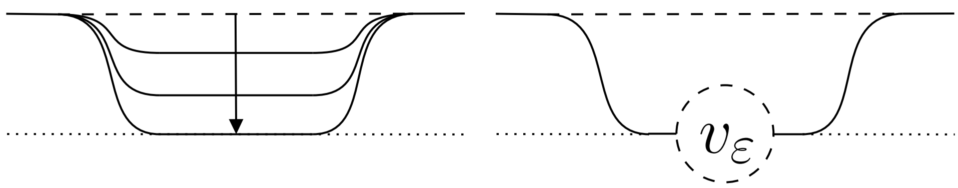

Figure 2.2: In both graphics above the lower thin dashed horizontal line depicts $\overline{M}$, the zero level set of the function v_ε, and the upper thick dashed horizontal line depicts the zero level set, $\Gamma(t_0)$, of $v_\varepsilon^{t_0} = v_\varepsilon^{t_0,1}$ which is deformed in the construction of the shifted functions. In the left-hand graphic the solid lines depict various zero level sets of the $v_\varepsilon^{t_0,s}$ as we vary s from 0 to 1. In the right-hand graphic the solid line depicts the zero level set of $v_\varepsilon^{t_0,0}$ and the thick dashed circle depicts the boundary of the ball, $B_{r_0}(p)$, in which $v_\varepsilon^{t_0,0} = v_\varepsilon$.

Next we construct a continuous path of functions, from v_ε to the local energy minimiser $g_\varepsilon \in \mathcal{A}_{\varepsilon,\frac{R}{2}}(p)$ (recall Step 1) for a fixed $R \in (0, r_0)$. This is done in such a way that we only alter the functions inside $B_R(p)$ (where it holds that $v_\varepsilon^{t_0,0} = v_\varepsilon$); thus, in the following description we will only consider functions in the ball $B_R(p)$. The radius $R > 0$ here is chosen sufficiently small based on the energy drop, $\eta > 0$, above achieved outside of $B_{r_0}(p)$, ensuring that the energy along the constructed path will remain a fixed amount below the min-max value.

In order to explicitly construct this local path, from v_ε to g_ε inside of $B_R(p)$, we utilise a sweep-out of the ball by images of Euclidean planes via a geodesic normal coordinate chart; depicted by the thin dashed curves in Figure 2.3. The hypersurfaces in this sweep-out are used to continuously transition from our constant mean curvature hypersurface and the

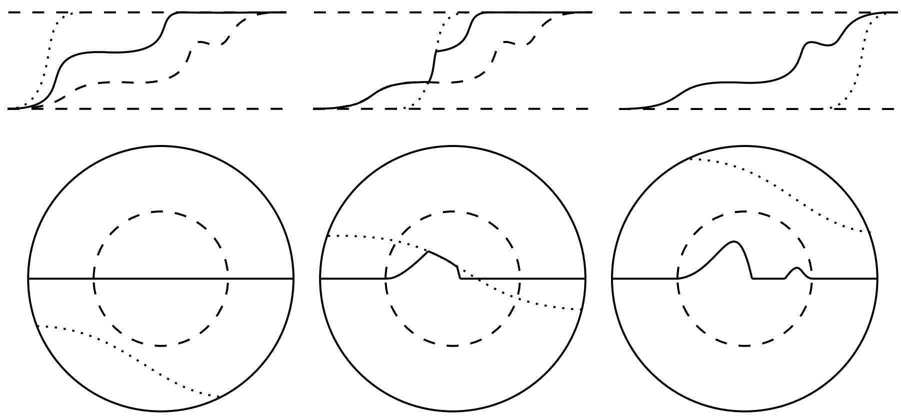

Figure 2.3: The graphics above depict various stages of the transition, in $W^{1,2}(N)$, between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$ inside $B_R(p)$. The first row depicts schematics of this transition and the second row depicts the geometry of the zero level sets of the local path functions; in this way, each column of the figure contains a schematic for the local path function with a corresponding depiction of the geometry of its zero level set below it. Further explanation of this construction is provided in Remark 2.5.

local $\mathcal{F}_\lambda$-minimiser at the diffuse level; precisely, the planes facilitate the construction of a path between the diffuse representatives v_ε and g_ε.

Remark 2.4. *In the setting of the Almgren–Pitts min-max one considers homotopy sweep-outs of N by cycles, as opposed to continuous paths in the Sobolev space $W^{1,2}(N)$ in the Allen–Cahn min-max. However, there does not necessarily exist a homotopy of cycles between our hypersurface and any local $\mathcal{F}_\lambda$-minimiser; for instance there exists no homotopy of cycles between a catenoidal neck and the union of two sufficiently close disks sharing a common boundary (provided by the two circles bounding the disks). We overcome this for the Allen–Cahn min-max by directly exploiting the topology of $W^{1,2}(N)$, showing that the local path may be seen as a diffuse analogue of [CLS22, Lemma 1.12], and illustrating why we guarantee local $\mathcal{F}_\lambda$-minimisation (see Remark 2.3).*

In the same manner as in the construction of v_ε, by placing $\overline{\mathbb{H}}^\varepsilon$ in the normal direction to the hypersurfaces in this "planar" sweep-out we construct a sweep-out at the diffuse level that is continuous in $W^{1,2}(N)$; the diffuse sweep-out thus acts as an approximation for the underlying "planar" sweep-out. This diffuse sweep-out of $B_R(p)$ is utilised twice, first for the construction of a path from v_ε to $\min\{g_\varepsilon, v_\varepsilon\}$, and second for the construction of a path from $\min\{g_\varepsilon, v_\varepsilon\}$ to g_ε. By taking a combination of maxima and minima of functions in the diffuse sweep-out, v_ε and g_ε, (which ensure the resulting functions are in $W^{1,2}(N)$) we are able to produce a local path from v_ε to g_ε; see Figure 2.3 for a depiction of this construction and Remark 2.5 below for further explanation of the portion of the local path between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$ in $B_R(p)$.

Remark 2.5. *We now explain in more detail the construction of the path transitioning between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$ as depicted by Figure 2.3: For the first row, in all three of the images the solid black line depicts the local path function in question (equal to v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$ in the left-hand and right-hand graphic respectively), the thick dashed curve depicts portions of $\min\{v_\varepsilon, g_\varepsilon\}$ that are not yet included in the path, the thin dashed curve depicts the diffuse sweep-out function in question, and the thick dashed upper and lower horizontal lines depict the functions ± 1 respectively. Notice that the local path function depicted in the middle graphic includes portions of v_ε that lie to the right of the diffuse sweep-out function, portions of $\min\{v_\varepsilon, g_\varepsilon\}$ that lie to the left of the diffuse sweep-out function and uses the diffuse sweep-out function itself to interpolate between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$. The explicit construction of the local path functions replicating the behaviour of these schematics involves taking various maxima and minima of the functions in the diffuse sweep-out, v_ε and g_ε; the diffuse sweep-out allows for this choice of maxima and minima (corresponding geometrically to a choice of zero level set) to be made continuously.*

For the second row, in all three graphics the outer solid circle depicts the boundary of $B_R(p)$ and the thick dashed inner circle depicts the boundary of $B_{\frac{R}{2}}(p)$. In the left-hand graphic the solid line depicts the zero level set of v_ε, which is $\overline{M}$, in the right-hand graphic the solid curve depicts the zero level set of $\min\{g_\varepsilon, v_\varepsilon\}$, and in the middle graphic the solid curve depicts the zero level set of a local path function in the transition between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$. The thin dashed curves in the three graphics depict a given hypersurface in the "planar" sweep-out of $B_R(p)$, each separating $B_R(p)$ into two open sets (one above and one below it). The zero level set of the function in the local path associated to this hypersurface is chosen to be the portions of the zero level set of $\min\{v_\varepsilon, g_\varepsilon\}$ that are beneath the hypersurface, the portions of $\overline{M}$ that are above the hypersurface, and when the the hypersurface lies between the nodal sets of v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$, the hypersurface itself is chosen as the zero level set.

Similar ideas as described in Remark 2.5 above are used for the construction of the remainder of the path from v_ε to g_ε in $B_R(p)$. Here instead one reverses the direction of the "planar" sweep-out and takes similar maxima and minima of $W^{1,2}(N)$ functions in order to construct the portion of the continuous path transitioning from $\min\{v_\varepsilon, g_\varepsilon\}$ to g_ε.

We note that $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(\min\{v_\varepsilon, g_\varepsilon\}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ (by local energy minimisation of g_ε) and that $R \in (0, r_0)$ is chosen based on $\eta > 0$ to ensure that the total energy contribution of the diffuse sweep-out functions in the ball is at most $\frac{\eta}{2}$. As a consequence of these two facts, the energy in $B_R(p)$ of any local path function can be estimated to be at most the energy of v_ε in $B_R(p)$ plus $\frac{\eta}{2}$; from there one obtains the energy estimate in the whole of N.

Specifically, we produce a continuous path of *local functions*,

$$s \in [-2, 2] \to g_\varepsilon^{t_0,s} \in W^{1,2}(N),$$

which satisfy the following properties for all $s \in [-2, 2]$

$$\begin{cases} g_\varepsilon^{t_0,-2} = v_\varepsilon^{t_0,0} \text{ on } N \\ g^{t_0,s} = v^{t_0,s} \text{ in } N \setminus B_R(p) \\ g^{t_0,2} = g_\varepsilon \text{ in } B_R(p) \end{cases}.$$

Furthermore, for each $s \in [-2, 2]$ we have the following upper energy bound

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t_0,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\eta}{2},$$

as depicted in Figure 2.1. In this manner we have exhibited a continuous path in $W^{1,2}(N)$ from $v_\varepsilon^{t_0,0}$ to a function $g_\varepsilon^{t_0,2}$, changing $v_\varepsilon^{t_0,0}$ only inside of $B_R(p)$ (from v_ε to the local energy minimiser g_ε).

In order to establish that E is locally $\mathcal{F}_\lambda$-minimising, we now argue by contradiction and assume that there exists a $\tau > 0$ such that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau \text{ for all } \varepsilon > 0 \text{ sufficiently small.} \tag{2.9}$$

Note that by the results discussed in Step 1, contradicting (2.9) will establish that M is $\mathcal{F}_\lambda$-minimising in $B_{\frac{R}{4}}(p)$ for $R > 0$ as chosen above (as (2.5) must then hold). Using the contradiction assumption, in addition to keeping the energy of the local path functions a fixed amount, $\frac{\eta}{2}$, below the min-max value, as $g_\varepsilon^{t_0,2} = g_\varepsilon$ in $B_R(p)$ we thus also have that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t_0,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta - \tau,$$

as depicted in Figure 2.1. We now directly exploit this extra energy drop afforded by the contradiction assumption in order to construct the next portion of the path.

To this end, we deform the rest of the super-level set $E(t_0)$ entirely

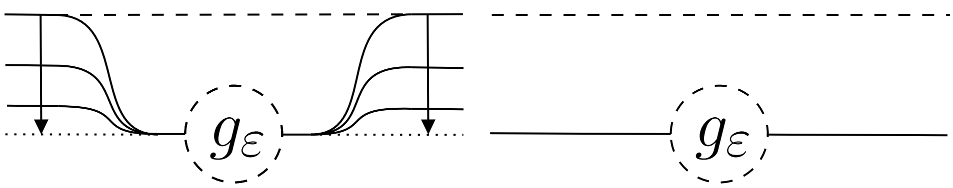

Figure 2.4: In both graphics above the lower horizontal lines depict $\overline{M}$, the upper horizontal lines depict the zero level set, $\Gamma(t_0)$, of $v_\varepsilon^{t_0} = v_\varepsilon^{t_0,1}$ which is deformed in the construction of the shifted functions and the thick dashed circle depicts the boundary of the ball, $B_{r_0}(p)$, in which the shifted functions are such that $g_\varepsilon^{t,2} = g_\varepsilon$. In the left-hand graphic the solid lines depict various zero level sets of the $g_\varepsilon^{t,2}$ as we vary t from t_0 to 0. In the right-hand graphic the solid line depicts the zero level set of $g_\varepsilon^{0,2} = g_\varepsilon$.

onto E outside of $B_{r_0}(p)$. This is done in such a way that the deformations fix the inside of $B_{r_0}(p)$, thus preserving a drop in $\mathcal{F}_\lambda$-energy under the assumption that E is not locally $\mathcal{F}_\lambda$-minimising. At the diffuse level this is replicated by placing $\overline{\mathbb{H}}^\varepsilon$ in the normal direction to these deformations, keeping the functions equal to the local energy minimiser g_ε (which yields an energy drop by assumption (2.9)) inside of $B_{r_0}(p)$ and producing a $W^{1,2}(N)$ continuous path of functions with controlled energy; this is depicted in Figure 2.4.

Specifically, we produce (under the assumption that (2.9) holds) a continuous path of *shifted functions*,

$$t \in [0, t_0] \to g_\varepsilon^{t,2} \in W^{1,2}(N),$$

which are equal to the local function $g_\varepsilon^{t_0,2}$ when $t = t_0$ (justifying notation), and satisfy the following properties for all $t \in [0, t_0]$

$$\begin{cases} g_\varepsilon^{0,2} = g_\varepsilon \text{ on } N \\ g_\varepsilon^{t,2} = g_\varepsilon \text{ in } B_{r_0}(p) \end{cases}.$$

Furthermore, for each $t \in [0, t_0]$ have the following upper energy bound

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) - \tau,$$

as depicted in Figure 2.1. In this manner we have exhibited a continuous path in $W^{1,2}(N)$ from $g_\varepsilon^{t_0,2}$ to the local energy minimiser g_ε, only changing $g_\varepsilon^{t_0,2}$ outside of $B_{r_0}(p)$.

To summarise all of the above, as the endpoints of each of the paths described agree with the start of the next, for all $\varepsilon > 0$ sufficiently small, we have exhibited a continuous paths in $W^{1,2}(N)$ connecting the stable critical point a_ε to the local energy minimiser g_ε. We also demonstrated that the energy along this path is bounded above by a value strictly below $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ (independently of ε), as depicted in Figure 2.1.

To complete the desired path between the stable critical points a_ε and b_ε it remains to construct the portion from the local energy minimiser g_ε to the stable critical point b_ε. By considering $-t_0$ instead of t_0 in each of the paths sketched above, we ensure that through symmetric (with respect to the underlying hypersurfaces) deformations of the super-level set $E(-t_0)$, the relevant symmetric portions of the path may constructed with identical upper energy bounds; this portion of the path is depicted in Figure 2.1, where the symmetry of the path with respect to g_ε is made apparent.

To conclude the proof of Theorem 2.3 we then concatenate the path from a_ε to g_ε and the path from g_ε to b_ε, completing the desired continuous path in $W^{1,2}(N)$ between the two stable critical points a_ε and b_ε. Under the assumption that (2.9) holds, the upper energy bounds along this path (as depicted by the solid curve in Figure 2.1) and (2.2) ensure that for $\varepsilon > 0$ sufficiently small we have that the

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_\lambda(E) - \min\left\{\frac{\eta}{4}, \frac{\tau}{2}\right\}.$$

As (2.1) holds, as mentioned in Subsection 2.1.2, and as the above path is admissible in the Allen–Cahn min-max construction of E, we must contradict the assumption that (2.9) holds. We thus conclude that (as (2.5) holds) any such E as produced by the Allen–Cahn min-max procedure in Ricci positive curvature must be such that E is locally $\mathcal{F}_\lambda$-minimising (in particular in $B_{\frac{R}{4}}(p)$ for the choice of $R > 0$ above), proving Theorem 2.3. We are then able to establish Theorems 2.1 and 2.2 by applying the surgery procedure described in Step 3.

3. **Surgery procedure:** As mentioned in Section 2.1, a local foliation around a hypersurface provides a natural way to perturb away an isolated singularity via a surgery procedure. Using the work of [HS85], in both [Sma93] and [CLS22], isolated singularities with regular tangent cone to locally area-minimising hypersurfaces are perturbed away by combining a "cut-and-paste" gluing along with a conformal change of metric.

We take a similar approach here. In [Les23] it is shown that near an isolated singularity with regular tangent cone of a locally $\mathcal{F}_\lambda$-minimising hypersurface of constant mean curvature λ there is a foliation around this

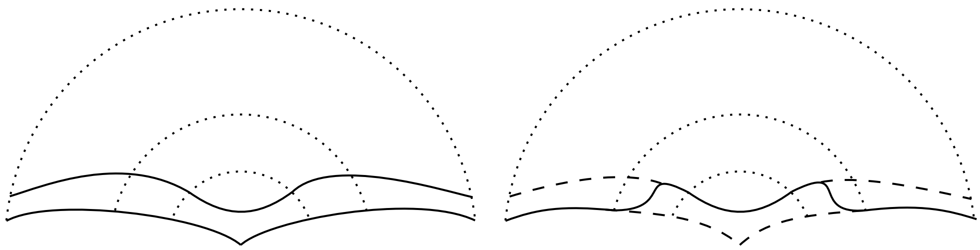

Figure 2.5: In both graphics the innermost two thinly dotted curves depict an annulus around an isolated singularity of a constant mean curvature hypersurface, and the outermost thinly dotted curves depict the boundary of the ball in which the foliations will be defined. In the left-hand graphic the lower solid curve depicts an isolated singularity with regular tangent cone of a constant mean curvature hypersurface and the upper solid curve depicts, under the assumption that the lower singular hypersurface is locally $\mathcal{F}_\lambda$-minimising, a smooth constant mean curvature hypersurface in the one-sided foliation provided by [Les23]. The solid curve in the right-hand graphic depicts the smooth hypersurface constructed by gluing both of the hypersurfaces in the left-hand graphic. This gluing is done in such a way that the hypersurface outside of the larger ball in the annulus agrees with the singular one and inside of the smaller ball in the annulus agrees with the hypersurface provided by the foliation; with the thick dashed lines depicting the pieces of the hypersurfaces in the left-hand graphic not included in the construction in the right. The resulting construction in the right-hand graphic is then, after a suitable metric perturbation, the desired smooth constant mean curvature hypersurface.

hypersurface (to either side) by smooth hypersurfaces of constant mean curvature λ. In Section 2.2, using this foliation, we establish a surgery procedure to perturb away isolated singularities with regular tangent cone of locally $\mathcal{F}_\lambda$-minimising constant mean curvature hypersurfaces. This is achieved by first constructing, via a “cut-and-paste” gluing, a smooth hypersurface close in Hausdorff distance to the original one inside a chosen ball (which may be taken arbitrarily small). This new smooth hypersurface has mean curvature equal to λ outside of an annulus in which the original hypersurface smooth; this construction is depicted and described in Figure 2.5.

It then remains to perturb the metric inside of this annulus so that the newly constructed hypersurface has constant mean curvature λ everywhere. This is achieved by an appropriate choice of function for conformal change of the original metric; with the resulting metric arbitrarily close, in the $C^{k,\alpha}$-norm, to the original. The result is then a smooth hypersurface of constant mean curvature λ with respect to the new metric, agreeing with the original hypersurface outside of the chosen ball.

2.1.4 Chapter structure and remarks

The remainder of chapter now proceeds as follows:

- Section 2.2 recalls the local foliation result of [Les23] and uses it to perturb away isolated singularities of constant mean curvature hypersurfaces with regular tangent cone. Here we also gather various remarks and provide an immediate application of the procedure to isoperimetric regions in dimension 8.
- Section 2.3 analyses the signed distance function and introduces the one-dimensional profile, before showing that it approximates the underlying hypersurface in a suitable sense. We also record some comments on the minimal ($\lambda = 0$) case of the approximation.
- Section 2.4 establishes a procedure to locally smooth Caccioppoli sets which are a priori assumed to be smooth in annular regions. This procedure is then used to relate local energy minimisation to the local geometric behaviour of the hypersurface by the construction of "recovery functions" for local $\mathcal{F}_\lambda$-minimisers around isolated singularities.
- Section 2.5 provides constructions of the various continuous paths in $W^{1,2}(N)$ along with calculations of upper bounds of the energy along these paths, as depicted in Figure 2.1.
- Section 2.6 ties together the results of the previous sections in order to prove Theorems 2.1, 2.2 and 2.3.
- Appendix 2.A concludes the chapter by providing alternative arguments in the minimal case (i.e. when $\lambda = 0$). In particular, we exhibit simplified computations of the upper energy bounds in Section 2.5 that provide a more direct route to establishing the main results in this case.

The reader is encouraged to keep in mind the following remarks throughout this chapter, which will be used implicitly in the sections that follow:

Remark 2.6. *As mentioned above, in [BW20b, Proposition B.1] it is shown that smooth constant mean curvature hypersurfaces are locally $\mathcal{F}_\lambda$-minimising. In proving Theorem 2.3 we aim to show that this also holds around isolated singularities for constant mean curvature hypersurfaces produced by the Allen–Cahn min-max in manifolds with positive Ricci* *curvature. We thus restrict to the dimensions in which these objects may be singular and therefore may work under the assumptions that N is of dimension $n+1 \geq 8$ and $\lambda \geq 0$ throughout the chapter.*

Remark 2.7. *The assumption of positive Ricci curvature is used only to ensure that the upper energy bounds on the paths constructed in Section 2.5 remain a fixed amount below the min-max value, $\mathcal{F}_\lambda(E)$. In particular, we note that the results of Sections 2.2, 2.3 and 2.4 as well as the paths constructed (but not their upper energy bounds) in Section 2.5 make no use of the assumption of positive Ricci curvature.*

Remark 2.8. *We make the choice of a positive upper bound on $\varepsilon > 0$ finitely many times throughout the construction of the paths in the proof of Theorem 2.3, ultimately constructing the paths for all $\varepsilon > 0$ smaller than a fixed positive constant. The specific choice of upper bound utilised in each instance may change, but we implicitly assume that a correct upper bound for which the desired property holds is used in each case. This remark will apply each time we choose $\varepsilon > 0$ sufficiently small.*

2.2 Surgery procedures

In this section we develop the surgery procedure described in Step 3 of Subsection 2.1.3, allowing for isolated singularities with regular tangent cone of constant mean curvature hypersurfaces that are locally $\mathcal{F}_\lambda$-minimising to be perturbed away. As an immediate application, we partially answer an open question of Lawson and obtain our first generic regularity result by smoothing boundaries of isoperimetric regions in dimension 8.

2.2.1 Perturbing isolated singularities

We show how the recent result of [Les23] can be combined with a local perturbation of the metric to regularise a hypersurface of constant mean curvature around an isolated singularity with area-minimising regular tangent cone. We first collect some notation and definitions, phrased in notation in keeping with this chapter, before stating the main theorem from [Les23] and using it to establish the surgery procedure.

We reset notation for this section, letting (N^{n+1}, g) be a Riemannian manifold with no curvature assumption. Throughout this section

$$T = (\partial[A]) \llcorner B_1(p)$$

will denote the n-current associated to a Caccioppoli set, $A \in \mathcal{C}(N)$, restricted to a ball $B_1(p)$ about a point $p \in N$, for which the following properties hold:

- $\mathrm{Spt}(T)$ is connected.
- $\mathrm{Sing}(T) = \{p\}$ (so that p is an isolated singularity of T).
- T has a regular tangent cone at p.

For a given $\lambda \in \mathbb{R}$ we fix a choice of $0 < r_1 < r_2 < 1$ sufficiently small so that the following properties hold:

- $\Gamma_0 = \partial(T \llcorner B_{r_1}(p))$ is a closed, embedded, connected, $(n-1)$-dimensional submanifold of $\partial B_{r_1}(p)$.
- $\mathrm{Spt}(T) \cap \partial B_{r_1}(p)$ is a transverse intersection.
- $\overline{A \cap B_{r_2}(p)}$ has connected complement in $B_{r_2}(p)$.

Let $\phi_j : \Gamma_0 \to \partial B_{r_1}(p)$ denote C^2 maps with

$$|\phi_j - i_{\Gamma_0}|_{C^2} \leq \frac{1}{j},$$

and for $\Gamma_j = (\phi_j)_* \Gamma_0$ we assume that $\Gamma_j \cap A^o \neq \emptyset$. Here we denote by i_{Γ_0} the identity map on Γ_0, $(\phi_j)_* \Gamma_0$ the push-forward of Γ_0 by the map ϕ_j and A^o the interior of A. We now recall a definition used in [Les23].

Definition 2.2. *For $\lambda \in \mathbb{R}$ we say that $T = (\partial[A]) \llcorner B_1(p)$ is one-sided minimising for $\mathcal{F}_\lambda$ in $B_1(p)$ if both of the following properties hold:*

- *A is a critical point of $\mathcal{F}_\lambda$ in $B_1(p)$.*
- *We have that*
$$\mathcal{F}_\lambda(A) \leq \mathcal{F}_\lambda(X).$$
for any $X \in \mathcal{C}(N)$ with $X \Delta A \subset \overline{A \cap B_1(p)}$ and $\partial(\partial[X] \llcorner B_1(p)) = \partial T$ (as currents).

Remark 2.9. *Note that this definition agrees with that of one-sided minimisation as introduced in [Les23], which involves an enclosed volume term, by interpreting the Caccioppoli sets as multiplicity one integral currents restricted to $B_1(p)$.*

Using the above notation and definition, [Les23] then proves the following local foliation result, generalising the results of [HS85] (in particular its one-sided extension due to [Liu19]) to the case of constant mean curvature hypersurfaces.

Theorem 2.4. *[Les23] Let $\lambda \in \mathbb{R}$, with T as defined above such that in addition T is one-sided minimising for $\mathcal{F}_\lambda$ in $B_1(p)$. Then, for every $j \geq 1$, there exist n-currents, S_j, that minimise $\mathcal{F}_\lambda$ in $\overline{A \cap B_{r_2}(p)}$, subject to the boundary condition $\partial S_j = \Gamma_j$, that satisfy the following properties:*

- $\mathrm{Spt}(S_j) \subset \overline{B_{r_1}(p)}$.
- *There exist sets of finite perimeter B_j, with $\overline{B_j} \subset \overline{A \cap B_{r_1}(p)}$, such that $S_j = \partial[B_j]\llcorner B_{r_1}(p)$.*
- $\Gamma_j = \mathrm{Spt}(S_j) \cap \partial B_{r_1}(p)$.
- *Each S_j is a critical point of $\mathcal{F}_\lambda$ in $B_{r_1}(p)$.*
- *For the measures associated to the supports of the S_j and T we have $\mu_{S_j} \to \mu_{T\llcorner B_{r_1}(p)}$. Thus, in particular on compact sets we have that $\mathrm{Spt}(S_j) \to \mathrm{Spt}(T)$ in the Hausdorff distance.*
- *The $\mathrm{Spt}(S_j)$ are smooth hypersurfaces.* □

Using Theorem 2.4 we now establish the desired surgery procedure, the proof of which is similar to [CLS22, Proposition 4.1].

Proposition 2.1. *Let (N^{n+1}, g) be a Riemannian manifold and, associated to $A \in \mathcal{C}(N)$, let $T = (\partial[A])\llcorner B_1(p)$ be an n-current with the both properties as stated above and satisfying the hypotheses of Theorem 2.4. Given $r \in (0, r_1)$, $k \geq 1$, $\alpha \in (0,1)$ and any $\varepsilon > 0$ there exists a current $\widetilde{T}$ and metric $\tilde{g}$ with the following properties:*

- $\mathrm{Sing}(\widetilde{T}) = \emptyset$.
- *$\widetilde{T}$ is a critical point of $\mathcal{F}_\lambda$ in $B_1(p)$ with respect to the metric $\tilde{g}$.*
- $\mathrm{Spt}(\widetilde{T}) \setminus B_r(p) = \mathrm{Spt}(T) \setminus B_r(p)$ *and* $\partial(\widetilde{T}\llcorner B_r(p)) = \partial(T\llcorner B_r(p))$.
- $d_{\mathcal{H}}(\mathrm{Spt}(T), \mathrm{Spt}(\widetilde{T})) < \varepsilon$, *where here $d_{\mathcal{H}}$ denotes the Hausdorff distance.*
- $||\tilde{g} - g||_{C^{k,\alpha}} < \varepsilon$ *with $g = \tilde{g}$ outside of $B_r(p)$.*

Proof. The case $\lambda = 0$ is precisely the content of [CLS22, Proposition 4.1]. We may thus consider the case $\lambda \in \mathbb{R} \setminus \{0\}$.

As $\mathrm{Sing}(T) = \{p\}$, for each $r \in (0, r_1)$ we have that

$$(\mathrm{Spt}(T) \cap B_r(p)) \setminus \{p\}$$

is smooth. We apply Theorem 2.4 to see that there exists some sequence, S_j, of smooth constant mean curvature hypersurfaces such that the S_j

converge as currents to $T \llcorner B_r(p)$. Allard's Theorem, (see [Sim84, Chapter 5] and [HS85, Lemma 1.14]) implies that we may write (for j sufficiently large) the intersection of the supports, $\mathrm{Spt}(S_j)$, with the annulus

$$A\left(p, \frac{r}{4}, \frac{3r}{4}\right) = B_{\frac{3r}{4}}(p) \setminus \overline{B_{\frac{r}{4}}(p)}$$

as a smooth graph over $\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4})$.

Explicitly, we let $u_j \in C^2(\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4}))$ denote the graphing function of $\mathrm{Spt}(S_j)$ over $\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4})$. Define φ to be a smooth cutoff function taking values in $[0,1]$ such that $\varphi = 1$ on $B_{\frac{3r}{8}}(p)$ and $\varphi = 0$ outside $B_{\frac{5r}{8}}(p)$. We then denote by $T + \varphi u_j$ the image of the normal graph of the function φu_j over $\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4})$.

We now define

$$\widetilde{T} = (T \setminus B_r(p)) \cup (S_j \cap B_{\frac{r}{4}}) \cup \left((T + \varphi u_j) \cap A\left(p, \frac{r}{4}, \frac{3r}{4}\right)\right)$$

for j large enough to ensure that $\mathrm{Spt}(S_j)$ is smooth and graphical as above. Notice that $\mathrm{Spt}(\widetilde{T})$ is smooth as $\mathrm{Spt}(S_j)$, $\mathrm{Spt}(T) \setminus B_r(p)$ and $(\mathrm{Spt}(T) + \varphi u_j) \cap A(p, \frac{r}{4}, \frac{3r}{4})$ are smooth; hence $\mathrm{Sing}(\widetilde{T}) = \emptyset$. By construction we ensure that

$$\begin{cases} \mathrm{Spt}(\widetilde{T}) \setminus B_r(p) = \mathrm{Spt}(T) \setminus B_r(p) \\ \partial(\widetilde{T} \llcorner B_r(p)) = \partial(T \llcorner B_r(p)) \end{cases}.$$

Note that for j sufficiently large we also ensure that we have

$$d_{\mathcal{H}}(\mathrm{Spt}(T), \mathrm{Spt}(\widetilde{T})) < \varepsilon$$

by the properties in the conclusion of Theorem 2.4.

Let $H_g(x)$ denote the mean curvature of the hypersurface $\widetilde{T}$ at a point $x \in \widetilde{T}$ with respect to the metric g. By construction, we note that H_g may not be equal to λ only on $A(p, \frac{r}{4}, \frac{3r}{4})$ (as both T and S_j are critical points of $\mathcal{F}_\lambda$). Moreover, as the graphing functions, u_j, smoothly converge to 0 (which in particular implies that $\widetilde{T}$ smoothly converges to T in $A(p, \frac{r}{4}, \frac{3r}{4})$), for each $x \in \widetilde{T}$ we have that

$$||H_g(x) - \lambda||_{C^{2,\alpha}} \to 0 \text{ as } j \to \infty, \tag{2.10}$$

Thus, we may choose j large enough to ensure that H_g and λ have the same sign (using the assumption that $\lambda \in \mathbb{R} \setminus \{0\}$), so that

$$\frac{H_g(x)}{\lambda} > 0 \tag{2.11}$$

for each $x \in \mathrm{Spt}(\widetilde{T})$. It remains to construct a new metric $\tilde{g}$, with $C^{k,\alpha}$ norm close to g, and show that $\widetilde{T}$ has constant mean curvature λ with respect to this new metric.

For some smooth function f on N to be determined we perform a conformal change of metric and set $\tilde{g} = e^{2f}g$. By standard results for conformal change of metric (e.g. see [Sak96, Chapter 2]) we then have that the mean curvature, $H_{\tilde{g}}$, of $\widetilde{T}$ with respect to the metric $\tilde{g}$ satisfies

$$H_{\tilde{g}}(x) = e^{-f}\left(H_g(x) + \frac{\partial f}{\partial \nu}\right)$$

at each point $x \in \widetilde{T}$, where here ν is the unit normal on $\widetilde{T}$ (agreeing with T outside of $A(p, \frac{r}{4}, \frac{3r}{4})$).

We define a smooth cutoff function, z, such that $z(x) \equiv 1$ if

$$\mathrm{dist}_N\left(y, \mathrm{Spt}(\widetilde{T}) \cap A\left(p, \frac{r}{4}, \frac{3r}{4}\right)\right) < \frac{r}{20}$$

and $z \equiv 0$ whenever

$$\mathrm{dist}_N\left(y, \mathrm{Spt}(\widetilde{T}) \cap A\left(p, \frac{r}{4}, \frac{3r}{4}\right)\right) > \frac{r}{10}.$$

We denote by $\Pi(y)$ the closest point projection of a point y to $\widetilde{T}$ in a tubular neighbourhood of

$$\mathrm{Spt}(\widetilde{T}) \cap A\left(p, \frac{r}{4}, \frac{3r}{4}\right),$$

so that $H_g(\Pi(y))$ is a well defined smooth function in this region. We now solve for $H_{\tilde{g}}(x) = \lambda$ for each $x \in \mathrm{Spt}(\widetilde{T})$ by setting

$$f(y) = \log\left(\frac{H_g(\Pi(y))}{\lambda}\right) z(y)$$

for each $y \in N$; this is well defined by (2.11) and the choice of tubular neighbourhood above. Note then by construction that the following properties hold

$$\begin{cases} \mathrm{Spt}(f) \subset A(p, \frac{r}{4}, \frac{3r}{4}) \\ \frac{\partial f}{\partial \nu} = 0 \text{ on } \mathrm{Spt}(\widetilde{T}) \cap A(p, \frac{r}{4}, \frac{3r}{4}) \end{cases},$$

where the second property follows as f is constant along normal geodesics to this region.

Thus we have that $H_{\tilde{g}}(\widetilde{T}) = \lambda$ for the choice of f as above. In particular, $\widetilde{T}$ is a critical point of $\mathcal{F}_\lambda$, with the conformal change in metric, $\tilde{g} = e^{2f}g$, occurring only in $A(p, \frac{r}{4}, \frac{3r}{4})$. Furthermore, by (2.10) we ensure that for each y in the tubular neighbourhood as chosen above we have that

$$||H_g(\Pi(y)) - \lambda||_{C^{2,\alpha}} \to 0 \text{ as } j \to \infty.$$

Therefore, for each $k \geq 1$ and $\alpha \in (0,1)$ we ensure by the smoothness of z that

$$||e^{2f} - 1||_{C^{k,\alpha}} = \left|\left| \left(\frac{H_g(\Pi(y))}{\lambda}\right)^{2z(y)} - 1 \right|\right|_{C^{k,\alpha}} \to 0 \text{ as } j \to \infty,$$

and so we guarantee that

$$||\tilde{g} - g||_{C^{k,\alpha}} \leq ||e^{2f} - 1||_{C^{k,\alpha}} ||g||_{C^{k,\alpha}} \to 0 \text{ as } j \to \infty.$$

Hence, by choosing j sufficiently large we ensure the metric change is smaller than $\varepsilon > 0$ in the $C^{k,\alpha}$ norm for the given values of $k \geq 1$ and $\alpha \in (0,1)$. □

2.2.2 Applications

We now record two effective applications of the surgery procedure:

1. **For Theorems 2.1 and 2.2:** A direct application of Proposition 2.1 is utilised in the proofs of Theorems 2.1 and 2.2 after establishing Theorem 2.3; this application is justified by the following remark.

Remark 2.10. *In the proof of Theorem 2.3 we will establish that the one-parameter Allen–Cahn min-max procedure of [BW20a] with constant prescribing function λ, in a compact Riemannian manifold of dimension 3 or higher with Ricci positive curvature, produces a closed embedded hypersurface, M, of constant mean curvature λ which is locally $\mathcal{F}_\lambda$-minimising around isolated singular points. For M as above, around isolated singularities with regular tangent cone, the results of [Sim83] imply that M is locally a graph over its unique tangent cone. Thus, M will satisfy all the required properties of T specified throughout this section in a sufficiently small ball around an isolated singular point with regular tangent cone. With this in hand, as any one-sided minimiser as in Definition 2.2 is required to be a Caccioppoli set (see [Liu19], [Les23] for more details), we may thus apply the results of Proposition 2.1 to M.*

2. **On an open question of Lawson:** In the second paragraph of [Bro86, Problem 5.7.], the following open question was proposed:

"Let Ω be a compact domain in a Riemannian manifold such that $\partial\Omega$ is of minimal area for the constrained volume. Can $\partial\Omega$ be approximated by a smooth hypersurface of positive mean curvature?" - B. Lawson

In the following discussion we show how Proposition 2.1 may be applied in order to answer this question affirmatively in the case that the boundary is assumed have non-zero mean curvature and contain only isolated singularities with regular tangent cone; this fully answers the above question in dimension 8 when $\partial\Omega$ is assumed to have non-zero mean curvature. In particular, the approximating hypersurfaces may be taken to have mean curvature with the same sign as the original boundary.

We consider a compact Riemannian manifold (N^{n+1}, g) with no curvature assumption, and re-normalise its volume (by re-scaling the metric g) so that $\text{Vol}_g(N) = 1$. We then fix a proportion of the volume, $\theta \in (0,1)$, and minimise boundary area amongst all Caccioppoli sets with enclosed volume equal to θ. In doing so we produce an *isoperimetric region*, $F \in \mathcal{C}(N)$, such that

$$\text{Per}_g(F) = \inf_{G \in \mathcal{C}(N)} \{\text{Per}_g(G) \mid \text{Vol}_g(G) = \theta\}.$$

The existence of such an isoperimetric region arising as a solution to the above variational problem is guaranteed by standard arguments (see e.g. [Mag12, Section 12.5]). It holds that boundaries of isoperimetric regions are constant mean curvature hypersurfaces which in particular are locally $\mathcal{F}_\lambda$-minimising (for further explanation, see the discussion at the beginning of [BW20b, Section 1.1]).

The regularity theory developed in [GMT83] (or the more general results in [BW20b, BW20c]) guarantee that the boundary, ∂F, of any isoperimetric region $F \in \mathcal{C}(N)$ is in fact a smoothly embedded constant mean curvature hypersurface away from a closed singular set of Hausdorff dimension at most $n-7$. In particular, when $n = 7$ the singular set consists of finitely many isolated singularities, each of which have regular tangent cone. Because the set of isolated singular points with regular tangent cone of the boundary must be discrete, but not necessarily closed when $n \geq 8$, it will suffice for repeated application of the construction in Proposition 2.1 to index the isolated singularities with regular tangent cone and make a small change to the underlying hypersurface around each point so that the Hausdorff distance of the resulting smooth construction is arbitrarily small.

We may then directly apply the “cut-and-paste” construction in the proof of Proposition 2.1 around each isolated singularity with regular tangent cone arising in the boundary of an isoperimetric region. This results in a constant mean curvature hypersurface (with constant equal to the mean curvature of ∂F) containing no isolated singularities with regular tangent cone that approximates the boundary of the isoperimetric region in Hausdorff distance. In particular, when $n = 7$ the approximating constant mean curvature hypersurfaces are entirely smooth. Furthermore, when ∂F is assumed to have non-zero constant mean curvature, the above smooth approximating hypersurfaces provide an affirmative answer to the open question above.

2.3 Signed distance and approximation

In this section we analyse the distance function to our hypersurface of constant mean curvature and use this in order to produce a suitable approximating function for the underlying hypersurface. By first establishing properties about the singular set of the distance function (points where it fails to be differentiable) we are able to control the behaviour of its level sets. This control is exploited in to construct a suitable (in the sense that (2.2) in Step 1 holds) approximating function, $v_\varepsilon \in W^{1,2}(N)$, for our underlying constant mean curvature hypersurface, around which the paths described in Step 2 of the proof strategy outlined Subsection 2.1.3 are ultimately constructed in Section 2.5.

2.3.1 Singular behaviour of the distance function

Recall that $M = \partial^* E \subset N$ is assumed to be a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set, denoted $\mathrm{Sing}(M) = \overline{M} \setminus M$, of Hausdorff dimension at most $n - 7$. We now adapt some of the analysis in [Bel23b, Section 3] to the setting of hypersurfaces of constant mean curvature.

Denoting by $S_{d_{\overline{M}}}$ the set of points in $N \setminus \overline{M}$ where the distance function $d_{\overline{M}}$ fails to be differentiable (precisely, the set of points $x \in N \setminus \overline{M}$ such that there exist two or more geodesics realising $d_{\overline{M}}(x)$), we then have that $d_{\overline{M}}$ is C^1 on $N \setminus (\overline{M} \cup \overline{S_{d_{\overline{M}}}})$ and that $S_{d_{\overline{M}}}$ is countably n-rectifiable, by the results of [Alb94]. We now show that $\overline{S_{d_{\overline{M}}}}$ is a countably n-rectifiable set; this fact will allow us to work solely with the smooth portions of the signed level sets of $d_{\overline{M}}$. To establish this we will need the following lemma; the proof of which is identical to that of [Bel23b, Lemma 3.1] (c.f. [Gro79],[Zho17]) interchanging the use of the sheeting theorem in [SS81] (or the more general version in [Wic14a]) for the one in [BW20c].

Lemma 2.1. *(Geodesic Touching) Let $x \in N \setminus \overline{M}$, then any minimising geodesic connecting x to $\overline{M}$ (i.e. a geodesic whose length realises $d_{\overline{M}}(x)$) with endpoint $y \in \overline{M}$ is such that $y \in M$ (i.e. is a regular point of M).*

Proof. Let γ be a geodesic from x to $\overline{M}$, realising $d_{\overline{M}}(x)$, with endpoint $y \in \overline{M}$. Fix z in the image of γ such that $\text{dist}_N(z, y) < inj(N)$. The geodesic ball, B, centred at z of radius $\text{dist}_N(z, y)$ is such that $B \cap \overline{M} = \emptyset$, else γ would not be minimising, and $y \in \partial B \cap \overline{M}$.

By the stationarity of M with respect to volume preserving deformations (in particular as a consequence of the monotonicity formula) there exist tangent cones to $\overline{M}$ at y. As $B \cap \overline{M} = \emptyset$, any tangent cone to $\overline{M}$ at y is supported in a half-space of $\mathbb{R}^{n+1}$ with boundary given by the tangent plane to B at y. By [Sim84, Chapter 7, Theorem 4.5/Remark 4.6] we have that any tangent cone to $\overline{M}$ is in fact the (possibly high multiplicity) tangent plane to B at y . The sheeting theorem provided by [BW20c, Theorem 5.1] then implies that $y \in M$. □

Let $F(y, t) = \exp_y(t\nu(y))$ where $y \in M$, $t \in \mathbb{R}$ and ν is the choice of unit normal to M pointing into E. We then have that $F(y, t)$ is a geodesic emanating from M orthogonally; here we interpret $F(y, t)$ for $t < 0$ by $\exp_y(t(\nu^-(y)))$ where ν^- is the unit normal to M pointing into $N \setminus E$. We define $\sigma^+(y), \sigma^-(y) \in \mathbb{R}$ with $\sigma^+(y) > 0$ and $\sigma^-(y) < 0$ chosen so that $F(y, t)$ is the minimising geodesic between its endpoint y, on M, and $F(y, t)$ for all $\sigma^-(y) \leq t \leq \sigma^+(y)$ but fails to be the minimising geodesic, between y and $F(y, t)$, for $t > \sigma^+(y)$ or $t < \sigma^-(y)$. With this definition we define the *cut locus* of M to be

$$\text{Cut}(M) = \{F(y, \sigma^\pm(y)) \, : \, y \in M, \sigma^\pm(y) < \infty\}. \tag{2.12}$$

Standard theory (see e.g. [Sak96, Chapter III]) for geodesics characterises the cut locus in the following manner: if $x = F(y, \sigma^\pm(y)) \in \text{Cut}(M)$ then either their exist (at least) two distinct geodesics realising $d_{\overline{M}}(x)$ or the map

$$F : M \times (0, \infty) \to N$$

is such that $dF(y, \sigma^\pm(y))$ is not invertible. We then see that

$$\overline{S_{d_{\overline{M}}}} \cap (N \setminus \overline{M}) = \text{Cut}(M)$$

(c.f. [MM02, Proposition 4.6]), and so in order to establish the countable n-rectifiability of $\overline{S_{d_{\overline{M}}}}$ it is sufficient (as $S_{d_{\overline{M}}}$ is countably n-rectifiable, as mentioned above) to show that $\mathrm{Cut}(M) \setminus S_{d_{\overline{M}}}$ is countably n-rectifiable in $N \setminus \overline{M}$; this fact is the analogue of [MM02, Proposition 4.9] in the case that the hypersurface M may contain singularities. Observe that we have $\overline{S_{d_{\overline{M}}}} \cap \overline{M} \subset \mathrm{Sing}(M)$ and, as noted above, $\mathrm{Sing}(M)$ has zero $\mathcal{H}^n$ measure, so this set does not affect the rectifiability.

The proof of [MM02, Proposition 4.9] may be adapted (exactly as in [Bel23b, Section 3]) to our setting by virtue of the fact that their arguments are local to points away from $\overline{M}$, and hence we may apply the arguments used in their proof to our situation without change. Thus we conclude that $\overline{S_{d_{\overline{M}}}}$ is countably n-rectifiable, and consequently its $\mathcal{H}^{n+1}$ measure is zero.

The level sets of $d_{\overline{M}}$ are smooth on $N \setminus (\overline{M} \cup \overline{S_{d_{\overline{M}}}})$ by virtue of the Implicit Function Theorem, invertibility of F and differentiability of $d_{\overline{M}}$ on this set. The arguments in the proof of [MM02, Proposition 4.6] show further that the map F restricts to a diffeomorphism,

$$F : \{(y,t) \,:\, y \in M, t \in (\sigma^-(y), \sigma^+(y))\} \to N \setminus (\overline{M} \cup \overline{S_{d_{\overline{M}}}}).$$

We then extend F to the set $M \times \{0\}$ by setting $F(y, 0) = y$ for each $y \in M$. The image of the extension of F is then $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$.

Finally, by defining the coordinates

$$V_M = \{(y,t) \mid y \in M, t \in \Big(\sigma^-(y), \sigma^+(y)\Big)\}, \tag{2.13}$$

we have that V_M is diffeomorphic to $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$.

Remark 2.11. *For each compact set, $K \subset M$, the continuity of the functions $\sigma^{\pm}(x)$ (which follows from [Sak96, Chapter III, Lemma 4.2]) on M implies that there exists some constant, $c_K 0$, such that*

$$0 < c_K < \min_{x \in K}\{\sigma^+(x), |\sigma^-(x)|\}.$$

For such a compact set K, as M itself is a two-sided hypersurface, there is a two-sided tubular neighbourhood of K when viewed as a subset of V_M (the coordinates defined in (2.13)), given by $K \times (-c_K, c_K)$ with its closure a subset of V_M. Furthermore, by the definition of c_K, the image under the map F of this two-sided tubular neighbourhood, $F(K \times (-c_K, c_K)) \subset N$, *is such that*

$$F(K \times (-c_K, c_K)) \cap (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}) = \emptyset.$$

Remark 2.12. *We are now able to define a projection to the hypersurface M on the set $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$. Precisely, for each $y \in N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$ there exists a unique geodesic in N with endpoint $x \in M$ realising $d_{\overline{M}}(y)$ (i.e. such that $d_{\overline{M}}(y) = d_N(x, y)$). We then denote by Π the smooth projection from a point in $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$ to its unique endpoint in M. Note that we may express this projection as*

$$\Pi(x) = F \circ \Pi_{V_M} \circ F^{-1}(x),$$

where Π_{V_M} is the smooth projection map to the first factor, defined in the coordinates V_M by sending the point $(x, s) \in V_M$ to $(x, 0) \in V_M$.

2.3.2 Level sets of the signed distance function

We now define the *signed distance* function on N, corresponding to our choice of unit normal, ν, to the hypersurface M pointing into E, by

$$d^{\pm}_{\overline{M}} = \begin{cases} +d_{\overline{M}}(x), \text{ if } x \in E \\ 0, \text{ if } x \in \overline{M} \\ -d_{\overline{M}}(x), \text{ if } x \in N \setminus E \end{cases};$$

so that a positive sign corresponds to our point lying in E. Denoting by $S_{d^{\pm}_{\overline{M}}}$ the set of points in $N \setminus \overline{M}$ where $d^{\pm}_{\overline{M}}$ fails to be differentiable, we then have that $\overline{S_{d^{\pm}_{\overline{M}}}} = \overline{S_{d_{\overline{M}}}}$ and thus, by the arguments in Subsection 2.3.1, $\overline{S_{d^{\pm}_{\overline{M}}}}$ is countably n-rectifiable with zero $\mathcal{H}^{n+1}$ measure.

We also denote the level sets, for each $s \in \mathbb{R}$, of the signed distance function, $d^{\pm}_{\overline{M}}$, by

$$\Gamma(s) = \{x \in N \mid d^{\pm}_{\overline{M}} = s\}.$$

These level sets are smooth in the open set $N \setminus (\overline{S_{d^{\pm}_{\overline{M}}}} \cup \overline{M})$ by the Implicit Function Theorem (as the signed distance is smooth and the exponential map is invertible on this open set). Note in particular that we have

$$\Gamma(s) = \emptyset \text{ for } |s| > d(N),$$

where $d(N)$ denotes the diameter of N.

We use the following notation to refer to the smooth parts of the level sets of $d^{\pm}_{\overline{M}}$, setting $\widetilde{\Gamma}(0) = M$, and for $s \neq 0$ we denote the smooth portion by

$$\widetilde{\Gamma}(s) = \Gamma(s) \setminus \overline{S_{d_M^\pm}}.$$

As $\overline{S_{d_M^\pm}}$ is countably n-rectifiable, and hence has vanishing $\mathcal{H}^{n+1}$ measure, we may apply the co-area formula (slicing with the function $d_M^\pm$, which is such that $|\nabla d_M^\pm| = 1$ a.e.) to conclude that

$$\text{for a.e. } s \in \mathbb{R} \text{ we have } \mathcal{H}^n\left(\Gamma(s) \cap \overline{S_{d_M^\pm}}\right) = 0. \tag{2.14}$$

Recall the diffeomorphism

$$F : V_M \to N \setminus (\mathrm{Sing}(M) \cup Cut(M))$$

defined by $F(y,t) = \exp_y(t\nu(y))$ and the coordinates, V_M, as defined in (2.13) on the set $N \setminus (\mathrm{Sing}(M) \cup Cut(M))$. We equip V_M with the pull-back metric via the map F, giving the usual induced metric for M on $M \times \{0\} \subset V_M$. Note that with these coordinates we have

$$F^{-1}(\widetilde{\Gamma}(s)) = M \times \{s\} \subset V_M.$$

We now work directly with the coordinates provided by V_M in order to establish various properties about the smooth portions of the level sets of $d_M^\pm$.

We first choose local coordinates, $(x_1, \ldots, x_n, s)$, on V_M so that the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ provide a local orthonormal frame around a given point $x_0 \in M$, and so that $\frac{\partial}{\partial s}$ is the unit speed of geodesics with constant base-point in M. The pullback metric (via the map F) then induces a volume form on V_M, and hence, at a point $(x_0, s_0) \in V_M$, an area element, $\theta(x,s)$, on the set $M \times \{s_0\} \subset V_M$. We trivially extend this area element to the entire set $M \times \mathbb{R}$ by setting $\theta(x,s) = 0$ for $(x,s) \in (M \times \mathbb{R}) \setminus V_M$.

From the structure of V_M and the map F we thus have for each $s \in \mathbb{R}$ that

$$\int_M \theta(x,s) dx_1 \cdots dx_n = \mathcal{H}^n(\widetilde{\Gamma}(s)). \tag{2.15}$$

In particular, recalling that the pull-back metric gives the induced metric for M on $M \times \{0\}$, we have that $\theta(x,0) = 1$ and thus

$$\int_M \theta(x,0) dx_1 \cdots dx_n = \mathcal{H}^n(M).$$

As the volume form is smooth on V_M, we have that the induced area element, $\theta(x,s)$, is continuous in both variables, in particular we have for any $x \in M$ that

$$\theta(x,s) \to \theta(x,0) \text{ as } s \to 0. \tag{2.16}$$

We may then compute derivatives of the area element and conclude, from [Gra04, Theorem 3.11], that

$$\frac{\partial}{\partial s}\log(\theta(x,s)) = -H(x,s), \tag{2.17}$$

which yields

$$\partial_s \theta(x,s) = -H(x,s)\theta(x,s). \tag{2.18}$$

In the above $H(x,s)$ denotes the scalar mean curvature, of the the pull-back of the level set $\Gamma(s)$ to V_M, at the point $(x,s) \in V_M$. Note that we thus have $H(x,0) = \lambda$ (as M is assumed to be a hypersurface of constant mean curvature λ). We also recall the Ricatti equation governing the evolution of the mean curvature along geodesics, from [Gra04, Corollary 3.6], in the following form

$$\frac{\partial}{\partial s}H(x,s) \geq \min_N \operatorname{Ric}_g. \tag{2.19}$$

Precisely, we set $\min_N \operatorname{Ric}_g = \min_{|X|=1} \operatorname{Ric}_g(X,X)$. Let us hereafter denote by $m = \min_N \operatorname{Ric}_g > 0$. By integrating the inequality (2.19) we then have that

$$\begin{cases} H(x,s) \geq \lambda + ms \text{ for } s > 0 \\ H(x,0) = \lambda \\ H(x,s) \leq \lambda + ms \text{ for } s < 0 \end{cases}. \tag{2.20}$$

We then combine (2.17) with (2.20) and apply the Fundamental Theorem of Calculus, in order to compute the following inequality for the area element which is valid for each $t \in \mathbb{R}$

$$\log(\theta(x,t)) \leq -\int_0^t (ms + \lambda)\, ds.$$

From this we directly conclude that

$$\theta(x,t) \leq e^{-t\left(\frac{mt}{2}+\lambda\right)}. \tag{2.21}$$

Noting that the quadratic $-t\left(\frac{mt}{2}+\lambda\right)$ is maximised for $t = -\frac{\lambda}{m}$, by plugging this into (2.21), we see that for each $s \in \mathbb{R}$

$$\theta(x,s) \leq e^{\frac{\lambda^2}{2m}}. \tag{2.22}$$

We may then apply the Dominated Convergence Theorem to see that, by (2.16) and (2.22), we have

$$\int_M \theta(x,s)dx_1\cdots dx_n \to \int_M \theta(x,0)dx_1\cdots dx_n.$$

In particular, by (2.15) and the fact that $\theta(x,0)=1$, we ensure that

$$\mathcal{H}^n(\widetilde{\Gamma}(s)) \to \mathcal{H}^n(M) \text{ as } s\to 0. \tag{2.23}$$

Finally, let us record for later use that (2.23) above implies that

$$\operatorname{ess\,inf}_{s\in[-2\varepsilon\Lambda_\varepsilon,2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\widetilde{\Gamma}(s)) \to \mathcal{H}^n(M) \text{ as } \varepsilon\to 0, \tag{2.24}$$

and

$$\operatorname{ess\,sup}_{s\in[-2\varepsilon\Lambda_\varepsilon,2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\widetilde{\Gamma}(s)) \to \mathcal{H}^n(M) \text{ as } \varepsilon\to 0. \tag{2.25}$$

2.3.3 Approximation with the one-dimensional profile

Denote by $\mathbb{H}$ the monotonically increasing solution to the following ODE

$$u''-W'(u)=0,$$

namely the one-dimensional Allen–Cahn equation, subject to the conditions that

$$\begin{cases}\mathbb{H}(0)=0\\ \lim_{s\to\pm\infty}\mathbb{H}(s)=\pm 1\end{cases}.$$

In particular, for the standard choice of potential, $W(u)=\frac{(1-u^2)^2}{4}$, we have explicitly that $\mathbb{H}(s)=\tanh(\frac{s}{\sqrt{2}})$. We note also that the re-scaled function $\mathbb{H}^\varepsilon(s)=\mathbb{H}(\frac{s}{\varepsilon})$ solves the ODE

$$\varepsilon u''-\frac{W'(u)}{\varepsilon}=0.$$

We now recall the construction of a smooth, increasing truncation of the one-dimensional heteroclinic solution to the re-scaled Allen–Cahn equation, $\overline{\mathbb{H}}^\varepsilon$, as used in [Bel23b] (this same identical truncation is also utilised in [WW19], [CM20] and [BW24]). For the construction of Allen–Cahn approximations, this truncation will be placed in the normal direction for the various underlying hypersurfaces sketched in Step 2 of the proof strategy outlined in Subsection 2.1.3.

The truncation we construct has the advantage that it will be constant, identically equal to either ±1, outside of an interval (depending on $\varepsilon > 0$) of the form $[-6\varepsilon|\log(\varepsilon)|, 6\varepsilon|\log(\varepsilon)|]$; thus outside of a tubular neighbourhood (of radius $6\varepsilon|\log(\varepsilon)|$) our approximating functions will contribute no energy (as $e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) = 0$ outside of the interval $[-6\varepsilon|\log(\varepsilon)|, 6\varepsilon\log(\varepsilon)]$).

We first perform the truncation for $\mathbb{H}$ and then re-scale to the ε level. First we let $\chi \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function such that the following properties hold

$$\begin{cases} \chi \equiv 1 \text{ on } (-1,1) \\ \chi \equiv 0 \text{ on } \mathbb{R} \setminus (-2,2) \\ \chi(-s) = \chi(s) \text{ for all } s \in \mathbb{R} \\ \chi'(s) \leq 0 \text{ for } s \geq 0 \end{cases}.$$

Denoting $\Lambda_\varepsilon = 3|\log(\varepsilon)|$ we then define the trunctation

$$\overline{\mathbb{H}}(s) = \begin{cases} \chi\left(\frac{s}{\Lambda_\varepsilon}\right)\mathbb{H}(s) + \left(1 - \chi\left(\frac{\Lambda_\varepsilon}{|s|}\right)\right) & \text{for } s > 0 \\ \chi\left(\frac{s}{\Lambda_\varepsilon}\right)\mathbb{H}(s) - \left(1 - \chi\left(\frac{\Lambda_\varepsilon}{|s|}\right)\right) & \text{for } s < 0 \end{cases}.$$

With this choice of truncation we then ensure that the following properties hold for the re-scaled truncation, $\overline{\mathbb{H}}^\varepsilon(s) = \overline{\mathbb{H}}(\frac{s}{\varepsilon})$,

$$\begin{cases} \overline{\mathbb{H}}^\varepsilon \equiv \mathbb{H}^\varepsilon \text{ on } (-\varepsilon\Lambda_\varepsilon, \varepsilon\Lambda_\varepsilon) \\ \overline{\mathbb{H}}^\varepsilon \equiv 1 \text{ on } (-\infty, -2\varepsilon\Lambda_\varepsilon] \\ \overline{\mathbb{H}}^\varepsilon \equiv 1 \text{ on } [2\varepsilon\Lambda_\varepsilon, \infty) \end{cases};$$

thus we have that $e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) = 0$ for $|s| \geq 2\varepsilon\Lambda_\varepsilon$, as desired.

Summarising the computations carried out in [Bel23b, Section 2.2], we then deduce that

$$\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \to 1 \text{ as } \varepsilon \to 0, \tag{2.26}$$

where the specific convergence is such that for fixed $\beta > 0$ and $\varepsilon > 0$ sufficiently small we have

$$1 - \beta\varepsilon^2 \leq \mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \leq 1 + \beta\varepsilon^2. \tag{2.27}$$

The above convergence properties for the energy of the re-scaled truncation guarantee that the functions we define from specific hypersurfaces for the path will have energy behaviour that is a diffuse reflection of their underlying geometry.

We may then finally proceed to define our one-dimensional profile by setting

$$v_\varepsilon(x) = \overline{\mathbb{H}}^\varepsilon(d^\pm_{\overline{M}}(x)).$$

As $\overline{\mathbb{H}}^\varepsilon$ is smooth and the distance function is Lipschitz, it follows that $v_\varepsilon \in W^{1,2}(N)$. We now prove that (2.2) holds, showing that v_ε acts as a suitable Allen–Cahn "approximation" of the hypersurface M in the sense that it recovers the $\varepsilon \to 0$ limit of the energies of the critical points obtained by the min-max in [BW20a]. We will exploit this fact directly in the construction of the paths in Section 2.5, eventually allowing us to establish Theorem 2.3.

By applying the co-area formula, slicing with $d^\pm_{\overline{M}}$ and noting again that $|\nabla d^\pm_{\overline{M}}| = 1$ a.e., we compute, similarly to [BW24, Section 3.6], that

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) &= \mathcal{E}_\varepsilon(v_\varepsilon) - \frac{\lambda}{2}\int_N v_\varepsilon \\
&= \frac{1}{2\sigma}\int_N \frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{W(v_\varepsilon)}{\varepsilon} - \frac{\lambda}{2}\int_N v_\varepsilon \\
&= \frac{1}{2\sigma}\int_{\mathbb{R}}\int_{\Gamma(t)} \frac{\varepsilon}{2}\left(\left(\overline{\mathbb{H}}^\varepsilon\right)'(t)\right)^2 + \frac{W(\overline{\mathbb{H}}^\varepsilon(t))}{\varepsilon} d\mathcal{H}^n dt \\
&\qquad - \frac{\lambda}{2}\int_{\mathbb{R}}\int_{\Gamma(t)} \overline{\mathbb{H}}^\varepsilon(t) d\mathcal{H}^n dt.
\end{aligned}$$

Using (2.14) and the definition of the energy density, e_ε, as in Subsection 2.1.1, we then have that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) = \frac{1}{2\sigma}\int_{\mathbb{R}} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(t))\mathcal{H}^n(\widetilde{\Gamma}(t))dt - \frac{\lambda}{2}\int_{\mathbb{R}} \overline{\mathbb{H}}^\varepsilon(t)\mathcal{H}^n(\widetilde{\Gamma}(t))dt.$$

From the properties of $\overline{\mathbb{H}}^\varepsilon$ as stated above we may obtain the following two bounds:

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \operatorname{ess\,inf}_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\widetilde{\Gamma}(s))\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) &- \frac{\lambda}{2}\int_{-2\varepsilon\Lambda_\varepsilon}^{\infty} \mathcal{H}^n(\widetilde{\Gamma}(t))dt \\
&+ \frac{\lambda}{2}\int_{-\infty}^{-2\varepsilon\Lambda_\varepsilon} \mathcal{H}^n(\widetilde{\Gamma}(t))dt,
\end{aligned}$$

and also

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \operatorname{ess\,sup}_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\widetilde{\Gamma}(s))\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) &- \frac{\lambda}{2}\int_{2\varepsilon\Lambda_\varepsilon}^{\infty} \mathcal{H}^n(\widetilde{\Gamma}(t))dt \\
&+ \frac{\lambda}{2}\int_{-\infty}^{2\varepsilon\Lambda_\varepsilon} \mathcal{H}^n(\widetilde{\Gamma}(t))dt.
\end{aligned}$$

Remark 2.13. *Whenever one is just considering minimal hypersurfaces, namely when $\lambda = 0$ and thus $\mathcal{F}_{\varepsilon,\lambda} = \mathcal{E}_\varepsilon$, one can now simply proceed to establish (2.2) (which in this case is simply the convergence of the Allen–Cahn energy of the one-dimensional profile, $\mathcal{E}_\varepsilon(v_\varepsilon)$, to the area of M,*

$\mathcal{H}^n(M)$) by utilising (2.24), (2.25) and (2.26) directly and ignoring the following volume bounds.

Further simplifications in this vein may be made for the energy calculations along the various paths sketched in Step 2 of the proof strategy in Subsection 2.1.3 and are explicitly outlined in Appendix 2.A. The main advantage in the minimal case is the lack of a volume term contribution in the energy which corresponds to the fact that the limiting functional one considers in this case is simply the area; hence does not record an enclosed volume term, unlike $\mathcal{F}_\lambda$.

In the general case however, we observe that

$$\begin{aligned}\mathrm{Vol}_g(E) = \mathcal{H}^{n+1}(\{x \in N \mid d^{\pm}_{\overline{M}} > 0\}) &= \int_0^\infty \mathcal{H}^n(\widetilde{\Gamma}(t))dt \\ &= \lim_{\varepsilon \to 0} \int_{\pm 2\varepsilon\Lambda_\varepsilon}^\infty \mathcal{H}^n(\widetilde{\Gamma}(t))dt,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Vol}_g(N \setminus E) = \mathcal{H}^{n+1}(\{x \in N \mid d^{\pm}_{\overline{M}} < 0\}) &= \int_{-\infty}^0 \mathcal{H}^n(\widetilde{\Gamma}(t))dt \\ &= \lim_{\varepsilon \to 0} \int_{-\infty}^{\pm 2\varepsilon\Lambda_\varepsilon} \mathcal{H}^n(\widetilde{\Gamma}(t))dt.\end{aligned}$$

Recalling (2.14) and combining these two identities with (2.24), (2.25) and (2.26) in the above two bounds on $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ we thus conclude that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \to \mathcal{H}^n(M) - \lambda \mathrm{Vol}_g(E) + \frac{\lambda}{2}\mathrm{Vol}_g(N) = \mathcal{F}_\lambda(E) \text{ as } \varepsilon \to 0,$$

establishing (2.2) as desired.

2.4 Relating local properties of the energy to the geometry

This section relates local behaviour of the energy of the one-dimensional profile, $v_\varepsilon = \overline{\mathbb{H}}^\varepsilon \circ d^{\pm}_{\overline{M}}$, to the local geometric properties of M. Recall that (as defined in Subsection 2.1.1) $M = \partial^* E \subset N$ is assumed to be a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set, denoted $\mathrm{Sing}(M) = \overline{M} \setminus M$, of Hausdorff dimension at most $n - 7$. Furthermore, M separates $N \setminus \overline{M}$ into open sets, E and $N \setminus E$, with common boundary given by $\overline{M}$ so that $E \in \mathcal{C}(N)$ with $M = \partial^* E$. For a given isolated singularity $p \in \overline{M}$ we fix throughout this section some $0 < r_1 < r_2 < \min\{R_p, R_l\}$ (the values of R_p and R_l were chosen in Subsection 2.1.1 based on the isolated singularity, p, and choice of metric, g on N respectively) such that $M \cap \overline{B_{r_2}(p)} \setminus B_{r_1}(p)$ is a smooth hypersurface.

2.4.1 Local smoothing of Caccioppoli sets

We first establish a procedure to locally perturb Caccioppoli sets, assumed to be smooth in an annular region, in order to ensure that they are smooth in the entire ball. This smoothing will be utilised for the construction of "recovery functions" that approximate local $\mathcal{F}_\lambda$-minimisers in Subsection 2.4.3. The following construction is technically involved but we emphasise that it boils down to a "cut-and-paste" gluing between a smooth level set of a mollified indicator function for a Caccioppoli set and its boundary.

Proposition 2.2. *Suppose that $F \in \mathcal{C}(N)$ is such that*

$$F \setminus B_{r_1}(p) = E \setminus B_{r_1}(p),$$

and fix some $\tilde{r}_2 \in (r_1, r_2)$. Then, for each $\delta > 0$, there exists $F_\delta \in \mathcal{C}(N)$ with the following properties:

- *∂F_δ is smooth in $B_{r_2}(p)$.*
- *$F_\delta \setminus B_{\tilde{r}_2}(p) = E \setminus B_{\tilde{r}_2}(p)$.*
- *$|\mathrm{Per}_g(F_\delta) - \mathrm{Per}_g(F)| \leq \delta$.*
- *$Vol_g(F_\delta \Delta F) \leq \delta$.*

In particular, we guarantee that F_δ agrees with F outside of $B_{\tilde{r}_2}(p)$ and is such that

$$|\mathcal{F}_\lambda(F_\delta) - \mathcal{F}_\lambda(F)| \leq (1 + \lambda)\delta.$$

Remark 2.14. *Though Proposition 2.2 is phrased in our setting to locally smooth a constant mean curvature hypersurface, the proof makes no use of the variational assumption on M. Thus, the same result holds for any Caccioppoli set which satisfies the same properties as M, without any condition on the mean curvature (i.e. simply for Caccioppoli sets that are only assumed to have smooth boundary in an annular region).*

Proof. Recall that, as $r_2 < R_l$, $B_{r_2}(p)$ is 2-bi-Lipschitz to the Euclidean ball $B_{r_2}^{\mathbb{R}^{n+1}}(0)$ via some a geodeisic normal coordinate chart, ϕ, with $\phi(p) = 0$ and such that

$$\frac{1}{2} \leq \sqrt{|g|} \leq 2 \text{ on } B_{r_2}(p). \tag{2.28}$$

We consider a radially symmetric mollifier $\rho \in C_c^\infty(B_1^{\mathbb{R}^{n+1}}(0))$ such that $\int_{\mathbb{R}^{n+1}} \rho = 1$ and re-scaling, for each $\theta > 0$, define $\rho_\theta(x) = \frac{1}{\theta^{n+1}}\rho(\frac{x}{\theta})$. Fixing $\tilde{r}_2 \in (r_1, r_2)$ we consider, for $\theta < r_2 - \tilde{r}_2$, the function

$$s_\theta = (\chi_{\phi(F)} * \rho_\theta) \circ \phi \in C^\infty(B_{\tilde{r}_2}(p)).$$

We now show that s_θ approximates the indicator function of the set F, χ_F, in the BV norm. First, by standard properties of mollifiers and application of (2.28) we have

$$\begin{aligned}||\chi_F - s_\theta||_{L^1(B_{\tilde{r}_2}(p)))} &= \int_{B_{\tilde{r}_2}(p)} |\chi_F - (\chi_{\phi(F)} * \rho_\theta) \circ \phi| d\mathcal{H}^n_g \\ &= \int_{\phi(B_{\tilde{r}_2}(p))} |\chi_{\phi(F)} - (\chi_{\phi(F)} * \rho_\theta)|\sqrt{|g|} d\mathcal{L}^n \\ &\leq 2||\chi_{\phi(F)} - (\chi_{\phi(F)} * \rho_\theta)||_{L^1(\phi(B_{\tilde{r}_2}(p)))} \to 0 \text{ as } \theta \to 0.\end{aligned}$$

Hence, lower semi-continuity of the perimeter yields

$$\mathrm{Per}_g(F) = |D_g\chi_F|(B_{\tilde{r}_2}(p)) \leq \liminf_{\theta\to 0} |D_g s_\theta|(B_{\tilde{r}_2}(p)). \tag{2.29}$$

For the reverse inequality we let $X \in \Gamma_c^1(TB_{\tilde{r}_2}(p))$ (a compactly supported C^1 vector field) with $|X|_g \leq 1$ and compute that

$$\begin{aligned}\int_{B_{\tilde{r}_2}(p)} s_\theta \mathrm{div}_g X d\mathcal{H}^n_g &= \int_{\phi(B_{\tilde{r}_2}(p))} (\chi_{\phi(F)} * \rho_\theta)\partial_i(\sqrt{|g|}\widehat{X}^i) d\mathcal{L}^n \\ &= \int_{\phi(B_{\tilde{r}_2}(p))} \chi_{\phi(F)}\partial_i(\rho_\theta * \sqrt{|g|}\widehat{X}^i) d\mathcal{L}^n \\ &= \int_{B_{\tilde{r}_2}(p)} \chi_F \mathrm{div}_g Y \leq |D_g\chi_F|(B_{\tilde{r}_2}(p)),\end{aligned}$$

where here $\widehat{X} = \widehat{X}^i\partial_i^\phi$ with $\widehat{X}^i = X^i \circ \phi^{-1}$, and $Y = Y^i\partial_i^\phi$ with

$$Y^i = \frac{1}{\sqrt{|g|}}\left(\rho_\theta * \sqrt{|g|}\widehat{X}^i\right) \circ \phi,$$

so that $|Y|_g \leq 1$. As the choice of X above was arbitrary in the above, we thus conclude, in combination with (2.29), that we have the desired convergence,

$$\lim_{\theta\to 0} |D_g s_\theta|(B_{\tilde{r}_2}(p)) = |D_g\chi_F|(B_{\tilde{r}_2}(p)).$$

Arguing identically as in the proof of [Mag12, Theorem 13.8], we conclude that the super-level sets of s_θ (which have smooth boundary for almost every $t \in (0,1)$ by Sard's Theorem), $L^t_\theta = \{s_\theta > t\}$, for a.e. $t \in (0,1)$, provide a sequence of open sets with smooth boundary such that both

$$\begin{cases} Vol_g(L^t_\theta \Delta F) \to 0 \text{ as } \theta \to 0 \\ Per(L^t_\theta; A) \to Per(F; A) \text{ as } \theta \to 0 \end{cases}, \tag{2.30}$$

for each open set $A \subset B_{\tilde{r}_2}(p)$, whenever $Per(F; \partial A) = 0$ (here we are using Δ to denote the symmetric difference of two sets).

As the exponential map is a radial isometry, we have that our geodesic normal coordinate chart, ϕ, is such that $B^{\mathbb{R}^{n+1}}_{r_2}(0) = \phi(B_{r_2}(p))$ and

$$S = \phi\left(M \cap (\overline{B}_{r_2}(p) \setminus B_{r_1}(p))\right)$$

is a smooth hypersurface in $B^{\mathbb{R}^{n+1}}_{r_2}(0) \setminus B^{\mathbb{R}^{n+1}}_{r_1}(0)$. In particular we may choose a pair of positive radii $r_1 < r_a < r_b < \tilde{r}_2$ so that, for some fixed $\delta > 0$, the set

$$T = \{x + b\nu_S(x) \mid x \in S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0)), b \in [-\delta, \delta]\},$$

is such that $T \subset\subset B^{\mathbb{R}^{n+1}}_{\tilde{r}_2}(0) \setminus B^{\mathbb{R}^{n+1}}_{r_1}(0)$. Here in the above we are denoting the unit normal induced on S from M by ν_S; note then that as ν is chosen to be the unit normal pointing into E and $F \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)$ we see that

$$F \cap \phi^{-1}(T) = \phi^{-1}(\{x + b\nu_S(x) \in T \mid b > 0\}). \tag{2.31}$$

We now consider normal derivatives of the function $\chi_{\widehat{F}} * \rho_\theta = s_\theta \circ \phi^{-1}$ for $\theta < \delta$ (where we denote $\widehat{F} = \phi(F)$) in the tubular neighbourhood T defined above. For $x + b\nu_S(x) \in T$ and $\theta > 0$ small enough so that

$$\nu_S(y) \cdot \nu_S(x) \geq \frac{1}{2} \text{ for any } y \in B_{2\theta}(x) \tag{2.32}$$

(S is smooth so ν_S varies continuously), we compute that for $b \in (-\theta, \theta)$ we have

$$\begin{aligned} \nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x) &= (\nabla\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_s(x)) \cdot \nu_S(x) \\ &= -((\mathcal{H}^n \llcorner S)\nu_S * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x) \\ &= -\left(\int_{S \cap B_\theta(x + b\nu_S(x))} \rho_\theta(x + b\nu_S(x) - y)\nu_S(y) \cdot \nu_S(x) d\mathcal{H}^n(y)\right) \\ &\leq -\frac{1}{2}\left(\int_{S \cap B_\theta(x + b\nu_S(x))} \rho_\theta(x + b\nu_S(x) - y) d\mathcal{H}^n(y)\right) < 0. \end{aligned}$$

Here we've used (2.32), that $D_g\chi_{\widehat{F}} = -(\mathcal{H}^n \llcorner S)\nu_S$ in T as S is smooth, and by the triangle inequality we have both $B_\theta(x + b\nu_S(x)) \subset B_{2\theta}(x)$ for

$b < \theta$ and that the final strict inequality holds as

$$S \cap B_{\theta-b}(x) \subset S \cap B_\theta(x + b\nu_S(x)).$$

Thus we have that both

$$\begin{cases} \nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_s(x)) \cdot \nu_S(x) < 0 \text{ for } b \in (-\theta, \theta) \\ \nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) = 0 \text{ for } b \notin (-\theta, \theta) \end{cases},$$

where in the second case we then have that $\chi_{\widehat{F}} * \rho_\theta \in \{0, 1\}$. We thus conclude that for each $t \in (0, 1)$ and $x \in S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ there exists a unique $b^t_\theta(x) \in (-\theta, \theta)$ such that

$$(\chi_{\widehat{F}} * \rho_\theta)(x + b^t_\theta(x)\nu_S(x)) = t,$$

which are such that $b^t_\theta \to 0$ point-wise on $S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ as $\theta \to 0$. Note also that by the structure of T we have

$$L^t_\theta \cap \phi^{-1}(T) = \phi^{-1}(\{x + b\nu_S(x) \in T \mid b < b^t_\theta(x)\}). \tag{2.33}$$

Consider now the function

$$h_\theta : S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0)) \times (-\theta, \theta) \to T$$

defined by setting

$$h^t_\theta(x, b) = (\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) - t.$$

By the above calculation we ensure that

$$\frac{\partial h_\theta}{\partial b}(x + b\nu_S(x)) = \nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x) < 0$$

and so by the Implicit Function Theorem (working in charts in which $S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ is locally a graph over its tangent plane) we have that the function b^t_θ is smooth on $S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ (noting that $h^t_\theta(x, b^t_\theta(x)) = 0$ for x in this set).

We now show that the directional derivatives of the functions b^t_θ converge to zero as $\theta \to 0$. At a point $x \in S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$, denoting directional derivatives by $\frac{\partial}{\partial x_i}$, we compute that by (2.32) we have

$$\left| \frac{\left(\frac{\partial h^t_\theta}{\partial x_i}\right)}{\left(\frac{\partial h^t_\theta}{\partial b}\right)} \right| = \left| \frac{\nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \frac{\partial}{\partial x_i}(x + b\nu_S(x))}{\nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x))} \right|$$

$$= \left| \frac{\int_{S\cap B_\theta(x+b\nu_S(x))} \rho_\theta(x+b\nu_S(x)-y)\nu_S(y)\cdot \frac{\partial}{\partial x_i}(x+b\nu_S(x))d\mathcal{H}^n(y)}{\int_{S\cap B_\theta(x+b\nu_S(x))} \rho_\theta(x+b\nu_S(x)-y)\nu_S(y)\cdot \nu_S(x)d\mathcal{H}^n(y)} \right|$$
$$\leq 2 \sup_{y\in S\cap B_\theta(x+b\nu_S(x))} \left| \nu_S(y)\cdot \frac{\partial}{\partial x_i}(x+b\nu_S(x)) \right| \to 0 \text{ as } \theta\to 0.$$

The last term in the above converges to zero as $\theta \to 0$ by using the triangle inequality, noting that $\nu_S(y)\cdot \frac{\partial}{\partial x_i} x \to 0$ by the continuity of ν_S and $\nu_S(y)\cdot b\frac{\partial \nu_S(x)}{\partial x_i} \leq C\theta$ for a constant C depending only on the choice of unit normal, ν_S, to S. We then have from the Implicit Function Theorem that

$$\left|\frac{\partial b^t_\theta}{\partial x_i}\right|(x) = \left| \frac{\left(\frac{\partial h^t_\theta}{\partial x_i}\right)}{\left(\frac{\partial h^t_\theta}{\partial b}\right)} \right| (x, b^t_\theta(x)) \to 0 \text{ as } \theta \to 0.$$

We conclude that the function b^t_θ defined on $S\cap(\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)}\setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ for $t\in(0,1)$ as above is smooth and in fact converges to zero uniformly in the C^1 norm as $\theta\to 0$. This convergence will aid us in ensuring perimeter control for the following constructions of the local smoothings.

We fix a further choice of $\tilde{r}_a \in (r_a, r_b)$ so that for $\theta < \tilde{r}_a - r_a$ we ensure (by the triangle inequality) that $x+b^t_\theta(x)\nu_S(x) \in B_{\tilde{r}_a}(p)$ whenever $x\in\partial B_{r_a}(p)$ (i.e. when $|x| = r_a$). Then we fix a further $\tilde{r}_b \in (\tilde{r}_a, r_b)$ so that for $\theta < r_b - \tilde{r}_b$ we ensure (again by the triangle inequality) that $x + b^t_\theta(x)\nu_S(x) \in N\setminus B_{\tilde{r}_b}(p)$ whenever $x\in\partial B_{r_b}(p)$ (i.e. when $|x| = r_b$).

We then choose a smooth radially symmetric cutoff function, $\eta \in C^\infty_c(\mathbb{R}^{n+1})$, with the following properties

$$\begin{cases} 0\leq \eta\leq 1 \text{ on } \mathbb{R}^{n+1} \\ |\nabla\eta| \leq C_\eta \text{ on } \mathbb{R}^{n+1} \\ \eta(x) = 1 \text{ if } |x|\leq \tilde{r}_a \\ \eta(x) = 0 \text{ if } |x|\geq \tilde{r}_b \end{cases},$$

for some fixed $C_\eta = C_\eta(\tilde{r}_a, \tilde{r}_b) > 0$.

Before proceeding to define the local smoothing we let

$$H^t_\theta = \phi^{-1}\left(\{x + b\nu_S(x)\in T \mid b < \eta(x)b^t_\theta(x)\}\right),$$

and, slightly abusing notation, we denote

$$\partial H^t_\theta = \phi^{-1}\left(\left\{x+\eta(x)b^t_\theta(x)\nu_S(x) \mid x\in S\cap(\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)}\setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))\right\}\right),$$

where the notation for boundary is justified by virtue of the fact that, as

M separates the region T, the "graph", given by

$$\phi^{-1}(x + \eta(x) b_\theta^t(x) \nu_S(x)),$$

also separates the region $\phi^{-1}(T)$.

We then consider the following hypersurface

$$\partial G_\theta^t = (M \setminus B_{\tilde{r}_b}(p)) \cup (\partial L_\theta^t \cap B_{\tilde{r}_a}(p)) \cup \Big(\partial H_\theta^t \cap (B_{\tilde{r}_b}(p) \setminus B_{\tilde{r}_a}(p))\Big)$$

which we ensure is smooth inside of $B_{r_b}(p)$ as $H_\theta^t \cap \partial B_{\tilde{r}_a}(p) \in \partial L_\theta^t$ and $H_\theta^t \cap \partial B_{\tilde{r}_b}(p) \in M$. We note also that as $\overline{M}$ satisfies the geodesic touching property from Lemma 2.1, by construction the boundary ∂G_θ^t also satisfies the conclusions of Lemma 2.1.

Furthermore, by (2.31) and (2.33), ∂G_θ^t arises as the boundary of the open set

$$G_\theta^t = (E \setminus ((B_{\tilde{r}_b}(p) \cup \phi^{-1}(T))) \cup (L_\theta^t \cap B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T)) \cup (H_\theta^t \cap \phi^{-1}(T)),$$

and hence

$$G_\theta^t \setminus B_{\tilde{r}_2}(p) = E \setminus B_{\tilde{r}_2}(p)$$

as $\phi^{-1}(T) \subset\subset B_{\tilde{r}_2}(p) \setminus B_{r_1}(p)$.

We may explicitly write the set $(L_\theta^t \Delta E) \cap \phi^{-1}(T)$ as

$$\phi^{-1}\left(\left\{ x + b\nu_S(x) \in T \;\middle|\; \begin{cases} 0 < b < b_\theta^t(x), \text{ if } b_\theta^t(x) > 0 \\ b_\theta^t < b < 0, \text{ if } b_\theta^t(x) < 0 \end{cases} \right\}\right).$$

As $0 \leq \eta \leq 1$, we know that $\eta(x) b_\theta^t(x)$ has the same sign as b_θ^t and by the choice of η that $|\eta(x) b_\theta^t(x)| \leq |b_\theta^t(x)|$. Therefore, by construction of H_θ^t, we have that

$$(G_\theta^t \Delta E) \cap \phi^{-1}(T) \subset (L_\theta^t \Delta E) \cap \phi^{-1}(T).$$

As the set G_θ^t agrees with either E or L_θ^t outside of $\phi^{-1}(T)$ we conclude that $G_\theta^t \Delta F \subset L_\theta^t \Delta F$ and hence by (2.30) we have

$$Vol_g(G_\theta^t \Delta F) \leq Vol_g(L_\theta^t \Delta F) \to 0 \text{ as } \theta \to 0.$$

Thus we may choose $\theta > 0$ sufficiently small to ensure that

$$Vol_g(G_\theta^t \Delta F) \leq \delta. \tag{2.34}$$

We note that we may always ensure that the four radii $r_a, \tilde{r}_a, \tilde{r}_b$ and r_b as chosen above are done so to ensure that

$$\mathrm{Per}_g(F; \partial(B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T))) = 0,$$

(this may be done without loss of generality by the assumption that F has finite perimeter) so that we guarantee

$$\mathrm{Per}_g(L^t_\theta; B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T)) \to \mathrm{Per}_g(F; B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T)) \text{ as } \theta \to 0.$$

In particular, for $\theta > 0$ sufficiently small we ensure that

$$|\mathrm{Per}_g(L^t_\theta; B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T)) - \mathrm{Per}_g(F; B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T))| \leq \frac{\delta}{2}.$$

As we have $G^t_\theta = E$ outside of $B_{\tilde{r}_b} \cup \phi^{-1}(T)$ we also have that

$$\mathrm{Per}_g(G^t_\theta; N \setminus (B_{\tilde{r}_b} \cup \phi^{-1}(T))) = \mathrm{Per}_g(E; N \setminus (B_{\tilde{r}_b} \cup \phi^{-1}(T))).$$

Note that as $|\nabla \eta| < C_\eta$ and $b^t_\theta \to 0$ uniformly in the C^1 norm we may choose $\theta > 0$ potentially smaller to ensure that

$$\sup_{y \in M \cap (B_{r_b}(p) \setminus B_{r_a}(p))} |J_{\phi^{-1} \circ (Id + (\eta b^t_\theta)) \circ \phi}(y) - 1| \leq \frac{\delta}{2\mathcal{H}^n(M \cap (B_{r_b}(p) \setminus B_{r_a}(p)))},$$

where here J_f denotes the Jacobian of a function f (which depends only on the C^1 norm of the function f), and so by the area formula we have

$$\begin{aligned} &|\mathcal{H}^n(\partial H^t_\theta \cap \phi^{-1}(T)) - \mathcal{H}^n(M \cap \phi^{-1}(T))| \\ &= \left| \int_{M \cap (B_{r_b}(p) \setminus B_{r_a}(p))} (J_{\phi^{-1} \circ (Id + (\eta b^t_\theta)) \circ \phi}(y) - 1) d\mathcal{H}^n(y) \right| \leq \frac{\delta}{2}. \end{aligned}$$

We may combine all the above facts to compute that, for $\theta > 0$ sufficiently small, we have

$$\begin{aligned} |\mathrm{Per}_g(G^t_\theta) - \mathrm{Per}_g(F)| &= |\mathrm{Per}_g(G^t_\theta; \phi^{-1}(T)) - \mathrm{Per}_g(F; \phi^{-1}(T)) \\ &\qquad + \mathrm{Per}_g(G^t_\theta; N \setminus \phi^{-1}(T)) - \mathrm{Per}_g(F; N \setminus \phi^{-1}(T))| \\ &\leq |\mathcal{H}^n(\partial H^t_\theta \cap \phi^{-1}(T)) - \mathcal{H}^n(M \cap \phi^{-1}(T))| \\ &\quad + |\mathrm{Per}_g(L^t_\theta; B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T)) - \mathrm{Per}_g(F; B_{\tilde{r}_b}(p) \setminus \phi^{-1}(T))| \leq \delta \end{aligned}$$

and thus guaranteeing, in combination with (2.34), that

$$|\mathcal{F}_\lambda(G^t_\theta) - \mathcal{F}_\lambda(F)| \leq (1 + \lambda)\delta.$$

Finally, by setting $F_\delta = G^t_\theta$ for one choice of almost any $t \in (0,1)$ and a choice of $\theta > 0$ sufficiently small as above then provides a Caccioppoli satisfying the desired conclusions. □

We conclude this subsection with a remark that will prove useful for controlling the energy of the "recovery functions" we construct to approximate a given local $\mathcal{F}_\lambda$-minimiser in Subsection 2.4.3.

Remark 2.15. *As ∂F_δ is smooth in $B_{r_2}(p)$ and agrees with M outside of $B_{\tilde{r}_2}(p)$, we have that ∂F_δ satisfies the conclusions of Lemma 2.1. More specifically, for any $x \in N \setminus \partial F_\delta$, any minimising geodesic connecting x to ∂F_δ has endpoint in a regular point of ∂F_δ. This fact will be exploited for the construction of the "recovery functions" in Lemma 2.3.*

2.4.2 Local energy minimisation

We now prove the existence of local energy minimisers that agree with our one-dimensional profile, $v_\varepsilon = d^{\pm}_{\overline{M}} \circ \overline{\mathbb{H}}^\varepsilon$, outside of a fixed ball in N. Minimisers of such problems will be used in Subsection 2.4.3 in order to conclude local geometric properties of the underlying constant mean curvature hypersurface, M, under appropriate assumptions on the behaviour of the energy of v_ε as $\varepsilon \to 0$. It is interesting to note that the boundary condition, imposed by v_ε, in the following minimisation problem depends on the approximating parameter $\varepsilon > 0$; our explicit control (namely (2.2)) of the function, v_ε, providing the boundary condition in the limit as $\varepsilon \to 0$ ensures that the minimisation problem reflects the underlying geometric behaviour of M.

Lemma 2.2. *Let $B_\rho(q) \subset N$ be a ball, of radius $\rho > 0$, centred at a point $q \in \overline{M}$ and define, for each $\varepsilon \in (0,1)$, the class of functions*

$$\mathcal{A}_{\varepsilon,\rho}(q) = \{u \in W^{1,2}(N) \mid |u| \leq 1, u = v_\varepsilon \text{ on } N \setminus B_\rho(q)\}.$$

Then, there exists $g_\varepsilon \in \mathcal{A}_{\varepsilon,\rho}(q)$ such that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_{\varepsilon,\rho}(q)} \mathcal{F}_{\varepsilon,\lambda}(u).$$

Proof. Let $I_\varepsilon = \inf_{u\in\mathcal{A}_{\varepsilon,\rho}(q)} \mathcal{F}_{\varepsilon,\lambda}(u)$ and consider $\{u_k\}_{k=1}^\infty \in \mathcal{A}_{\varepsilon,\rho}(q)$ such that $\mathcal{F}_{\varepsilon,\lambda}(u_k) \to I_\varepsilon$ as $k \to \infty$. Note that for any $u \in \mathcal{A}_{\varepsilon,\rho}(q)$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(u) = \frac{1}{2\sigma}\int_N \varepsilon\frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} - \frac{\lambda}{2}\int_N u \geq \frac{1}{2\sigma}\int_N \varepsilon\frac{|\nabla u|^2}{2} - \frac{\lambda}{2}\mathrm{Vol}_g(N),$$

using the fact that $|u| \leq 1$ $\mathcal{H}^{n+1}$-a.e. and $W \geq 0$. Hence, for k sufficiently large we have that

$$\int_N |\nabla u_k|^2 \leq \frac{4\sigma}{\varepsilon}\left(I_\varepsilon + 1 + \frac{\lambda}{2}\mathrm{Vol}_g(N)\right),$$

so $\sup_k ||\nabla u_k||_{L^2(N)} < \infty$; note here that our bound is in fact dependent on the fixed choice of ε. Using the triangle and Poincaré inequality (as $u_k = v_\varepsilon$ outside $B_\rho(q)$ so $u_k - v_\varepsilon \in W_0^{1,2}(B_\rho(q))$ for each $k \in \mathbb{N}$) we have

$$\begin{aligned}||u_k||_{L^2(N)} &= ||u_k - v_\varepsilon + v_\varepsilon||_{L^2(N)} \leq ||u_k - v_\varepsilon||_{L^2(N)} + ||v_\varepsilon||_{L^2(N)} \\ &\leq C_P||\nabla u_k - \nabla v_\varepsilon||_{L^2(N)} + ||v_\varepsilon||_{L^2(N)} \\ &\leq C_P(\sup_k ||\nabla u_k||_{L^2(N)} + ||\nabla v_\varepsilon||_{L^2(N)}) + ||v_\varepsilon||_{L^2(N)},\end{aligned}$$

(where C_P is the Poincaré constant for N) so $\sup_k ||u_k||_{L^2(N)} < \infty$. We thus ensure that $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $W^{1,2}(N)$ and hence, by Rellich-Kondrachov compactness, there exists $u \in W^{1,2}(N)$ and a sub-sequence of the $\{u_k\}_{k=1}^\infty$ (which we have not relabelled) such that u_k converges to u, weakly in $W^{1,2}(N)$, strongly in $L^2(N)$ (hence strongly in $L^1(N)$) and $\mathcal{H}^{n+1}$-*a.e.* in N. We then have that $|u| \leq 1$ and $u = v_\varepsilon$ on $N \setminus B_\rho(q)$ at $\mathcal{H}^{n+1}$-*a.e.* point, thus $u \in \mathcal{A}_{\varepsilon,\rho}(q)$. Combining the weak convergence of the u_k in $W^{1,2}(N)$, almost everywhere convergence of the u_k with Fatou's lemma (using the continuity of W and the fact that $W \geq 0$) and the strong $L^1(N)$ convergence of the u_k, we conclude that $\mathcal{F}_{\varepsilon,\lambda}(u) = I_\varepsilon$.

The above arguments show that, for all $\varepsilon \in (0,1)$, the energy minimisation problem in the class $\mathcal{A}_{\varepsilon,\rho}(q)$ is well posed. Hence, for each $\varepsilon \in (0,1)$ we produce a function, $g_\varepsilon \in W^{1,2}(N)$, with the following properties

$$\begin{cases} |g_\varepsilon| \leq 1 \text{ on } N \\ g_\varepsilon = v_\varepsilon \text{ on } N \setminus B_\rho(q) \\ \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_\varepsilon} \mathcal{F}_{\varepsilon,\lambda}(u) \end{cases}.$$

Thus we conclude that $g_\varepsilon \in \mathcal{A}_{\varepsilon,\rho}(q)$ and solves the energy minimisation as desired. In particular, we note that $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$. □

2.4.3 Recovery functions for local geometric properties

Using the local smoothing developed in Subsection 2.4.1, our approximation from Subsection 2.3.3 and the function side minimisation from Subsection 2.4.2, we now establish the following "recovery function" type

Lemma relating the energy of v_ε to local geometric properties of constant mean curvature hypersurfaces; in particular producing an approximating function for local $\mathcal{F}_\lambda$-minimisers. This lemma will be utilised in the proof of Theorem 2.3 in order to exploit a relation, namely (2.5) as sketched in Subsection 2.1.3, between the $\varepsilon \to 0$ energy behaviour of v_ε to the local $\mathcal{F}_\lambda$-minimisation of M.

Recall, from the beginning of this section, that for an isolated singularity $p \in \overline{M}$ we have fixed $0 < r_1 < r_2 < \min\{R_p, R_l\}$ such that $M \cap \overline{B_{r_2}(p)} \setminus B_{r_1}(p)$ is a smooth hypersurface.

Lemma 2.3. *Let $F \in \mathcal{C}(N)$ be such that $F \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)$ and*

$$\mathcal{F}_\lambda(F) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(G) \mid G \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)\}.$$

Then, for $\varepsilon > 0$ sufficiently small, there exist functions $f_\varepsilon \in \mathcal{A}_{\varepsilon, r_2}(p)$, so that $|f_\varepsilon| \leq 1$ and $f_\varepsilon = v_\varepsilon$ on $N \setminus B_{r_2}(p)$, such that

$$\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F).$$

Furthermore, if the functions v_ε are such that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \text{ for some sequence } \tau_\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

where the $g_\varepsilon \in \mathcal{A}_{\varepsilon, r_2}(p)$ are defined as in Lemma 2.2 for $B_{r_2}(p)$, then E solves the above minimisation problem. Namely, we have that

$$\mathcal{F}_\lambda(E) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(G) \mid G \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)\}.$$

Remark 2.16. *The existence of solutions to the variational problem given in Lemma 2.3 is guaranteed by standard arguments involving the direct method (see e.g. [Mag12, Section 12.5]).*

Proof. Let $F \in \mathcal{C}(N)$ solve the minimisation problem as in the assumptions of the lemma and consider, for each $\varepsilon, \delta > 0$, the functions

$$f_{\varepsilon,\delta} = \overline{\mathbb{H}}^\varepsilon \circ d^{\pm}_{\partial F_\delta}.$$

Here the set $F_\delta \in \mathcal{C}(N)$ is a local smoothing of F resulting from an application of Proposition 2.2 with the above choice of δ, and we define the signed distance function to ∂F_δ by

$$d^{\pm}_{\partial F_\delta} = \begin{cases} +d_{\partial F_\delta}(x), \text{ if } x \in F_\delta \\ 0, \text{ if } x \in \partial F_\delta \\ -d_{\partial F_\delta}(x), \text{ if } x \in N \setminus F_\delta \end{cases},$$

where here $d_{\partial F_\delta}$ is the usual distance function to the set ∂F_δ. Note that the functions $f_{\varepsilon,\delta} \in \mathcal{A}_{\varepsilon,r_2}(p)$ for $\varepsilon > 0$ small enough so that $2\varepsilon\Lambda_\varepsilon < r_2 - \tilde{r}_2$, where $\tilde{r}_2$ is chosen as in Proposition 2.2 (for a concrete choice one can take $\tilde{r}_2 = r_1 + \frac{r_2 - r_1}{2}$). To see this we note that as $F_\delta = E$ outside of $B_{\tilde{r}_2}(p)$ by its construction, we have that $d_{\partial F_\delta} = d_{\overline{M}}$ on the set

$$(N \setminus B_{r_2}(p)) \cap \{d_{\partial F_\delta} \leq 2\varepsilon\Lambda_\varepsilon\}$$

and hence $f_{\varepsilon,\delta} = v_\varepsilon$ on $N \setminus B_{r_2}(p)$ whenever ε is chosen appropriately small as above.

By Remark 2.15, for each δ chosen as above the hypersurfaces given by ∂F_δ satisfy the conclusions of Lemma 2.1. We may thus repeat the analysis for each ∂F_δ (of the cut locus, level sets etc.) as we did for M in Subsection 2.3.3 in order to conclude that

$$\mathcal{F}_{\varepsilon,\lambda}(f_{\varepsilon,\delta}) \to \mathcal{F}_\lambda(F_\delta) \text{ as } \varepsilon \to 0.$$

By the construction of the F_δ in Proposition 2.2 we ensure that

$$|\mathcal{F}_\lambda(F_\delta) - \mathcal{F}_\lambda(F)| \leq (1+\lambda)\delta$$

and thus by setting $f_\varepsilon = f_{\varepsilon,\frac{\varepsilon}{1+\lambda}}$ we ensure that this choice of sub-sequence is such that

$$\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F) \text{ as } \varepsilon \to 0$$

as desired.

For the second portion of the lemma, under the assumption that the functions v_ε are such that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \text{ for some sequence } \tau_\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

and the fact that $f_\varepsilon \in \mathcal{A}_{\varepsilon,r_2}(p)$, we see that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \leq \mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) + \tau_\varepsilon.$$

Here in the above we are using the fact that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_{\varepsilon,r_2}(p)} \mathcal{F}_{\varepsilon,\lambda}(u) \leq \mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon)$$

by the construction of g_ε in Lemma 2.2. Hence, by (2.2) (as established in Subsection 2.3.3) and the fact that $\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F)$ as $\varepsilon \to 0$ from the above, letting $\varepsilon \to 0$ we conclude that

$$\mathcal{F}_\lambda(E) \leq \mathcal{F}_\lambda(F).$$

In particular we have that

$$\mathcal{F}_\lambda(E) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(G) \mid G \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)\},$$

as desired. □

Remark 2.17. *By combining Lemma 2.3 with the results of [Mag12, Section 28.2] we deduce that if the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon$, for some sequence $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$, (which by the conclusion of Lemma 2.3 implies that M is locally $\mathcal{F}_\lambda$-minimising in a ball around p) then any tangent cone to the hypersurface M at a point of $B_{r_1}(p) \cap \partial F$ is area-minimising (in the sense that the cone is a perimeter minimiser in $\mathbb{R}^{n+1}$) and of multiplicity one.*

2.5 Construction of paths

In order to prove Theorem 2.3 we will work under the contradiction assumption (namely (2.9) as sketched in Subsection 2.1.3) that the one-dimensional profile, $v_\varepsilon = \overline{\mathbb{H}}^\varepsilon \circ d^\pm_{\overline{M}}$, as defined in Subsection 2.3.3 is such that for all $\varepsilon > 0$ sufficiently small we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau \text{ for some } \tau > 0.$$

Here the g_ε are defined, as in Subsection 2.4.2, to be the local energy minimisers agreeing with the one-dimensional profile outside of a ball centred on an isolated singularity of M. We then exhibit a continuous path in $W^{1,2}(N)$, from a_ε to b_ε, with energy along the path bounded a fixed amount below $\mathcal{F}_\lambda(E)$, independent of ε (this is the path that was sketched in Subsection 2.1.3). In this manner we will have exhibited, see Section 2.6, an admissible path in the min-max procedure of [BW20a] which contradicts the assumption that M arose from this construction, thus establishing Theorem 2.3. In this section we explicitly construct the separate pieces of our desired path, showing that the maximum energy along these pieces are bounded a fixed amount below $\mathcal{F}_\lambda(E)$, independently of ε. We also provide here, in Appendix 2.A, alternative computations in the minimal case, namely when $\lambda = 0$ and thus $\mathcal{F}_{\varepsilon,\lambda} = \mathcal{E}_\varepsilon$, which serve as a more straightforward route to prove that Theorem 2.3 holds in this setting.

2.5.1 Choosing radii for local properties

We first fix various radii, dependent only on the geometry of M around an isolated singularity, in order to later define functions for our path. Let $p \in \operatorname{Sing}(M)$ be an isolated singularity of M, there then exists an $R_p > 0$ (as defined in Subsection 2.1.1) such that $\overline{B_{R_p}(p)} \cap \operatorname{Sing}(M) = \{p\}$, i.e. such that $M \cap \overline{B_{R_p}(p)} \setminus \{p\}$ is smooth.

The fact that M has bounded (constant) mean curvature (specifically as the monotonicity formula holds at all points of $\overline{M}$) provides Euclidean volume growth in sufficiently small balls. Namely we guarantee that there exists two constants, $C_p, r_p > 0$, both depending on the point $p \in \operatorname{Sing}(M)$ such that, for all $r < r_p$, we have

$$\mathcal{H}^n(M \cap B_{r_p}(p)) \leq C_p r^n. \tag{2.35}$$

We fix a further choice, $r_0 \in \frac{1}{4}(0, \min\{r_p, R_p\})$, such that

$$\mathcal{H}^n(M \setminus B_{4r_0}(p)) > \frac{3\mathcal{H}^n(M)}{4}. \tag{2.36}$$

Set $\widehat{K} = M \cap (\overline{B_{R_p}(p)} \setminus B_{r_0}(p))$, which is compact in M, so by the reasoning in Remark 2.11 we ensure that there exists $c_{\widehat{K}} > 0$ such that

$$F(\widehat{K} \times (-c_{\widehat{K}}, c_{\widehat{K}})) \cap (\operatorname{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}) = \emptyset.$$

Define on $\widehat{K} \times (-c_{\widehat{K}}, c_{\widehat{K}}) \subset V_M$ the function

$$h(x,a) = \operatorname{dist}_N(F(\Pi_{V_M}(x,a)), p), \tag{2.37}$$

where we use the definition of the projection, Π_{V_M}, as in Remark 2.12. We thus ensure that, by the definitions in Remark 2.12, for $(x,a) \in K \times (-c_{\widehat{K}}, c_{\widehat{K}}) \subset V_M$ we have, for some constant $C_h > 0$ dependent only on $\widehat{K}$, N and g, that

$$|\nabla h(x,a)|_{(x,a)} \leq C_h. \tag{2.38}$$

Here in the above we are computing the gradient, ∇, on V_M with respect to the pullback metric F^*g.

We now fix a choice of $0 < r < \frac{3}{4}r_0$ sufficiently small to ensure that for the annulus $\mathcal{A} = M \cap (B_{r_0+3r}(p) \setminus B_{r_0+2r})$ we have, recalling that we denote $m = \min_N \mathrm{Ric}_g = \min_{|X|=1} \mathrm{Ric}_g(X, X)$ as in Subsection 2.3.2,

$$\frac{\mathcal{H}^n(\mathcal{A})}{r^2} \leq \left(\frac{me^{-\frac{\lambda^2}{2m}}\mathcal{H}^n(M)}{8C_h^2}\right). \tag{2.39}$$

Here in the above we are using the Euclidean volume growth of the mass of M (which follows again from the fact that the monotonicity formula holds at all points of M) to ensure that $\mathcal{H}^n(\mathcal{A})$ is of order r^n, combined with the assumption that, by Remark 2.6, we may assume $n \geq 7$. The reason for the above precise choice of r will become clear when calculating the energy of the shifted functions, $v_\varepsilon^{t,s}$, in Subsection 2.5.3.

Finally, for ease of notation we define the balls $B_i = B_{r_0+ir}(p)$ for $i \in \{1, 2, 3, 4\}$ which are such that $B_1 \subset\subset B_2 \subset\subset B_3 \subset\subset B_4 \subset\subset B_{4r_0}(p)$; with this notation we have that $\mathcal{A} = M \cap (\overline{B_3} \setminus B_2)$.

2.5.2 Defining the shifted functions

We define, for $s \in [0, 1]$, $t \in [-t_1, t_1]$ where $t_1 > 0$ is to be chosen and all $\varepsilon > 0$ sufficiently small, functions $v_\varepsilon^{t,s} \in W^{1,2}(N)$ with both paths

$$\begin{cases} t \in [-t_1, t_1] \to v_\varepsilon^{t,s} \in W^{1,2}(N) \\ s \in [0, 1] \to v_\varepsilon^{t,s} \in W^{1,2}(N) \end{cases}$$

continuous in $W^{1,2}(N)$. The functions $v_\varepsilon^{t,s}$ will be defined so that the following properties hold for all $s \in [0, 1]$, $t \in [-t_1, t_1]$ and $\varepsilon > 0$ sufficiently small

$$\begin{cases} v_\varepsilon^{t,1} = \overline{\mathbb{H}}^\varepsilon \circ (d_{\overline{M}}^{\pm} - t) \text{ on } N \\ v_\varepsilon^{0,s} = v_\varepsilon \text{ on } N \\ v_\varepsilon^{t,0} = v_\varepsilon \text{ in } B_{r_0}(p) \subset B_1 \end{cases}, \tag{2.40}$$

and in such a way that, for some $E(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon).$$

Furthermore, we will show in Lemma 2.4 of Subsection 2.5.3 that there exists $0 < t_0 < t_1$ and $\eta > 0$ such that for all $s \in [0, 1]$ and $\varepsilon > 0$ sufficiently small we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon).$$

Finally, in Lemma 2.5 of Subsection 2.5.4 we will show that for $t > t_0$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,1}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon).$$

These upper bounds on the energy along the paths provided by the functions $v_\varepsilon^{t,s}$ will be calculated explicitly in Subsection 2.5.3. Note that these paths of functions, along with their upper energy bounds, are precisely the shifted functions sketched in Step 2 of Subsection 2.1.3.

Thus, by assuming the above holds, when these paths are combined with those provided in Subsection 2.5.5, which change the shifted functions $v_\varepsilon^{t,0}$ only in $B_{r_0}(p)$ to decrease their energy, we construct a path from a_ε to b_ε in $W^{1,2}(N)$ whose energy along the path remains bounded strictly below $\mathcal{F}_\lambda(E)$. We now proceed to define the $v_\varepsilon^{t,s}$ explicitly.

Consider the balls $B_1 \subset\subset B_2 \subset\subset B_3 \subset\subset B_4$ centred at $p \in M$ as specified in Subsection 2.5.1. Let $K = M \cap \overline{B_4} \setminus B_1$, which is compact in M, and fix $c_K = c_{M\cap\overline{B_4}\setminus B_1} > 0$ by the reasoning in Remark 2.11; we then have that $K \times (-c_K, c_K) \subset V_M$ is a two-sided tubular neighbourhood of $K = M \cap \overline{B_4} \setminus B_1$ in N under the map F. Note that we have

$$F\left(K \times (-c_K, c_K)\right) \cap \left(\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}\right) = \emptyset$$

by the choice of $c_K > 0$, and as $K \subset \widehat{K}$ we ensure that $c_K \leq c_{\widehat{K}}$ where $c_{\widehat{K}} > 0$ is as chosen in Subsection 2.5.1.

For the value of $r > 0$, balls, $B_i = B_{r_0+ir}(p)$ and annulus, $\mathcal{A} = M \cap (\overline{B_3} \setminus B_1)$, as chosen in subsection 2.5.1 we define the following Lipschitz function on M

$$f(y) = \begin{cases} 1 \text{ if } y \in M \setminus \overline{B_3} \\ 0 \text{ if } y \in M \cap \overline{B_2} \\ \frac{1}{r}(\mathrm{dist}_N(y,p) - r_0 - 2r) \text{ if } y \in \mathcal{A} \end{cases} ; \qquad (2.41)$$

thus we ensure that $0 \leq f \leq 1$ and proceed to set

$$f_s(x) = s + (1-s)f(x).$$

Recall, from Remark 2.12, the definition of the smooth projection, Π, to M defined on the set $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$. Fix a choice of $t_1 > 0$ and $\tilde{\varepsilon} > 0$ sufficiently small to ensure that $2\varepsilon\Lambda_\varepsilon + t_1 < \min\{c_K, \frac{r}{2}\}$, for all $0 < \varepsilon < \tilde{\varepsilon}$. We now define, for $t \in [-t_1, t_1]$, $s \in [0,1]$ and $0 < \varepsilon < \tilde{\varepsilon}$, the functions

$$v_\varepsilon^{t,s}(x) = \begin{cases} \overline{\mathbb{H}}^\varepsilon(d_{\overline{M}}^{\pm}(x) - tf_s(\Pi(x)))) \text{ if } x \in F(K \times [-c_K, c_K]) \\ \overline{\mathbb{H}}^\varepsilon(d_{\overline{M}}^{\pm}(x) - ts) \text{ if } x \in B_{r_0 + \frac{3r}{2}}(p) \\ \overline{\mathbb{H}}^\varepsilon(d_{\overline{M}}^{\pm}(x) - t) \text{ if } x \in N \setminus B_{r_0+\frac{7r}{2}}(p) \\ 1 \text{ if } x \in E \cap \{d_{\overline{M}}^{\pm} > 2\varepsilon\Lambda_\varepsilon + t_1\} \\ -1 \text{ if } x \in (N \setminus E) \cap \{d_{\overline{M}}^{\pm} < -2\varepsilon\Lambda_\varepsilon - t_1\} \end{cases}.$$

Note then that by the above definition of the functions $v_\varepsilon^{t,s}$ we ensure that (2.40) holds as desired.

We now show that the $v_\varepsilon^{t,s}$ are well defined Lipschitz functions on N so that in particular, as

$$F(\mathcal{A} \times [-2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} - t_1, 2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} + t_1]) \subset\subset \overline{B_{r_0+\frac{7r}{2}}(p)} \setminus B_{r_0+\frac{3r}{2}}(p)$$

(which is guaranteed as we ensure $2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} + t_1 < \frac{r}{2}$ by our choices of $t_1 > 0$ and $\tilde{\varepsilon} > 0$) we may express them simply as

$$v_\varepsilon^{t,s}(x) = \overline{\mathbb{H}}^\varepsilon(d_{\overline{M}}^{\pm}(x) - tf_s(\Pi(x))) \text{ on } N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}). \tag{2.42}$$

Note here that we are using the fact that the set $\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}$ has zero $\mathcal{H}^{n+1}$ measure by the definition of M and the rectifiability arguments for $\overline{S_{d_{\overline{M}}}}$ as contained in Subsections 2.1.1 and 2.3.2 respectively. We will exploit this characterisation of the functions $v_\varepsilon^{t,s}$ directly in order to compute their upper energy bounds in Subsection 2.5.3.

Note that if $x \in \overline{B_{r_0+\frac{7r}{2}}(p)} \setminus B_{r_0+\frac{3r}{2}}(p)$ and $|d_{\overline{M}}^{\pm}| \leq 2\varepsilon\Lambda_\varepsilon + t_1$ then by the triangle inequality we see that $x \in F(K \times [-c_K, c_K])$. We have that both

$$\begin{cases} v_\varepsilon^{t,s} \equiv 1 \text{ in } F(K \times [2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} + t_1, c_K]) \\ v_\varepsilon^{t,s} \equiv -1 \text{ in } F(K \times [-c_K, -(2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} + t_1)]) \end{cases}$$

Combining these two facts with the fact that f is Lipschitz we see that the $v_\varepsilon^{t,s}$ are in fact Lipschitz functions in the set $\overline{B_{r_0+\frac{7r}{2}}(p)} \setminus B_{r_0+\frac{3r}{2}}(p)$.

Noting that

$$F(\mathcal{A} \times [-2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} - t_1, 2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} + t_1]) \subset\subset \overline{B_{r_0+\frac{7r}{2}}(p)} \setminus B_{r_0+\frac{3r}{2}}(p)$$

as $2\tilde{\varepsilon}\Lambda_{\tilde{\varepsilon}} + t_1 < \frac{r}{2}$, we have by definition of f that

$$\begin{cases} v_\varepsilon^{t,s} = \overline{\mathbb{H}}^\varepsilon(d^\pm_{\overline{M}}(x) - ts) \text{ if } x \in B_{r_0+\frac{3r}{2}}(p) \cap F(K \times [-c_K, c_K]) \\ v_\varepsilon^{t,s} = \overline{\mathbb{H}}^\varepsilon(d^\pm_{\overline{M}}(x) - t) \text{ if } x \in (N \setminus B_{r_0+\frac{7r}{2}}(p)) \cap F(K \times [-c_K, c_K]) \end{cases}.$$

Combining the above two paragraphs, we conclude that the $v_\varepsilon^{t,s}$ are indeed well defined continuous functions on N. It only remains to prove that they are Lipschitz.

As noted in the above paragraph, the functions $v_\varepsilon^{t,s}$ are Lipschitz in $\overline{B_{r_0+\frac{7r}{2}}(p)} \setminus B_{r_0+\frac{3r}{2}}(p)$. Furthermore, in the sets $N \setminus B_{r_0+\frac{7r}{2}}(p)$ and $B_{r_0+\frac{3r}{2}}(p)$ the $v_\varepsilon^{t,s}$ are Lipschitz functions by definition (as on these sets $v_\varepsilon^{t,s} = \overline{\mathbb{H}}^\varepsilon(d^\pm_{\overline{M}}(x) - t)$ and $v_\varepsilon^{t,s} = \overline{\mathbb{H}}^\varepsilon(d^\pm_{\overline{M}}(x) - ts)$ respectively). In conclusion we have that the $v_\varepsilon^{t,s}$ are well defined continuous functions on N that are Lipschitz on each of the three sets $\overline{B_{r_0+\frac{7r}{2}}(p)} \setminus B_{r_0+\frac{3r}{2}}(p)$, $N \setminus B_{r_0+\frac{7r}{2}}(p)$ and $B_{r_0+\frac{3r}{2}}(p)$; concluding that the $v_\varepsilon^{t,s}$ are indeed well defined Lipschitz functions on N. In particular, we have that the $v_\varepsilon^{t,s} \in W^{1,2}(N)$. We now proceed to prove upper energy bounds and show that these shifted functions form continuous paths in $W^{1,2}(N)$.

2.5.3 Energy and continuity of the shifted functions

We now calculate an upper bound on the energy of the shifted functions $v_\varepsilon^{t,s}$ for $t \in [t_0, t_0]$, where $0 < t_0 \leq t_1$ is to be chosen, all $s \in [0,1]$ and $\varepsilon > 0$ sufficiently small. The calculation method employed here is similar to those in [BW24, Sections 4, 6.1 and 7.1]. Though these calculations appear technically involved, they really are the diffuse analogue of the bounds sketched geometrically for the shifted functions in Step 2 of Subsection 2.1.3.

Lemma 2.4. *There exists $t_0 \in (0, t_1)$ so that for all $t \in [-t_0, t_0]$, $s \in [0,1]$ and $\varepsilon > 0$ sufficiently small we have that*

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon)$$

where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$. Furthermore, we have that, for all $\varepsilon > 0$ sufficiently small there exists

$$\eta = \frac{m}{8}\mathcal{H}^n(M)t_0^2 > 0$$

such that for all $s \in [0,1]$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta.$$

Proof. We denote, for $(x,a) \in V_M$,

$$\hat{v}_\varepsilon^{t,s}((x,a)) = v_\varepsilon^{t,s}(F(x,a))$$

so that, as $\Pi \circ F = F \circ \Pi_{V_M}$ from the constructions in Remark 2.12, we have by the expression (2.42) that

$$\hat{v}_\varepsilon^{t,s}(x,a) = \overline{\mathbb{H}}^\varepsilon(a - tf_s(F(\Pi_{V_M}(x,a)))) \text{ on } V_M. \tag{2.43}$$

Note that by the definitions in (2.37) and (2.41) and the fact that f is constant outside of $\mathcal{A}$ we have for points $(x,a) \in V_M$ that

$$\begin{cases} \nabla\Big(f_s(F(\Pi_{V_M}(x,a)))\Big) = \frac{1-s}{r}\nabla h(x,a) \text{ if } x \in \mathcal{A} \\ \nabla\Big(f_s(F(\Pi_{V_M}(x,a)))\Big) = 0 \text{ if } x \in M \setminus \mathcal{A} \end{cases}.$$

We then compute by the generalised Gauss' Lemma, see for example [Gra04, Chapter 2.4], that the quantity $|\nabla \hat{v}_\varepsilon^{t,s}(x,a)|^2_{(x,a)}$ is equal to

$$\Big((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\Big)^2 \left(1 + \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\chi_{\mathcal{A}\times\mathbb{R}}(x,a)\right), \tag{2.44}$$

where here $\chi_{\mathcal{A}\times\mathbb{R}}$ is the indicator function for the set $\mathcal{A}\times\mathbb{R}$ on V_M.

Using the co-area formula (slicing with a) and Fubini's Theorem we have, noting that we have multiplied the energy by the constant 2σ for convenience and working in the coordinates V_M, that

$$\begin{aligned}
2\sigma\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) &= \int_{V_M} e_\varepsilon(\hat{v}_\varepsilon^{t,s}) - \sigma\lambda\hat{v}_\varepsilon^{t,s} d\mathcal{H}^{n+1}_{F^*g} \\
&= \int_{V_M} \frac{\varepsilon}{2}|\nabla\hat{v}_\varepsilon^{t,s}|^2 + \frac{W(\hat{v}_\varepsilon^{t,s})}{\varepsilon} - \sigma\lambda\hat{v}_\varepsilon^{t,s} d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&= \int_{V_M}\left[\frac{\varepsilon}{2}\Big((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\Big)^2 + \frac{W(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))}{\varepsilon}\right. \\
&\qquad\qquad \left. - \sigma\lambda\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\right] d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&\quad + \int_{V_M}\left[\frac{\varepsilon}{2}\Big((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\Big)^2\frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\right. \\
&\qquad\qquad \left. \cdot\chi_{\mathcal{A}\times\mathbb{R}}(x,a)\right] d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&= \int_M\int_{\sigma^-(x)}^{\sigma^+(x)}\Big[e_\varepsilon\Big(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\Big) - \sigma\lambda\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\Big]\theta(x,a)da\, d\mathcal{H}^n(x) \\
&\quad + \int_{\mathcal{A}}\int_{\sigma^-(x)}^{\sigma^+(x)}\left[\frac{\varepsilon}{2}\Big((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\Big)^2\frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\right. \\
&\qquad\qquad \left. \cdot\theta(x,a)\right] da\, d\mathcal{H}^n(x).
\end{aligned}$$

Then, as $v_\varepsilon = v_\varepsilon^{0,s}$ for any $s \in [0,1]$ by (2.40), we have from the above expression that we may write the difference in the two energies, $2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon))$ as

$$\int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\right) - e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a)\right)\right]\theta(x,a)da\, d\mathcal{H}^n(x) \tag{2.45}$$

$$- \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda\left[\overline{\mathbb{H}}^\varepsilon(a - tf_s(x)) - \overline{\mathbb{H}}^\varepsilon(a)\right]\theta(x,a)da\, d\mathcal{H}^n(x) \tag{2.46}$$

$$+ \int_{\mathcal{A}} \int_{\sigma^-(x)}^{\sigma^+(x)} \left[\frac{\varepsilon}{2}\left((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\right)^2 \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\right. \tag{2.47}$$

$$\left.\cdot\, \theta(x,a)\right] da\, d\mathcal{H}^n(x). \tag{2.48}$$

We now analyse each of the terms appearing in the above energy difference separately. By massaging and rewriting each of the terms above we will eventually be able to deduce our desired energy bound as simple consequences of the underlying geometry of M.

First, we consider the term (2.45). Applying the Fundamental Theorem of Calculus, Fubini's Theorem and integrating by parts then, by setting

$$\theta(x, \sigma^\pm(x)) = \lim_{a \to \sigma^\pm(x)} \theta(x,a)$$

and recalling (2.18), we calculate for the term (2.45) that

$$\begin{aligned}
&\int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\right) - e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a)\right)\right]\theta(x,a)da\, d\mathcal{H}^n(x) \\
&= -\int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon'\left(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x))\right)\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&= \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\partial_a\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&- \int_0^t \int_M f_s(x) e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x) - rf_s(x))\right)\theta(x,\sigma^+(x))da\, d\mathcal{H}^n(x)\, dr \\
&+ \int_0^t \int_M f_s(x) e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x) - rf_s(x))\right)\theta(x,\sigma^-(x))da\, d\mathcal{H}^n(x)\, dr \\
&= \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} \Big[e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))(\lambda - H(x,a)) \\
&\qquad \cdot\, \theta(x,a)\Big] da\, d\mathcal{H}^n(x)\, dr \\
&- \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&- \int_0^t \int_M f_s(x) e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x) - rf_s(x))\right)\theta(x,\sigma^+(x))da\, d\mathcal{H}^n(x)\, dr
\end{aligned}$$

$$+\int_0^t \int_M f_s(x) e_\varepsilon \left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x) - r f_s(x))\right) \theta(x, \sigma^-(x)) da\, d\mathcal{H}^n(x)\, dr.$$

Note that in the last equality above we have both added and subtracted the following term,

$$\int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - r f_s(x)))\theta(x, a) da\, d\mathcal{H}^n(x)\, dr,$$

in order to introduce the quantity $\lambda - H(x, a)$ to the calculation.

We now consider the term

$$\int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - r f_s(x)))(\lambda - H(x, a))\theta(x, a) da\, d\mathcal{H}^n(x)\, dr \tag{2.49}$$

separately, which we may control by the assumption of positive Ricci curvature.

The fact that the $\sigma^\pm$ on M as defined in Subsection 2.3.1 and the volume elements, θ, on V_M as defined in Subsection 2.3.2 are all continuous functions, as well as the choice of radii (namely that $B_4 = B_{r_0+4r}(p) \subset B_{4r_0}(p)$ by the choice of $r < \frac{3}{4} r_0$) ensuring that (2.36) holds, ensure that we may fix a compact set $L \subset (M \setminus B_4)$ along with a constant $l > 0$ sufficiently small so that $|\sigma^\pm(x)| > l$ for $x \in L$ and such that

$$\mathcal{H}^n(\{(x, l) \in V_M \mid x \in L\}) > \frac{\mathcal{H}^n(M)}{2}. \tag{2.50}$$

Note that as $L \cap B_4 = \emptyset$ we have that $f(x) = 1$ (and thus $f_s(x) = 1$ by definition) for all $x \in L$.

The relation (2.18) combined with (2.20) for the volume elements, $\theta(x, s)$, on V_M in Subsection 2.3.2 ensure that for small positive values of s the volume elements are strictly decreasing (here we are implicitly using the assumption that $\lambda \geq 0$ by Subsection 2.1.2/Remark 2.6). Therefore we ensure that for some fixed $0 < t_0 \leq \min\{l, t_1, 2d(N)\}$ sufficiently small we have that $\theta(x, a) \geq \theta(x, l)$ for all $a \in [-t_0, t_0]$.

Again by (2.20) (relying on the positive Ricci curvature assumption) and the above paragraph we thus have, for $x \in L$, that

$$\begin{cases} (\lambda - H(x, a))\theta(x, a) \leq -m\, a\, \theta(x, l) \text{ if } a \in [0, t_0] \\ (\lambda - H(x, a))\theta(x, a) \geq -m\, a\, \theta(x, l) \text{ if } a \in [-t_0, 0] \end{cases}.$$

With these inequalities and (2.50) we compute for $t \in [-t_0, t_0]$ it holds that, using (2.27), $f_s \leq 1$ and the fact that $e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - r)) \neq 0$ only for values $a \in (r - 2\varepsilon\Lambda_\varepsilon, r + 2\varepsilon\Lambda_\varepsilon)$, we have

$$\begin{aligned}(2.49) &\leq -\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon)\left[\frac{m\sigma}{2}\mathcal{H}^n(M)t^2 - 2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)t\right]\\ &\leq -(1-\beta\varepsilon^2)\left[\frac{m\sigma}{2}\mathcal{H}^n(M)t^2 - 2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)t\right].\end{aligned}$$

Second, consider the term (2.46). By applying the Fundamental Theorem of Calculus and Fubini's Theorem we rewrite the term as

$$\begin{aligned}(2.46) &= -\int_M\int_{\sigma^-(x)}^{\sigma^+(x)}\sigma\lambda\left[\overline{\mathbb{H}}^\varepsilon(a - tf_s(x)) - \overline{\mathbb{H}}^\varepsilon(a)\right]\theta(x,a)da\,d\mathcal{H}^n(x)\\ &= \int_0^t\int_M f_s(x)\int_{\sigma^-(x)}^{\sigma^+(x)}\sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a)da\,d\mathcal{H}^n(x)\,dr.\end{aligned}$$

Third, we focus on the terms (2.47)/(2.48). Using (2.22), (2.27), (2.38), (2.39) and noting that $s \in [0,1]$, we see that

$$(2.47)/(2.48) \leq 2\sigma t^2\left(\frac{\mathcal{H}^n(\mathcal{A})C_h^2 e^{\frac{\lambda^2}{2m}}}{r^2}\right)\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \leq \frac{m\sigma}{4}\mathcal{H}^n(M)t^2(1+\beta\varepsilon^2).$$

We may now simplify the terms (2.45) to (2.48) appearing in the difference of the energies as computed above to see that for $t \in [-t_0, t_0]$ we have that $2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon))$ is

$$\leq -(1-\beta\varepsilon^2)\left[\frac{m\sigma}{4}\mathcal{H}^n(M)t^2 - 2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)t\right] + \frac{\beta\varepsilon^2 m\sigma}{2}\mathcal{H}^n(M)t^2 \tag{2.51}$$

$$+\int_0^t\int_M f_s(x)\int_{\sigma^-(x)}^{\sigma^+(x)}\sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a)da\,d\mathcal{H}^n(x)\,dr \tag{2.52}$$

$$-\int_0^t\int_M f_s(x)\int_{\sigma^-(x)}^{\sigma^+(x)}\lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a)da\,d\mathcal{H}^n(x)\,dr \tag{2.53}$$

$$-\int_0^t\int_M f_s(x)e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x) - rf_s(x))\right)\theta(x,\sigma^+(x))da\,d\mathcal{H}^n(x)\,dr \tag{2.54}$$

$$+\int_0^t\int_M f_s(x)e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x) - rf_s(x))\right)\theta(x,\sigma^-(x))da\,d\mathcal{H}^n(x)\,dr. \tag{2.55}$$

The objective now is to show that the sum of the terms (2.52) to (2.55) are errors uniformly small in ε. Loosely, we will show that the sum of these terms is bounded by a function $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ in order to obtain our desired upper bound for the difference in energies.

We first we treat (2.52) and (2.53) together. To this end we define the function $m_\varepsilon(rf_s(x), x)$ by

$$\max_{a\in[rf_s(x)-2\varepsilon\Lambda_\varepsilon, rf_s(x)+2\varepsilon\Lambda_\varepsilon]}\theta(x,a) - \min_{a\in[rf_s(x)-2\varepsilon\Lambda_\varepsilon, rf_s(x)+2\varepsilon\Lambda_\varepsilon]}\theta(x,a),$$

so that by (2.22), $0 \leq m_\varepsilon(rf_s(x), x) \leq \max_{a\in\mathbb{R}} \theta(x,a) \leq e^{\frac{\lambda^2}{2m}}$ and by the continuity of the volume elements we have that

$$m_\varepsilon(rf_s(x), x) \to 0 \text{ as } \varepsilon \to 0.$$

Hence, by applying the Dominated Convergence Theorem we see that

$$M_\varepsilon(t) = 2\sigma\lambda \int_M \int_0^t m_\varepsilon(rf_s(x), x) d\mathcal{H}^n(x)\, dr \to 0 \text{ as } \varepsilon \to 0.$$

With this definition we ensure that the functions $M_\varepsilon(t)$ are continuous.

Noting that whenever $0 < \tilde{\varepsilon} < \varepsilon$ we have that

$$0 \leq m_{\tilde{\varepsilon}}(rf_s(x), x) \leq m_\varepsilon(rf_s(x), x)$$

we may apply Dini's Theorem (as the $M_\varepsilon(t)$ are thus increasing/decreasing, hence monotone, in ε for positive/negative values of $t \in \mathbb{R}$ respectively) to the functions $M_\varepsilon(t)$ to conclude that

$$M_\varepsilon \to 0 \text{ uniformly on compact subsets of } \mathbb{R} \tag{2.56}$$

Using the above we compute that for (2.52) and (2.53), as $|f_s|, |\overline{\mathbb{H}}^\varepsilon| \leq 1$, (2.27) and by Fubini's Theorem, we have, for $t \in [-t_0, t_0]$, that

$$\begin{aligned}
&\Bigg| \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a) \\
&\qquad - \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a) da\, d\mathcal{H}^n(x)\, dr \Bigg| \\
&\leq \int_0^t \int_M \Bigg| \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a) \\
&\qquad - \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a) da \Bigg| d\mathcal{H}^n(x)\, dr \\
&\leq \int_0^t \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \Bigg[\sigma\lambda\Big(\max_{a\in[rf_s(x)-2\varepsilon\Lambda_\varepsilon, rf_s(x)+2\varepsilon\Lambda_\varepsilon]} \theta(x,a)\Big) (\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x)) da \\
&- \lambda\Big(\min_{a\in[rf_s(x)-2\varepsilon\Lambda_\varepsilon, rf_s(x)+2\varepsilon\Lambda_\varepsilon]} \theta(x,a)\Big) e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x))) da \Bigg] d\mathcal{H}^n(x)\, dr \\
&\leq 2\sigma\lambda \int_0^t \int_M m_\varepsilon(rf_s(x), x) d\mathcal{H}^n(x)\, dr + 2\sigma\lambda\mathcal{H}^n(M) e^{\frac{\lambda^2}{2m}} t_0 \beta\varepsilon^2 \\
&\leq \max_{t\in[-2d(N), 2d(N)]} M_\varepsilon(t) + 2\sigma\lambda\mathcal{H}^n(M) e^{\frac{\lambda^2}{2m}} t_0 \beta\varepsilon^2.
\end{aligned}$$

From this we conclude the following bound

$$|(2.52) + (2.53)| \leq \max_{t\in[-2d(N), 2d(N)]} M_\varepsilon(t) + 2\sigma\lambda\mathcal{H}^n(M) e^{\frac{\lambda^2}{2m}} t_0 \beta\varepsilon^2. \tag{2.57}$$

Finally we work on (2.54) and (2.55). If $t \geq 0$ then as each of the functions f_s, θ and e_ε are non-negative, we ensure that (2.54) ≤ 0. Similarly, if $t \leq 0$ then we may argue analogously to ensure that (2.55) ≤ 0. As $|f_s| \leq 1$ by definition, $\theta \leq e^{\frac{\lambda^2}{2m}}$ by (2.22) and $e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(\sigma^\pm(x) - rf_s(x)) \neq 0$ only when $\sigma^\pm(x) \in (rf_s(x) - 2\varepsilon\Lambda_\varepsilon, rf_s(x) + 2\varepsilon\Lambda_\varepsilon)$ by definition of $\overline{\mathbb{H}}^\varepsilon$, we conclude that

$$(2.54) \leq (2\sigma + \beta\varepsilon^2)t_0 e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^+(x) \leq 2\varepsilon\Lambda_\varepsilon\} \tag{2.58}$$

and

$$(2.55) \leq (2\sigma + \beta\varepsilon^2)t_0 e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^-(x) \geq -2\varepsilon\Lambda_\varepsilon\}. \tag{2.59}$$

Now, as $\mathcal{H}^n(\{x \in M | \sigma^\pm(x) = 0\}) = 0$ by the Dominated Convergence Theorem we are able to conclude that both

$$\mathcal{H}^n(\{x \in M \mid \sigma^+(x) \leq 2\varepsilon\Lambda_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0 \tag{2.60}$$

and

$$\mathcal{H}^n(\{x \in M \mid \sigma^-(x) \geq -2\varepsilon\Lambda_\varepsilon\} \text{ as } \varepsilon \to 0. \tag{2.61}$$

We can combine all of our above bounds to conclude that for $t \in [-t_0, t_0]$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{m}{8}\mathcal{H}^n(M)t^2 + E(\varepsilon).$$

Here in the above we are utilising (2.51), (2.57), (2.58), (2.59) and the fact that $t_0 \leq 2d(N)$ in order to define

$$\begin{aligned} 2\sigma E(\varepsilon) = \frac{3m\sigma}{4}&\mathcal{H}^n(M)(2d(N))^2\beta\varepsilon^2 + (1-\beta\varepsilon^2)\left[2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)(2d(N))\right] \\ &+ \max_{t\in[-2d(N),2d(N)]} M_\varepsilon(t) + 2\sigma\lambda\mathcal{H}^n(M)e^{\frac{\lambda^2}{2m}}(2d(N))\beta\varepsilon^2 \\ &\quad + (2\sigma+\beta\varepsilon^2)(2d(N))e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^+(x) \leq 2\varepsilon\Lambda_\varepsilon\} \\ &\qquad + (2\sigma+\beta\varepsilon^2)(2d(N))e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^-(x) \geq -2\varepsilon\Lambda_\varepsilon\}. \end{aligned}$$

We ensure that we have $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ by virtue of (2.56), (2.60) and (2.61). This concludes the desired bound

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon)$$

which holds for all $t \in [-t_0, t_0]$, $s \in [0,1]$ and $\varepsilon > 0$ sufficiently small.

For the fixed energy drop for the functions $v_\varepsilon^{\pm t_0,s}$ we set once and for all

$$\eta = \frac{m}{8}\mathcal{H}^n(M)t_0^2.$$

Thus we conclude, taking $\varepsilon > 0$ sufficiently small so that $E(\varepsilon) < \eta$, the desired energy drop for these functions. $\square$

We now conclude this subsection by showing that both of the paths

$$\begin{cases} t \in [-t_1, t_1] \to v_\varepsilon^{t,s} \in W^{1,2}(N) \\ s \in [-1, 1] \to v_\varepsilon^{t,s} \in W^{1,2}(N) \end{cases}$$

are continuous in $W^{1,2}(N)$ for fixed $s \in [0,1]$ and $t \in [-t_1, t_1]$ respectively. Recall that the functions F, Π_{V_M} and f_s are all continuous. For each $t \in [-t_1, t_1]$ consider that for values $s, \tilde{s} \in [0,1]$ we have, by the Dominated Convergence Theorem, the fact that $|\overline{\mathbb{H}}^\varepsilon| \leq 1$ and the expression in (2.43) that

$$\begin{aligned} ||v_\varepsilon^{t,s} - v_\varepsilon^{t,\tilde{s}}||^2_{L^2(N)} &= \int_{V_M} |\hat{v}_\varepsilon^{t,s} - \hat{v}_\varepsilon^{t,\tilde{s}}|^2 d\mathcal{H}^{n+1}_{F^*g} \\ &= \int_{V_M} |\overline{\mathbb{H}}^\varepsilon(a - tf_s(F(\Pi_{V_M}(x,a)))) \\ &\qquad - \overline{\mathbb{H}}^\varepsilon(a - tf_{\tilde{s}}(F(\Pi_{V_M}(x,a))))|^2 d\mathcal{H}^{n+1}_{F^*g}(x,a) \\ &\to 0 \text{ as } \tilde{s} \to s. \end{aligned}$$

Furthermore, we see that by the Dominated Convergence Theorem, the fact that $\overline{\mathbb{H}}^\varepsilon$ is smooth, the bound (2.38), as well as the expressions in (2.43) and (2.44) we have that

$$\begin{aligned} &||\nabla v_\varepsilon^{t,s} - \nabla v_\varepsilon^{t,\tilde{s}}||^2_{L^2(N)} = \int_{V_M} |\nabla \hat{v}_\varepsilon^{t,s} - \nabla \hat{v}_\varepsilon^{t,\tilde{s}}|^2 d\mathcal{H}^{n+1}_{F^*g} \\ &\leq \int_{V_M} \Big((\overline{\mathbb{H}}^\varepsilon)'\Big(a - tf_s(F(\Pi_{V_M}(x,a)))\Big) \\ &\qquad - (\overline{\mathbb{H}}^\varepsilon)'\Big(a - tf_{\tilde{s}}(F(\Pi_{V_M}(x,a)))\Big)\Big)^2 d\mathcal{H}^{n+1}_{F^*g}(x,a) \\ &\quad + \int_{V_M} t^2 \Big|\nabla\Big(f_{\tilde{s}}(F(\Pi_{V_M}(x,a)))\Big)(\overline{\mathbb{H}}^\varepsilon)'\Big(a - tf_{\tilde{s}}(F(\Pi_{V_M}(x,a)))\Big) \\ &\qquad - \nabla\Big(f_s(F(\Pi_{V_M}(x,a)))\Big)(\overline{\mathbb{H}}^\varepsilon)'\Big(a - tf_s(F(\Pi_{V_M}(x,a)))\Big)\Big|^2 d\mathcal{H}^{n+1}_{F^*g}(x,a) \\ &\to 0 \text{ as } \tilde{s} \to s. \end{aligned}$$

Hence, for fixed $t \in [-t_1, t_1]$, the path

$$s \in [0,1] \to v_\varepsilon^{t,s} \text{ is continuous in } W^{1,2}(N). \tag{2.62}$$

Analogous arguments to those above show that, for fixed $s \in [0,1]$, the

path
$$t \in [-t_1, t_1] \to v_\varepsilon^{t,s} \text{ is continuous in } W^{1,2}(N). \tag{2.63}$$

2.5.4 Sliding the one-dimensional profile

We now define, for $t \in \mathbb{R}$, the functions $v_\varepsilon^t \in W^{1,2}(N)$ by setting

$$v_\varepsilon^t(x) = \overline{\mathbb{H}}^\varepsilon(d_{\overline{M}}^{\pm}(x) - t).$$

Note then that, by (2.40), for $t \in [-t_1, t_1]$ we have $v_\varepsilon^t = v_\varepsilon^{t,1}$. By the continuity of translations on L^p for $1 \leq p < \infty$ we have that $v_\varepsilon^t \in W^{1,2}(N)$ for all $t \in \mathbb{R}$, the path

$$t \in \mathbb{R} \to v_\varepsilon^t \text{ is continuous in } W^{1,2}(N)$$

and that $v_\varepsilon^0 = v_\varepsilon$. By choosing $\varepsilon > 0$ sufficiently small to ensure that $2\varepsilon\Lambda_\varepsilon < d(N)$ we have both

$$\begin{cases} v_\varepsilon^t = -1 \text{ on } N \text{ for } t \geq 2d(N) \\ v_\varepsilon^t = +1 \text{ on } N \text{ for } t \leq -2d(N). \end{cases}$$

We now show that, using the assumption of positive Ricci curvature, the path provided by the functions v_ε^t, along with the energy reducing paths from -1 and $+1$ provided by negative gradient flow of the energy to a_ε and b_ε respectively, provides a "recovery path" for the value $\mathcal{F}_\lambda(E)$; this path connects a_ε to b_ε, passing through v_ε, with the maximum value of the energy along this path approximately $\mathcal{F}_\lambda(E)$. Note that these paths of functions, along with their upper energy bounds, are precisely the sliding functions sketched in Step 2 of Subsection 2.1.3.

Lemma 2.5. *For all $t \in [-2d(N), 2d(N)]$ we have that*

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon)$$

where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ is as in Lemma 2.4. Furthermore, we have that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm\tilde{t}}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t}) + E(\varepsilon),$$

whenever $\tilde{t} \geq t \geq 0$; thus, in particular for $t > t_0$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon).$$

Proof. We compute in an identical manner to the proof of Lemma 2.4, writing

$$2\sigma\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) = \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-t)\right) - \sigma\lambda\overline{\mathbb{H}}^\varepsilon(a-t)\right]\theta(x,a)da\, d\mathcal{H}^n(x).$$

Assuming that either $\tilde{t} \geq t > 0$ or $\tilde{t} \leq t < 0$, we compute the difference of energies as

$$\begin{aligned}
&2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\tilde{t}}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t)) \\
&= \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-\tilde{t})\right) - e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-t)\right)\right]\theta(x,a)da\, d\mathcal{H}^n(x) \\
&\quad - \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda\left[\overline{\mathbb{H}}^\varepsilon(a-\tilde{t}) - \overline{\mathbb{H}}^\varepsilon(a-t)\right]\theta(x,a)da\, d\mathcal{H}^n(x).
\end{aligned}$$

Similarly to Lemma 2.4 this yields, noting that $\lambda - H(x,a) \leq 0$ for all $a \in \mathbb{R}$ by the assumption of positive Ricci curvature (2.20), the expression

$$\begin{aligned}
&2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\tilde{t}}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t)) \\
&\quad\leq \int_t^{\tilde{t}} \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a-r)\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&\quad\quad - \int_t^{\tilde{t}} \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a-r))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&\quad\quad - \int_t^{\tilde{t}} \int_M e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x)-r)\right)\theta(x,\sigma^+(x))da\, d\mathcal{H}^n(x)\, dr \\
&\quad\quad + \int_t^{\tilde{t}} \int_M e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x)-r)\right)\theta(x,\sigma^-(x))da\, d\mathcal{H}^n(x)\, dr.
\end{aligned}$$

Near identical computations to those in the proof of Lemma 2.4 (precisely those computations for the error terms (2.52) and (2.53) giving the bound (2.57), as well the bounds on (2.58) and (2.59) giving (2.60), (2.61) respectively) yield upper bounds on the the above four terms and show that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\tilde{t}}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) + E(\varepsilon),$$

where the expression for $E(\varepsilon)$ is identical to that of Lemma 2.4 (here we are exploiting the fact that $t \in [-2d(N), 2d(N)]$). Finally, using the fact that $v_\varepsilon^0 = v_\varepsilon$, $v_\varepsilon^{\pm t_0} = v_\varepsilon^{\pm t_0,1}$ and the upper energy bounds from Lemma 2.4, we conclude the various desired upper bounds. □

2.5.5 Paths to local energy minimisers

Recall that for the one-dimensional profile, $v_\varepsilon = \overline{\mathbb{H}}^\varepsilon \circ d_{\overline{M}}^{\pm}$ we have, by Lemma 2.4 and (2.40), for a fixed $\eta > 0$ and some $r_0 > 0$, as chosen in Subsection 2.5.1, that there exists a $t_0 > 0$ and functions $v_\varepsilon^{t,0} \in W^{1,2}(N)$ for $t \in [-t_0, t_0]$, with the following properties:

- $v_\varepsilon^{t,0} = v_\varepsilon$ in $B_{r_0}(p)$.
- $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon)$, where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$.
- $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,0}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta$.

We now produce the local functions as sketched geometrically in Step 2 of Subsection 2.1.3. These functions are defined so that they remain constant outside of $B_{r_0}(p)$ and provide a continuous path in $W^{1,2}(N)$ from v_ε to g_ε in the ball $B_{r_0}(p)$ such that the maximum energy along this path increases by at most $\frac{\eta}{2}$ above $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0})$; loosely speaking, this may be seen as a diffuse analogue of [CLS22, Lemma 1.12] which concerns the construction of homotopy sweep-outs with controlled area.

Proposition 2.3. *For $\eta > 0$ as above there exists, for some $R \in (0, r_0)$ and $\varepsilon > 0$ sufficiently small, functions $g_\varepsilon^{t,s} \in W^{1,2}(N)$, for each $t \in [-t_0, t_0]$ and $s \in [-2, 2]$, such that the following properties hold:*

- *For each $t \in [-t_0, t_0]$ we have $g_\varepsilon^{t,-2} = v_\varepsilon^{t,0}$ on N and $g_\varepsilon^{t,s} = v_\varepsilon^{t,0}$ on $N \setminus B_R(p)$ for all $s \in [-2, 2]$.*
- *For each $t \in [-t_0, t_0]$ we have $g_\varepsilon^{t,2} = g_\varepsilon$ in $B_R(p)$ where $g_\varepsilon \in \mathcal{A}_{\varepsilon,\frac{R}{2}}(p)$ arises from a choice of minimiser in Lemma 2.2 for $\rho = \frac{R}{2}$ and $q = p$.*
- *For each $s \in [-2, 2]$, $t \in [-t_0, t_0] \to g_\varepsilon^{t,s}$ is a continuous path in $W^{1,2}(N)$.*
- *For each $t \in [-t_0, t_0]$, $s \in [-2, 2] \to g_\varepsilon^{t,s}$ is a continuous path in $W^{1,2}(N)$.*
- *For each $t \in [-t_0, t_0]$ and $s \in [-2, 2]$ we have*

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{\eta}{2}.$$

Furthermore, if the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$ for some $\tau > 0$, we have for each $t \in [-t_0, t_0]$ that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) - \tau.$$

Remark 2.18. *The paths exhibited in Proposition 2.3 are "local" in the sense that they remain constant outside of $B_R(p)$, and additionally do not require the assumption of positive Ricci curvature of N. Specifically, in any ambient manifold (without any curvature assumption), given an $\eta > 0$ (not necessarily as fixed by Lemma 2.4) and $f \in W^{1,2}(N)$ such* *that $|f| \leq 1$ and with f equal to v_ε in a ball bi-Lipschitz diffeomorphic to the Euclidean ball of the same radius, a path as in Proposition 2.3 may be constructed (replacing $v_\varepsilon^{t,0}$ by f in the conclusions) for a sufficiently small $R > 0$.*

Proof. Given some $\eta > 0$ we choose $R > 0$ sufficiently small to ensure that

$$2^{n+2}R^n\omega_n + C_pR^n + 2\lambda \mathrm{Vol}_g(B_{\frac{R}{2}}(p)) < \frac{\eta}{4}, \tag{2.64}$$

where here ω_n is the volume of the n-dimensional unit ball. We then potentially re-choose a smaller $0 < R \leq R_l$ (with R_l as defined in Subsection 2.1.1), so that the closed ball $\overline{B}_R(p)$ is 2-bi-Lipschitz diffeomorphic, via some smooth geodeisic normal coordinate chart, ψ, to the Euclidean ball, $\overline{B}_R^{\mathbb{R}^{n+1}}(0)$ of radius R, with $\psi(p) = 0$.

We consider a sweep-out of $\overline{B}_R^{\mathbb{R}^{n+1}}(0)$ by the horizontal planes, Π_l, defined for $l \in [-1, 1]$ by

$$\Pi_l = \left\{ y \in \overline{B}_R^{\mathbb{R}^{n+1}}(0) \,\middle|\, y = (y_1, \dots, y_n, lR) \right\}.$$

For each $l \in [-1, 1]$, the plane Π_l is the intersection of $\overline{B}_R^{\mathbb{R}^{n+1}}(0)$ with the plane $\{y_{n+1} = lR\}$. We consider the images, $P_l = \psi(\Pi_l)$ in N. These images, P_l, then sweep-out the closure of $B_R(p)$, $\overline{B}_R(p)$ in the sense that

$$\begin{cases} \overline{B}_R(p) = \cup_{l\in[-1,1]} P_l \\ P_l \cap P_s = \emptyset \text{ for } l \neq s \end{cases}.$$

Note that each plane Π_l divides the ball $\overline{B}_R^{\mathbb{R}^{n+1}}(0)$ into two disjoint connected regions given by

$$\begin{cases} \left\{ y \in \overline{B}_R^{\mathbb{R}^{n+1}}(0) \mid y_{n+1} > lR \right\} \\ \left\{ y \in \overline{B}_R^{\mathbb{R}^{n+1}}(0) \mid y_{n+1} < lR \right\} \end{cases}.$$

We then denote the images of these sets under the diffeomorphism ψ in $\overline{B}_R(p)$ as

$$\begin{cases} E_l = \psi\left(\left\{ y \in \overline{B}_R^{\mathbb{R}^{n+1}}(0) \mid y_{n+1} > lR \right\} \right) \\ F_l = \psi\left(\left\{ y \in \overline{B}_R^{\mathbb{R}^{n+1}}(0) \mid y_{n+1} < lR \right\} \right) \end{cases}$$

respectively. Note then that $\overline{B}_R(p) = E_l \cup F_l \cup P_l$, where the unions are all mutually disjoint.

We now define, for $l \in [-1, 1]$, the functions $p_\varepsilon^l \in W^{1,2}(B_R(p))$ given by

$$p_\varepsilon^l = \overline{\mathbb{H}}^\varepsilon(d_{P_l}^\pm),$$

where here, for d_{P_l} the usual distance function on N to the set P_l, we define the Lipschitz signed distance to P_l by

$$d_{P_l}^\pm = \begin{cases} +d_{P_l}, \text{ if } x \in E_l \\ 0, \text{ if } x \in P_l \\ -d_{P_l}, \text{ if } x \in F_l \end{cases}.$$

As the Lipschitz images of the planes, P_l, vary continuously in the Hausdorff distance as $l \in [-1,1]$ varies, the functions p_ε^l vary continuously in $W^{1,2}(B_R(p))$ with respect to $l \in [-1,1]$ (c.f. [Gua18, Proposition 9.2]).

Applying Lemma 2.2, for the choices $\rho = \frac{R}{2}$ and $q = p$, for each $\varepsilon > 0$ sufficiently small we produce a local energy minimiser $g_\varepsilon \in W^{1,2}(N)$. In particular, this function agrees with the one-dimensional profile, v_ε, outside of $B_{\frac{R}{2}}(p)$ and is such that $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$.

For $l \in [-1,1]$ we define on the ball $B_R(p)$ the functions

$$\check{h}_\varepsilon^l(x) = \max\Big\{ \min\{v_\varepsilon(x), p_\varepsilon^l(x)\}, \min\{g_\varepsilon(x), v_\varepsilon(x)\}\Big\},$$

and

$$\hat{h}_\varepsilon^l(x) = \max\Big\{ \min\{g_\varepsilon(x), p_\varepsilon^l(x)\}, \min\{g_\varepsilon(x), v_\varepsilon(x)\}\Big\}.$$

With these definitions, the above two functions replicate the transition behaviour described by Remark 2.5 and depicted in Figure 2.3 in Step 2 of Subsection 2.1.3. One can then readily check that these functions satisfy, for $\varepsilon > 0$ sufficiently small so that $2\varepsilon\Lambda_\varepsilon < \frac{R}{2}$, the following properties

$$\begin{cases} \check{h}_\varepsilon^{-1} = v_\varepsilon \\ \check{h}_\varepsilon^1 = \hat{h}_\varepsilon^1 = \min\{g_\varepsilon, v_\varepsilon\} \\ \hat{h}_\varepsilon^{-1} = g_\varepsilon \\ |\check{h}_\varepsilon^l| \leq 1 \\ |\hat{h}_\varepsilon^l| \leq 1 \end{cases}.$$

For two functions $f, g \in W^{1,2}(B_R(p))$ we may write

$$\begin{cases} \max\{f,g\} = \max\{f-g,0\} + g = \frac{1}{2}(|f-g| + (f-g)) + g \\ \min\{f,g\} = \min\{f-g,0\} + g = \frac{1}{2}(|f-g| - (f-g)) + g \end{cases},$$

from which we conclude that $\max\{f,g\}, \min\{f,g\} \in W^{1,2}(N)$. We also note that if $t \to f_t$ is a continuous path in $W^{1,2}(B_R(p))$, then the paths $t \to \max\{f_t, 0\}$ and $t \to \min\{f_t, 0\}$ are continuous in $W^{1,2}(B_R(p))$. To see this we write

$$\max\{f_t, 0\} - \max\{f_s, 0\} = \frac{1}{2}((|f_t| - |f_s|) + (f_t - f_s))$$

which, by applying reverse triangle inequality, may be controlled by the $W^{1,2}(B_R(p))$ norm of $f_t - f_s$; showing continuity of the path in $W^{1,2}(B_R(p))$. The proof for the $W^{1,2}(B_R(p))$ continuity of the path $t \to \min\{f_t, 0\}$ is identical. With the above in mind, we have that $\check{h}^l_\varepsilon, \hat{h}^l_\varepsilon \in W^{1,2}(B_R(p))$ for each $l \in [-1, 1]$, and that both paths

$$\begin{cases} l \in [-1,1] \to \check{h}^l_\varepsilon \\ l \in [-1,1] \to \hat{h}^l_\varepsilon \end{cases}$$

are continuous in $W^{1,2}(B_R(p))$.

We have that $v_\varepsilon = g_\varepsilon$ on $B_R(p) \setminus \overline{B_{\frac{R}{2}}(p)}$ (as by construction we ensure that $v_\varepsilon = g_\varepsilon$ on $N \setminus B_{\frac{R}{2}}(p)$) and hence $\hat{h}^l_\varepsilon = \check{h}^l_\varepsilon = v_\varepsilon$ on $B_R(p) \setminus \overline{B_{\frac{R}{2}}(p)}$ also. Thus we may extend these functions on $B_R(p)$ to two functions on the whole of N, $\check{g}^{t,l}_\varepsilon, \hat{g}^{t,l}_\varepsilon \in W^{1,2}(N)$ for $t \in [-t_0, t_0]$ and $l \in [-1, 1]$ by defining

$$\check{g}^{t,l}_\varepsilon(x) = \begin{cases} v^{t,0}_\varepsilon(x), & \text{if } x \in N \setminus B_R(p) \\ \check{h}^l_\varepsilon(x), & \text{if } x \in B_R(p) \end{cases},$$

and

$$\hat{g}^{t,l}_\varepsilon(x) = \begin{cases} v^{t,0}_\varepsilon(x), & \text{if } x \in N \setminus B_R(p) \\ \hat{h}^l_\varepsilon(x), & \text{if } x \in B_R(p) \end{cases}.$$

This is well defined as we have that $v^{t,0}_\varepsilon = v_\varepsilon$ on the set $B_R(p)$, hence in defining the functions above we have only edited the functions $v^{t,0}_\varepsilon$ in $B_R(p)$ and thus keep them in $W^{1,2}(N)$. By the above arguments, as both paths

$$\begin{cases} l \in [-1,1] \to \check{h}^l_\varepsilon \\ l \in [-1,1] \to \hat{h}^l_\varepsilon \end{cases}$$

are continuous in $W^{1,2}(B_R(p))$, we ensure that the four paths

$$\begin{cases} t \in [-t_0, t_0] \to \check{g}^{t,l}_\varepsilon \\ l \in [-1,1] \to \check{g}^{t,l}_\varepsilon \\ t \in [-t_0, t_0] \to \hat{g}^{t,l}_\varepsilon \\ l \in [-1,1] \to \hat{g}^{t,l}_\varepsilon \end{cases}$$

are continuous in $W^{1,2}(N)$ (here we are implicitly using the fact that the path, $t \in [-t_0, t_0] \to v_\varepsilon^{t,0}$, of shifted functions is continuous in $W^{1,2}(N)$, as shown in Subsection 2.5.3)

Note that then as $\check{h}_\varepsilon^1 = \hat{h}_\varepsilon^1 = \min\{g_\varepsilon, v_\varepsilon\}$ we have for each $t \in [-t_0, t_0]$ that $\check{g}_\varepsilon^{t,1} = \hat{g}_\varepsilon^{t,-1} = \min\{g_\varepsilon, v_\varepsilon\}$ in $B_R(p)$. We thus define, for $s \in [-2, 2]$, the functions

$$g_\varepsilon^{t,s} = \begin{cases} \check{g}^{t,s+1}, \text{ if } s \in [-2, 0] \\ \hat{g}_\varepsilon^{t,s-1}, \text{ if } s \in [0, 2] \end{cases},$$

so that in $B_R(p)$ we have

$$\begin{cases} g_\varepsilon^{t,0} = \check{g}_\varepsilon^{t,1} = \hat{g}_\varepsilon^{t,-1} = \min\{g_\varepsilon, v_\varepsilon\} \\ g_\varepsilon^{t,-2} = \check{g}_\varepsilon^{0,-1} = v_\varepsilon^{t,0} = v_\varepsilon \\ g_\varepsilon^{0,2} = \hat{g}_\varepsilon^{0,1} = g_\varepsilon \end{cases}.$$

Furthermore, by the continuity of the paths mentioned above we have that both of the paths

$$\begin{cases} s \in [-2, 2] \to g_\varepsilon^{t,s} \\ t \in [-t_0, t_0] \to g_\varepsilon^{t,s} \end{cases}$$

are continuous in $W^{1,2}(N)$.

We now show that we may bound the energy of all the functions $g_\varepsilon^{t,s}$ above by $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0})$ plus errors depending only on the geometry of the ball $B_R(p)$. As $\{g_\varepsilon \neq v_\varepsilon\} \subset B_{\frac{R}{2}}(p)$ we note that $g_\varepsilon^{t,s} = v_\varepsilon^{t,0}$ on $N \setminus B_{\frac{R}{2}}(p)$. We then compute that, as $\check{h}_\varepsilon^{s+1}$ is always equal to one of $g_\varepsilon, v_\varepsilon$ or p_ε^{s+1} in $B_{\frac{R}{2}}(p)$, for $s \in [-2, 0]$ we have

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) &= \frac{1}{2\sigma} \int_N e_\varepsilon(g_\varepsilon^{t,s}) - \frac{\lambda}{2} \int_N g_\varepsilon^{t,s} \\ &= \frac{1}{2\sigma} \int_{N \setminus B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma} \int_{B_{\frac{R}{2}}(p)} e_\varepsilon(\check{h}_\varepsilon^{s+1}) \\ &\quad - \frac{\lambda}{2} \int_{N \setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2} \int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\ &= \frac{1}{2\sigma} \int_{N \setminus B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma} \int_{B_{\frac{R}{2}}(p) \cap \{\check{h}_\varepsilon^{s+1} = v_\varepsilon\}} e_\varepsilon(v_\varepsilon) \\ &\quad + \frac{1}{2\sigma} \int_{B_{\frac{R}{2}}(p) \cap \{\check{h}_\varepsilon^{s+1} = g_\varepsilon\}} e_\varepsilon(g_\varepsilon) \\ &\quad + \frac{1}{2\sigma} \int_{B_{\frac{R}{2}}(p) \cap \{\check{h}_\varepsilon^{s+1} = p_\varepsilon^{s+1}\}} e_\varepsilon(p_\varepsilon^{s+1}) \end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda}{2}\int_{N\setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\
&\leq \frac{1}{2\sigma}\int_{N\setminus B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon) \\
&\quad + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s+1}) \\
&\quad - \frac{\lambda}{2}\int_{N\setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\
&= \mathcal{E}_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s+1}) \\
&\quad - \frac{\lambda}{2}\int_{N\setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\
&= \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s+1}) \\
&\quad + \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} (v_\varepsilon^{t,0} - \check{h}_\varepsilon^{s+1}),
\end{aligned}
$$

where in the final line we have both added and subtracted the term $\frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0}$. Similarly for $s \in [0,2]$ we may compute that

$$
\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) &+ \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) \\
&+ \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s-1}) + \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} (v_\varepsilon^{t,0} - \hat{h}_\varepsilon^{s-1}).
\end{aligned}
$$

Thus, as for all $l \in [-1,1]$ we have $|\check{h}_\varepsilon^l| \leq 1$, $|\hat{h}_\varepsilon^l| \leq 1$ and $|v_\varepsilon^{t,0}| \leq 1$, we see that for each $t \in [-t_0, t_0]$ and $s \in [-2,2]$

$$
\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) \tag{2.65}
$$

$$
+ \sup_{l\in(-1,1)} \left(\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^l) \right) + \lambda \mathrm{Vol}_g(B_{\frac{R}{2}}(p)). \tag{2.66}
$$

Similarly, we note that as $v_\varepsilon = g_\varepsilon$ on $N \setminus B_{\frac{R}{2}}(p)$, $|g_\varepsilon|, |v_\varepsilon| \leq 1$ on N and $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ by construction of g_ε, we have that

$$
\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) \leq \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon) + \lambda \mathrm{Vol}_g(B_{\frac{R}{2}}(p)).
$$

and so by a similar co-area formula calculation to Subsection 2.3.3 (slicing with $d^{\pm}_{\overline{M}}$) and using (2.27) we see that

$$
\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon) \leq (1+\beta\varepsilon^2) \operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} \mathcal{H}^n(\{d^{\pm}_{\overline{M}} = a\} \cap B_R(p))
$$

$$\to \mathcal{H}^n(M \cap B_R(p)) \text{ as } \varepsilon \to 0.$$

By the Euclidean volume growth of M in (2.35) we conclude that, for sufficiently small $\varepsilon > 0$, we have

$$\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) \leq C_p R^n + \lambda \mathrm{Vol}_g(B_{\frac{R}{2}}(p)) + \frac{\eta}{8}. \tag{2.67}$$

We now compute upper $\mathcal{E}_\varepsilon$ bounds on the other terms appearing in (2.65)/(2.66), namely those involving the functions p^l_ε, in terms of the geometry of N. Once again by applying the co-area formula, slicing with $d^\pm_{P_l}$, and using (2.27) we see that for each $l \in (-1, 1)$ we have

$$\begin{aligned}
\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p^l_\varepsilon) &= \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(d^\pm_{P_l})) \\
&= \frac{1}{2\sigma}\int_{\mathbb{R}}\int_{\{d^\pm_{P_l}=a\}\cap B_{\frac{R}{2}}(p)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a))d\mathcal{H}^n\, da \\
&= \frac{1}{2\sigma}\int_{\mathbb{R}} \mathcal{H}^n(\{d^\pm_{P_l} = a\}\cap B_{\frac{R}{2}}(p))e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a))\, da \\
&\leq (1+\beta\varepsilon^2)\operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\{d^\pm_{P_l} = a\}\cap B_{\frac{R}{2}}(p)).
\end{aligned}$$

We now focus on bounding the term

$$\operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\{d^\pm_{P_l} = a\}\cap B_{\frac{R}{2}}(p))$$

from above independently of ε. Recall that $P_l = \psi(\Pi_l)$ where ψ is a smooth 2-bi-Lipschitz map from $\overline{B}^{\mathbb{R}^{n+1}}_R(0)$ to $\overline{B}_R(p)$ and

$$\Pi_l = \left\{ y \in \overline{B}^{\mathbb{R}^{n+1}}_R(0) \,\middle|\, y = (y_1, \dots, y_n, lR)\right\}.$$

For $l \in (-1, 1)$ we have that Π_l is a smooth embedded n-dimensional submanifold of $\overline{B}^{\mathbb{R}^{n+1}}_R$ and hence its image, P_l, in N is a smooth embedded n-dimensional submanifold of $\overline{B}_R(p) \subset N$.

We define the tubular hypersurface at distance s from P_l to be the set

$$P_l(s) = \{x \in N \mid \exists \text{ a geodesic of length } s \text{ meeting } P_l \text{ orthogonally}\}.$$

Here we choose $\varepsilon > 0$ possibly smaller to ensure that $2\varepsilon\Lambda_\varepsilon < \frac{R}{2}$, so that we guarantee $\{|d^\pm_{P_l}| = a\} \cap B_{\frac{R}{2}}(p) \subset P_l(a)$ for all $l \in (-1, 1)$ such that $B_{\frac{R}{2}}(p) \cap P_l \neq \emptyset$. We then apply the formula for the volume of a tubular hypersurface (e.g. see [Gra04, Lemma 3.12/8.2]) to compute that, for each $l \in (-1,1)$ we have

$$\mathcal{H}^n(P_l(s)) \leq \int_{P_l}\int_{S^0} \theta_u(q,s)du\, d\mathcal{H}^n(q).$$

In the above we are adopting the notation that S^0 is the 0 dimensional unit sphere, and for $u \in S^0$ we define $\theta_u(q,a)$ to be the Jacobian of the exponential map, $\exp_q$, at the point $\exp_q(au)$. Note then that for each $q \in P_l$ we have that $\theta_u(q,0) = 1$ and $\theta_u(q,a) \to 1$ as $a \to 0$. For all $\varepsilon > 0$ sufficiently small we may ensure that $\theta_u(q,a) \leq 2$ for any $q \in \overline{B}_R(p)$; this may be seen by noticing that as the exponential map is smooth, its Jacobian varies continuously in each of its variables and hence its maximum, for a fixed $a \in \mathbb{R}$ on $\overline{B}_R(p) \times \mathbb{R}$, is achieved and converges to 1 as $a \to 0$. Thus, we have that

$$\mathcal{H}^n(P_l(a)) \leq 4\mathcal{H}^n(P_l).$$

Note that as ψ is 2-bi-Lipschitz we may control the measure of its image (e.g. by applying [EG15, Theorem 2.8]) to see that

$$\mathcal{H}^n(P_l) \leq 2^n\mathcal{H}^n(\Pi_l) \leq 2^nR^n\omega_n,$$

where ω_n is the volume of the n-dimensional unit ball. Hence, for all $l \in [-1,1]$ and $\varepsilon > 0$ sufficiently small we have that

$$\mathcal{H}^n(P_l(s)) \leq 2^{n+2}R^n\omega_n.$$

We thus conclude that as $\{|d^{\pm}_{P_l}| = a\} \cap B_{\frac{R}{2}}(p) \subset P_l(a)$ for all $l \in (-1,1)$ such that $B_{\frac{R}{2}}(p) \cap P_l \neq \emptyset$ we have

$$\operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\{d^{\pm}_{P_l} = a\} \cap B_{\frac{R}{2}}(p)) \leq 2^{n+2}R^n\omega_n,$$

and hence we deduce that

$$\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p^l_\varepsilon) \leq (1+\beta\varepsilon^2)2^{n+2}R^n\omega_n. \tag{2.68}$$

Choosing $\varepsilon > 0$ again possibly smaller we ensure that

$$2^{n+2}R^n\omega_n\beta\varepsilon^2 < \frac{\eta}{8}.$$

Recall that, given some $\eta > 0$, we chose $R > 0$ in (2.64) sufficiently small to ensure that

$$2^{n+2}R^n\omega_n + C_pR^n + 2\lambda\mathrm{Vol}_g(B_{\frac{R}{2}}(p)) < \frac{\eta}{4}.$$

Combining the above two bounds with (2.65), (2.66), (2.67) and (2.68) we conclude that for each $t \in [-t_0, t_0]$ and $s \in [-2, 2]$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{\eta}{2},$$

for $\varepsilon > 0$ sufficiently small as desired.

We now establish the final part of the proposition. As $\{g_\varepsilon \neq v_\varepsilon\} \subset B_{\frac{R}{2}}(p)$ we have $g_\varepsilon^{t,s} = v_\varepsilon^{t,0}$ on $N \setminus B_{\frac{R}{2}}(p)$, which combined with the fact that in $B_R(p)$ we have both $v_\varepsilon^{t,0} = v_\varepsilon$ and $g_\varepsilon^{t,2} = g_\varepsilon$, we thus observe that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + (\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)).$$

Using this, if the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$ for some $\tau > 0$, we therefore conclude that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) - \tau,$$

as desired. □

2.6 Proof of Theorems 2.1, 2.2 and 2.3

We recall our setup before combining the results of the previous subsections to prove the main results of this chapter. Let (N, g) be a smooth compact Riemannian manifold of dimension 3 or higher with positive Ricci curvature. We then consider $M \subset N$ a closed embedded hypersurface of constant mean curvature $\lambda \in \mathbb{R}$, smooth away from a closed singular set of Hausdorff dimension at most $n - 7$, as produced by the one-parameter Allen–Cahn min-max in [BW20a], with constant prescribing function λ. In particular, the properties of M as stated in Subsection 2.1.1 hold with M arising as the reduced boundary of a Caccioppoli set, $E \subset N$.

2.6.1 Proof of Theorem 2.3

Proof of Theorem 2.3. For each isolated singularity $p \in \mathrm{Sing}(M)$ we may choose the radii r_0, r and R as in Subsection 2.5.1 and Proposition 2.3 in order to define the various paths constructed in Subsections 2.5.2, 2.5.4 and 2.5.5. We begin by assuming for contradiction that for $g_\varepsilon \in \mathcal{A}_{\varepsilon,\frac{R}{2}}(p)$, defined as in Lemma 2.2 (by setting $\rho = \frac{R}{2}$ and $q = p$), there exists a $\tau > 0$ such that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau,$$

for all $\varepsilon > 0$ sufficiently small. Then, using the various path constructions in Section 2.5 as mentioned above, we define the following nine paths in $W^{1,2}(N)$:

- First, a path from $+1$ to the constant b_ε through constant functions,

$$s \in [1, b_\varepsilon] \to s,$$

  which, by the construction of the stable critical point b_ε through the negative gradient flow of $\mathcal{F}_{\varepsilon,\lambda}$ in Subsection 2.1.2, has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(+1).$$

- Second, a path from $+1$ to $v_\varepsilon^{-t_0} = v_\varepsilon^{-t_0,1}$,

$$t \in [-2d(N), -t_0] \to v_\varepsilon^t,$$

  which by Lemma 2.5 has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v^{-t_0,1}) \leq \mathcal{F}_\lambda(E) - \eta + E(\varepsilon).$$

  This path varies continuously by the reasoning in Subsection 2.5.4.

- Third, a path from $v_\varepsilon^{-t_0} = v_\varepsilon^{-t_0,1}$ to $v_\varepsilon^{-t_0,0} = g_\varepsilon^{-t_0,-2}$,

$$s \in [0, 1] \to v_\varepsilon^{-t_0,s},$$

  which by Lemma 2.4 has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta.$$

  This path varies continuously by (2.62).

- Fourth, a path from $g_\varepsilon^{-t_0,-2}$ to $g_\varepsilon^{-t_0,2}$,

$$s \in [-2, 2] \to g_\varepsilon^{-t_0,s},$$

  which by Proposition 2.3 and Lemma 2.4 has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\eta}{2}.$$

This path varies continuously by Proposition 2.3.

- Fifth, a path from $g_\varepsilon^{-t_0,2}$ to $g_\varepsilon^{t_0,2}$,

$$t \in [-t_0, t_0] \to g_\varepsilon^{t,2},$$

which by Proposition 2.3 and Lemma 2.4 has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \tau + E(\varepsilon).$$

This path varies continuously by Proposition 2.3.

- Sixth, a path from $g_\varepsilon^{t_0,-2} = v_\varepsilon^{t_0,0}$ to $g_\varepsilon^{t_0,2}$,

$$s \in [-2, 2] \to g_\varepsilon^{t_0,s},$$

which by by Proposition 2.3 and Lemma 2.4 has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\eta}{2}.$$

This path varies continuously by Proposition 2.3.

- Seventh, a path from $g_\varepsilon^{t_0,-2} = v_\varepsilon^{t_0,0}$ to $v_\varepsilon^{t_0,1} = v_\varepsilon^{t_0}$,

$$s \in [0, 1] \to v_\varepsilon^{t_0,s},$$

which by Lemma 2.4 has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta.$$

This path varies continuously by (2.62).

- Eighth, a path from $v_\varepsilon^{t_0,1} = v_\varepsilon^{t_0}$ to -1,

$$t \in [t_0, 2d(N)] \to v_\varepsilon^t,$$

which by Lemma 2.5 has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon).$$

This path varies continuously by the reasoning in Subsection 2.5.4.

- Ninth, a path from -1 to the constant a_ε through constant functions,

$$s \in [-1, a_\varepsilon] \to s,$$

which, by the construction of the stable critical point a_ε through the negative gradient flow of $\mathcal{F}_{\varepsilon,\lambda}$ in Subsection 2.1.2, has

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(-1).$$

Consider the above paths in the following order: first (reversed), second, third (reversed), fourth, fifth, sixth (reversed), seventh, eighth and ninth; this is the path depicted in Figure 2.1 in Subsection 2.1.3. In the order just given, the endpoint of each partial path matches the starting point of the next, therefore their composition in the same order provides a continuous path in $W^{1,2}(N)$, for all $\varepsilon > 0$ sufficiently small, from the constant a_ε to the constant b_ε with

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \min\left\{\frac{\eta}{2}, \tau\right\} + E(\varepsilon),$$

by Subsection 2.1.2, Lemma 2.4, Lemma 2.5 and Proposition 2.3. By (2.2) and the fact that $E(\varepsilon) \to 0$ by Lemma 2.4, by choosing $\varepsilon > 0$ sufficiently small we ensure that we have

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\lambda}(E) - \min\left\{\frac{\eta}{4}, \frac{\tau}{2}\right\}.$$

Note that (2.1) holds, by (2.2) and the path provided in Lemma 2.5, so we have explicitly that $\mathcal{F}_{\varepsilon,\lambda}(u_{\varepsilon_j}) \to \mathcal{F}_\lambda(E)$ as $\varepsilon_j \to 0$. Thus, as the above path is admissible in the min-max construction of [BW20a], we contradict the initial assumption that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$ for some $\tau > 0$. We therefore conclude that for any such M as produced by the Allen–Cahn min-max procedure in Ricci positive curvature must be such that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \text{ for some sequence } \tau_\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

With the above in mind we may apply Lemma 2.3, by setting $r_1 = \frac{R}{4}$ and $r_2 = \frac{R}{2}$, in order to establish that E satisfies

$$\mathcal{F}_\lambda(E) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(G) \mid G \setminus B_{\frac{R}{4}}(p) = E \setminus B_{\frac{R}{4}}(p)\},$$

thus E is locally $\mathcal{F}_\lambda$-minimising, as desired. In particular, by Remark 2.17, we note that every tangent cone at an isolated singularity of M is thus area-minimising. Applying the above reasoning for each isolated singularity of M then concludes the proof of Theorem 2.3. □

2.6.2 Proof of Theorems 2.1 and 2.2

Proof of Theorem 2.2. To prove Theorem 2.2 we exploit the surgery procedure developed in Section 2.2. We exploit the fact that, by Remark 2.10, the proof of Theorem 2.3 ensures M is one-sided $\mathcal{F}_\lambda$-minimising in the sense of Definition 2.2. Because the set of isolated singular points of M with regular tangent cone is discrete, but not necessarily closed when $n \geq 8$, it suffices for our purposes to index the isolated singularities with regular tangent cone and make a small change to the metric around each point, ensuring that the sum of the resulting perturbations is arbitrarily small.

With the above in hand, we now make an arbitrarily small change to the metric at each isolated singular point, $p \in \mathrm{Sing}(M)$, with regular tangent cone. By applying Proposition 2.1 in a ball, $B_\rho(p)$, for some $\rho > 0$ sufficiently small, there exists both:

- A metric, $\tilde{g}$, arbitrarily close to g in the $C^{k,\alpha}$ norm for each $k \geq 1$ and $\alpha \in (0,1)$, agreeing with g on $N \setminus B_\rho(p)$.
- A closed embedded hypersurface, $\widetilde{M}$, of constant mean curvature λ, which is smooth in $B_\rho(p)$, and agrees with M on $N \setminus B_\rho(p)$.

In this manner we are able to locally smooth M up to an arbitrarily small perturbation of the metric g. Thus we have shown that for each $k \geq 1$ and $\alpha \in (0,1)$ there exists a dense set of metrics, $\mathcal{G}_k \subset \mathrm{Met}^{k,\alpha}_{\mathrm{Ric}_g>0}$, such that for each $h \in \mathcal{G}_k$, (N,h) admits a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set of Hausdorff dimension at most $n-7$, containing no singularities with regular tangent cone. By taking the intersection over all the sets $\mathcal{G}_k$ (c.f. [Whi15, Theorem 2.10]) there thus exists a dense set, $\mathcal{G}$, of the smooth metrics with Ricci positive curvature such that for each $h \in \mathcal{G}$, (N,h) admits a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set of Hausdorff dimension at most $n-7$, containing no singularities with regular tangent cone. This concludes the proof of Theorem 2.2. □

Proof of Theorem 2.1. In ambient dimension 8, all singularities of M are isolated with regular tangent cone. Hence, for each $g \in \mathcal{G}$, as produced in the proof of Theorem 2.2, there exists a smooth hypersurface of constant mean curvature. The fact that $\mathcal{G}$ as above is then open in dimension 8 follows from an adaptation of a bumpy metric theorem for constant mean curvature hypersurfaces contained in [Whi91, Section 7]. Precisely, one may apply [IMN18, Proposition 2.3] to assume without loss that each constant mean curvature hypersurface associated to each $g \in \mathcal{G}$ is nondegenerate, after which one can apply the above bumpy metric theorem directly to deduce that $\mathcal{G}$ is open. Combined with the fact that $\mathcal{G}$ is dense as shown above, this concludes the proof of Theorem 2.1. □

2.A The minimal case

In this appendix we outline several alterations and simplifications that can be made to the arguments of this chapter when the underlying hypersurface is assumed to be minimal (i.e. when $\lambda = 0$). In most cases, setting $\lambda = 0$ in the above sections is sufficient to streamline many of the arguments. Thus, we will mainly summarise the changes made to the arguments in each section in the minimal case and only fully explain those arguments and calculations that take an entirely different approach. These changes are detailed for each section below:

Section 2.1: For this section one may simply set $\lambda = 0$ except for in Subsection 2.1.2, where one may use solely the Allen–Cahn min-max procedure in [Gua18]; note in particular here that the stable critical points of $\mathcal{E}_\varepsilon$ on N are then simply given by the constant functions ± 1. The fact that the properties of M as stated in Subsection 2.1.1 then hold for any minimal hypersurface arising from the Allen–Cahn min-max procedure of [Gua18] follow from combining the results of [Gua18, Theorem A] and [Bel23b, Theorem 1.8]. Note that the embeddedness of M is immediate by an application of the maximum principle.

Section 2.2: As mentioned at the beginning of the proof of Proposition 2.1, for the surgery procedure one can simply use [CLS22, Proposition 4.1] and thus only require the foliation due to [HS85] (in particular its one-sided extension due to [Liu19]). Note here however that for the construction of the smoothed hypersurfaces in the proof of [CLS22, Proposition 4.1], one is not able to guarantee any sign control on the mean curvature (as the underlying hypersurface has zero mean curvature).

Section 2.3: Firstly, in the proof of Lemma 2.1 one may, as in the proof of [Bel23b, Lemma 3.1], simply use the sheeting theorem for minimal hypersurfaces available in [SS81] (or the more general version in [Wic14a]). The remainder of the arguments in Subsection 2.3.1 go through via the same reasoning as in [Bel23b, Section 3].

In Subsection 2.3.2 we have our first major alteration stemming from the fact that (2.20) becomes

$$\begin{cases} H(x,s) \geq ms \text{ for } s > 0 \\ H(x,0) = 0 \\ H(x,s) \leq ms \text{ for } s < 0 \end{cases} . \tag{2.69}$$

Combined with (2.18), (2.69) then gives the following upper area bound for the volume elements of the level sets

$$\theta(x,b) \leq \theta(x,a)e^{-\frac{m(b^2-a^2)}{2}}, \tag{2.70}$$

whenever $|b| \geq |a| > 0$ and both a and b have the same sign. Thus we have in particular that for each $s \in \mathbb{R} \setminus \{0\}$

$$\theta(x,s) \leq e^{-\frac{ms^2}{2}}\theta(x,0) \leq \theta(x,0) = 1. \tag{2.71}$$

By integrating over M we also conclude that

$$\mathcal{H}^n(\widetilde{\Gamma}(b)) \leq e^{-\frac{m(b^2-a^2)}{2}}\mathcal{H}^n(\widetilde{\Gamma}(a)) \leq \mathcal{H}^n(\widetilde{\Gamma}(a)). \tag{2.72}$$

In particular, for each $s \in \mathbb{R} \setminus \{0\}$

$$\mathcal{H}^n(\widetilde{\Gamma}(s)) \leq e^{-\frac{ms^2}{2}}\mathcal{H}^n(M) \leq \mathcal{H}^n(M). \tag{2.73}$$

For Subsection 2.3.3, as indicated in Remark 2.13, we have a simpler computation to show that (2.2) holds when $\lambda = 0$. Precisely, we use the co-area formula, slicing with $d^{\pm}_{\overline{M}}$ and using $|\nabla d^{\pm}_{\overline{M}}| = 1$, to compute that by (2.73) we have

$$\begin{aligned} 2\sigma\mathcal{E}_\varepsilon(v_\varepsilon) &= \int_N e_\varepsilon(v_\varepsilon) = \int_N e_\varepsilon(v_\varepsilon)|\nabla d^{\pm}_{\overline{M}}| = \int_{\mathbb{R}} \left(\int_{\Gamma(s)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) d\mathcal{H}^n \right) ds \\ &= \int_{\mathbb{R}} \mathcal{H}^n(\Gamma(s)) e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) ds = \int_{\mathbb{R}} \mathcal{H}^n(\widehat{\Gamma}(s)) e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) \\ &\leq 2\sigma\mathcal{H}^n(M)\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon). \end{aligned}$$

so in particular we have

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leq \mathcal{H}^n(M)\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon).$$

For a lower bound we similarly see that

$$ess \inf_{s \in [-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} \mathcal{H}^n(\widehat{\Gamma}(s))\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \leq \mathcal{E}_\varepsilon(v_\varepsilon).$$

By applying (2.24) and (2.26) for the above bounds on $\mathcal{E}_\varepsilon(v_\varepsilon)$ we thus conclude that

$$\mathcal{E}_\varepsilon(v_\varepsilon) \to \mathcal{H}^n(M) \text{ as } \varepsilon \to 0; \tag{2.74}$$

this is the analogue of (2.2) when $\lambda = 0$, as desired.

Section 2.4: For this section one may simply set $\lambda = 0$ in order to streamline the arguments.

Section 2.5: We will utilise the same $W^{1,2}(N)$ functions that are constructed throughout this section but provide entirely alternative upper $\mathcal{E}_\varepsilon$ energy bound computations in the case that our underlying hypersurface is minimal. In particular, we exhibit an alternative method to upper bound the $\mathcal{E}_\varepsilon$ energy of the shifting and sliding functions. These calculations are aided by virtue of the fact that one does not have to keep track of the enclosed volume term that appears in the energy when $\lambda \neq 0$. Consequently, the computations we exhibit here are both more straightforward and reflect the underlying geometry in a clear manner. As we will later explain, the alternative upper bounds computed here may be used directly in place of those obtained in Subsections 2.5.3 and 2.5.4 in order to prove Theorem 2.3 (for the minimal case) just as in Subsection 2.6.1.

First we provide alternative bounds to those in Lemma 2.5 for the sliding functions, $v_\varepsilon^t \in W^{1,2}(N)$. One may simply compute as we did above for v_ε that

$$\begin{aligned}
2\sigma\mathcal{E}_\varepsilon(v_\varepsilon^t) &= \int_N e_\varepsilon(v_\varepsilon^t) = \int_N e_\varepsilon(v_\varepsilon^t)|\nabla(d_M^\pm - t)| \\
&= \int_{\mathbb{R}} \left(\int_{\Gamma(s+t)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) d\mathcal{H}^n \right) ds \\
&= \int_{\mathbb{R}} \mathcal{H}^n(\Gamma(s+t)) e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) ds = \int_{\mathbb{R}} \mathcal{H}^n(\widetilde{\Gamma}(s+t)) e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)),
\end{aligned}$$

where here we have used the co-area formula to slice with $(d_M^\pm - t)$, using $|\nabla(d_M^\pm - t)| = 1$. Note that by (2.72) we then in fact have that

$$\mathcal{E}_\varepsilon(v_\varepsilon^{\pm\tilde{t}}) \leq \mathcal{E}_\varepsilon(v_\varepsilon^{\pm t}) \tag{2.75}$$

whenever $\tilde{t} \geq t \geq 0$. In particular this shows that the sliding functions provide a "recovery path" for the value $\mathcal{H}^n(M)$; this path connects $+1$ to -1, passing through v_ε, with the maximum value of the energy along this path approximately $\mathcal{H}^n(M)$.

We now proceed to compute alternative upper energy bounds analogous to those in Lemma 2.4 for the shifting functions, $v_\varepsilon^{t,s} \in W^{1,2}(N)$.

These bounds, along with the energy comparison for the sliding functions above, will serve as adequate analogues to replace the bounds obtained in Subsections 2.5.3 and 2.5.4 when working in the minimal case. Observe that, similarly to the calculation at the beginning of the proof of Lemma 2.4, by (2.43), the co-area formula (slicing with a) and Fubini's Theorem we have

$$\begin{aligned}
&2\sigma\mathcal{E}_\varepsilon(v_\varepsilon^{t,s}) = \int_{V_M} e_\varepsilon(\hat{v}_\varepsilon^{t,s})d\mathcal{H}^{n+1}_{F^*g} \\
&= \int_{V_M} \frac{\varepsilon}{2}|\nabla \hat{v}_\varepsilon^{t,s}|^2 + \frac{W(\hat{v}_\varepsilon^{t,s})}{\varepsilon} d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&= \int_{V_M} \frac{\varepsilon}{2}\left((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\right)^2 + \frac{W(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))}{\varepsilon} d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&\quad + \int_{V_M} \left[\frac{\varepsilon}{2}\left((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\right)^2 \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\right. \\
&\qquad\qquad\qquad\qquad \left.\cdot\, \chi_{\mathcal{A}\times\mathbb{R}}(x,a)\right] d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&= \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\right)\theta(x,a)da\, d\mathcal{H}^n(x) \\
&\quad + \int_{\mathcal{A}} \int_{\sigma^-(x)}^{\sigma^+(x)} \left[\frac{\varepsilon}{2}\left((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\right)^2 \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\right. \\
&\qquad\qquad\qquad\qquad \left.\cdot\, \theta(x,a)\right] da\, d\mathcal{H}^n(x).
\end{aligned}$$

Using (2.27), (2.38), (2.39) (setting $\lambda = 0$), (2.71) and noting that $s \in [0,1]$, we see that the second term in the final equality is

$$\leq 2\sigma\left(\frac{t^2C_h^2\mathcal{H}^n(\mathcal{A})}{r^2}\right)\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \leq \frac{m\sigma}{4}\mathcal{H}^n(M)t^2\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon)$$

We also note that by (2.41) and (2.71) we have

$$\begin{aligned}
&\int_{M\cap\overline{B_3}} \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\right)\theta(x,a)da\, d\mathcal{H}^n(x) \\
&\qquad\leq \int_{M\cap\overline{B_3}} \mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon)\theta(x,0)d\mathcal{H}^n(x) = 2\sigma\mathcal{H}^n(M\cap\overline{B_3})\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon),
\end{aligned}$$

and that

$$\begin{aligned}
&\int_{M\setminus\overline{B_3}} \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\right)\theta(x,a)da\, d\mathcal{H}^n(x) \\
&\quad= \int_{M\setminus\overline{B_3}} \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - t)\right)\theta(x,a)da\, d\mathcal{H}^n(x) \\
&\quad\leq \max_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} e^{-\frac{m(s+t)^2}{2}} \int_{M\setminus\overline{B_3}} \mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon)\theta(x,0)d\mathcal{H}^n(x) \\
&\quad= \max_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} e^{-\frac{m(s+t)^2}{2}} 2\sigma\mathcal{H}^n(M\setminus\overline{B_3})\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon).
\end{aligned}$$

Combining the above three bounds, dividing by 2σ and by adding and subtracting the term $\mathcal{H}^n(M \setminus \overline{B_3})\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon)$ we see that

$$\mathcal{E}_\varepsilon(v_\varepsilon^{t,s}) \leq \mathcal{H}^n(M)\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) + \frac{m}{8}\mathcal{H}^n(M)t^2\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \\ + \left(\max_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} e^{-\frac{m(s+t)^2}{2}} - 1\right)\mathcal{H}^n(M \setminus \overline{B_3})\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon).$$

For fixed $t \in \mathbb{R}$ we consider the Taylor series expansion as follows

$$e^{-\frac{m(s^2+2st)}{2}} = 1 - \frac{m}{2}(s^2 + 2st) + R(s)$$

where $R(s)$ is the remainder term from Taylor's theorem of order $O(s^2)$. Thus for $\varepsilon > 0$ sufficiently small we have that

$$\max_{s\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} e^{-\frac{m(s^2+2st)}{2}} \leq 1 + \gamma\varepsilon^2,$$

for some constant $\gamma > 0$ (depending on a fixed upper bound for ε and the upper bound $|t| \leq 2d(N)$). We thus see that by (2.27) we have

$$\left(\max_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} e^{-\frac{m(s+t)^2}{2}} - 1\right)\mathcal{H}^n(M \setminus \overline{B_3})\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \\ \leq (e^{-\frac{mt^2}{2}} - 1)\mathcal{H}^n(M \setminus \overline{B_3}) + \gamma\varepsilon^2,$$

potentially increasing the constant $\gamma > 0$ (in particular through multiplication by $\beta > 0$). Again combining this with (2.27) we see that

$$\mathcal{E}_\varepsilon(v_\varepsilon^{t,s}) \leq \mathcal{H}^n(M) + \frac{m}{8}\mathcal{H}^n(M)t^2 + (e^{-\frac{mt^2}{2}} - 1)\mathcal{H}^n(M \setminus \overline{B_3}) + \gamma\varepsilon^2,$$

potentially increasing the constant $\gamma > 0$ again. We now proceed to show that, by setting

$$h(t) = \frac{m}{8}\mathcal{H}^n(M)t^2 + (e^{-\frac{mt^2}{2}} - 1)\mathcal{H}^n(M \setminus \overline{B_3}),$$

we have $h(\pm t_0) < 0$ and $h(t) \leq 0$ for all $t \in [-t_0, t_0]$ for some $t_0 > 0$ to be chosen. These bounds on $h(t)$ then directly imply analogous bounds to those in the conclusion Lemma 2.4, from which the remaining bound in the minimal case for the conclusion of Lemma 2.5 directly follows when combined with (2.75) as deduced above.

First note that $h(0) = 0$. We compute

$$h'(t) = \frac{m}{4}\mathcal{H}^n(M)t - mte^{-\frac{mt^2}{2}}\mathcal{H}^n(M \setminus \overline{B_3})$$

from which we see that $h'(t) = 0$ for $t = 0$ and $t = \pm t_2$ where

$$t_2^2 = -\frac{2}{m}\log\left(\frac{\mathcal{H}^n(M)}{4\mathcal{H}^n(M\setminus\overline{B_3})}\right);$$

note that $t_2 > 0$ is then well defined and strictly positive by virtue of (2.36). Furthermore, we note that

$$h''(t) = \frac{m}{4}\mathcal{H}^n(M) - me^{-\frac{mt^2}{2}}\mathcal{H}^n(M\setminus\overline{B_3}) + m^2t^2e^{-\frac{mt^2}{2}}\mathcal{H}^n(M\setminus\overline{B_3})$$

so that $h''(0) < 0$ by virtue of (2.36) again. We thus conclude that as $h(0) = h'(0) = 0$ and $h''(0) < 0$ we have and $h(t) < 0$ for all $t \in [-t_0, t_0]\setminus\{0\}$ for some sufficiently small $t_0 \in (0, t_2)$. This thus directly implies, for all $\varepsilon > 0$ sufficiently small, that as

$$\mathcal{E}_\varepsilon(v_\varepsilon^{t,s}) \leq \mathcal{H}^n(M) + h(t) + \gamma\varepsilon^2,$$

we have

$$\mathcal{E}_\varepsilon(v_\varepsilon^{t,s}) \leq \mathcal{E}_\varepsilon(v_\varepsilon) + E(\varepsilon) \tag{2.76}$$

where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$, where we are using (2.74). Moreover, we conclude that

$$\mathcal{E}_\varepsilon(v_\varepsilon^{\pm t_0,s}) \leq \mathcal{E}_\varepsilon(v_\varepsilon) - \eta \tag{2.77}$$

where we may choose for example $\eta = \frac{h(t_0)}{2} < 0$. These are the desired analogues in the minimal case of the general (i.e. $\lambda \neq 0$) energy bounds obtained in Lemma 2.4.

For Subsection 2.5.5 one may simply replicate the calculations and set $\lambda = 0$, ignoring any volume terms that appear in the energy. Note that this also means that the choice of $R \in (0, r_0)$ made in the proof of Proposition 2.3 need only satisfy

$$2^{n+2}R^n\omega_n + C_pR^n < \frac{\eta}{4}$$

in place of (2.64) and so that $0 < R < R_l$ (where R_l is defined as in Subsection 2.1.1). Thus the choice of R in the statement of Proposition 2.3 may be taken to be strictly larger than in the general $\lambda \neq 0$ case.

Section 2.6: For the proof of Theorem 2.3 in Subsection 2.6.1 we first note that in the minimal case we do not need to include the first of ninth portions of the path that flow to stable critical points (as ± 1 are the stable critical points of $\mathcal{E}_\varepsilon$ in this case). The remaining seven portions of the whole path (listed as the second to eighth paths in Subsection 2.6.1) remain identical save for the fact that the upper energy bounds provided by (2.75), (2.76) and (2.77) may be used in the minimal case in place of the respective applications of Lemmas 2.4 and 2.5 for the general ($\lambda \neq 0$) case.

For the proof of Theorem 2.2 in Subsection 2.6.2 we may, as mentioned in the adaptations for Section 2.2 above, simply apply the surgery procedure given by [CLS22, Proposition 4.1]. For the openness conclusion in proof of Theorem 2.1 one may appeal instead to the results of [Whi91, Section 2] in the minimal case.

Bibliography

[AA76] W. K. Allard and F. J. Almgren. The structure of stationary one dimensional varifolds with positive density. *Inventiones mathematicae*, 34(2):8397, June 1976.

[Alb94] Giovanni S. Alberti. On the structure of singular sets of convex functions. *Calculus of Variations and Partial Differential Equations*, 2:17–27, 1994.

[All72] William K. Allard. On the first variation of a varifold. *Annals of Mathematics*, 95(3):417, May 1972.

[All75] William K. Allard. On the first variation of a varifold: Boundary behavior. *The Annals of Mathematics*, 101(3):418, May 1975.

[Alm65] Fredrick J. Almgren. *The theory of varifolds : a variational calculus in the large for the k-dimensional area integrand.* Mimeographed notes, Princeton University, 1965.

[Alm00] Fredrick J. Almgren. *Almgren's big regularity paper : Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2.* World Scientific, 2000.

[BDG69] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. *Inventiones mathematicae*, 7(3):243268, September 1969.

[Bel23a] Costante Bellettini. Extensions of schoen–simon–yau and schoen–simon theorems via iteration à la de giorgi. `https://arxiv.org/abs/2310.01340`, 2023.

[Bel23b] Costante Bellettini. Multiplicity-1 minmax minimal hypersurfaces in manifolds with positive ricci curvature. *Communications on Pure and Applied Mathematics*, 77(3):20812137, October 2023.

[Bir17] George D. Birkhoff. Dynamical systems with two degrees of freedom. *Transactions of the American Mathematical Society*, 18(2):199300, 1917.

[BM23] Costante Bellettini and Kobe Marshall-Stevens. On isolated singularities and generic regularity of min-max CMC hypersurfaces. `https://arxiv.org/abs/2307.10388`, 2023.

[Bou15] Theodora Bourni. Allard-type boundary regularity for $C^{1,\alpha}$ boundaries. *Advances in Calculus of Variations*, 9(2):143161, February 2015.

[Bro86] John E. Brothers. *Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute*, volume 44. Geometric Measure Theory and the Calculus of Variations, Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 1986.

[BW20a] Costante Bellettini and Neshan Wickramasekera. The inhomogeneous Allen–Cahn equation and the existence of prescribed-mean-curvature hypersurfaces. `https://arxiv.org/abs/2010.05847`, 2020.

[BW20b] Costante Bellettini and Neshan Wickramasekera. Stable cmc integral varifolds of codimension 1: regularity and compactness. `https://arxiv.org/abs/1802.00377`, 2020.

[BW20c] Costante Bellettini and Neshan Wickramasekera. Stable prescribed-mean-curvature integral varifolds of codimension 1: regularity and compactness. `https://arxiv.org/abs/1902.09669`, 2020.

[BW24] Costante Bellettini and Myles Workman. Embeddedness of minmax cmc hypersurfaces in manifolds with positive ricci curvature. *Nonlinear Differential Equations and Applications NoDEA*, 31(2), February 2024.

[Cac27] R. Caccioppoli. Sulle quadratura delle superficie piane e curve. *Atti Accad. Naz. Lincei, Rend., VI. Ser.*, 6:142–146, 1927.

[CCMS20] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic initial data. `https://arxiv.org/abs/2003.14344`, 2020.

[CCMS22] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic low-entropy initial data. `https://arxiv.org/abs/2102.11978`, 2022.

[Cho19] Otis Chodosh. Geometric features of the allen-cahn equation. `http://web.stanford.edu/~ochodosh/AllenCahnSummerSchool2019`, 2019.

[CL24] Otis Chodosh and Chao Li. Generalized soap bubbles and the topology of manifolds with positive scalar curvature. *Annals of Mathematics*, 199(2), March 2024.

[CLS22] Otis Chodosh, Yevgeny Liokumovich, and Luca Spolaor. Singular behavior and generic regularity of min-max minimal hypersurfaces. *Ars Inveniendi Analytica*, 2022.

[CM05] Tobias Colding and William Minicozzi. Estimates for the extinction time for the ricci flow on certain 3-manifolds and a question of perelman. *Journal of the American Mathematical Society*, 18(3):561569, April 2005.

[CM11] Tobias Holck Colding and William P Minicozzi. A course in minimal surfaces. *American Mathematical Society*, April 2011.

[CM20] Otis Chodosh and Christos Mantoulidis. Minimal surfaces and the allen–cahn equation on 3-manifolds: index, multiplicity, and curvature estimates. *Annals of Mathematics*, 191(1), January 2020.

[CM23] Otis Chodosh and Christos Mantoulidis. The p-widths of a surface. *Publications mathématiques de lIHÉS*, 137(1):245342, May 2023.

[CMS23a] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Generic regularity for minimizing hypersurfaces in dimensions 9 and 10. `https://arxiv.org/abs/2302.02253`, 2023.

[CMS23b] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Improved generic regularity of codimension-1 minimizing integral currents. `https://arxiv.org/abs/2306.13191`, 2023.

[DG61] Ennio De Giorgi. Frontiere orientate di misura minima. *Seminario di Matematica della Scuola Normale Superiore di Pisa*, 76:1–56, 1961.

[DMS24] Camillo De Lellis, Paul Minter, and Anna Skorobogatova. The fine structure of the singular set of area-minimizing integral currents iii: Frequency 1 flat singular points and $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones. https://arxiv.org/abs/2304.11553, 2024.

[Dou31] Jesse Douglas. Solution of the problem of plateau. *Transactions of the American Mathematical Society*, 33(1):263321, 1931.

[DS11] Camillo De Lellis and Emanuele Spadaro. Q-valued functions revisited. *Memoirs of the American Mathematical Society*, 211(991):00, 2011.

[DS14] Camillo De Lellis and Emanuele Spadaro. Regularity of area minimizing currents I: gradient L^p estimates. *Geometric and Functional Analysis*, 24(6):18311884, November 2014.

[DS16a] Camillo De Lellis and Emanuele Spadaro. Regularity of area minimizing currents II: center manifold. *Annals of Mathematics*, page 499575, March 2016.

[DS16b] Camillo De Lellis and Emanuele Spadaro. Regularity of area minimizing currents III: blow-up. *Annals of Mathematics*, page 577617, March 2016.

[EG15] Lawrence C. Evans and Ronald F. Gariepy. Measure theory and fine properties of functions, revised edition. *Chapman and Hall/CRC*, 2015.

[EM24] Nick Edelen and Paul Minter. Uniqueness of regular tangent cones for immersed stable hypersurfaces. https://arxiv.org/abs/2401.15301, 2024.

[Eva10] Lawrence Evans. *Partial Differential Equations*. American Mathematical Society, March 2010.

[Fed70] Herbert Federer. The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bulletin of the American Mathematical Society*, 76:767–771, 1970.

[Fed96] Herbert Federer. *Geometric Measure Theory*. Springer Berlin Heidelberg, 1996.

[FF60] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *The Annals of Mathematics*, 72(3):458, November 1960.

[FRS20] Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra. Generic regularity of free boundaries for the obstacle problem. *Publications mathématiques de l'IHÉS*, 132(1):181–292, 2020.

[Gas19] Pedro Gaspar. The second inner variation of energy and the Morse index of limit interfaces. *The Journal of Geometric Analysis*, 30(1):6985, January 2019.

[GG18] Pedro Gaspar and Marco A. M. Guaraco. The AllenCahn equation on closed manifolds. *Calculus of Variations and Partial Differential Equations*, 57(4), June 2018.

[GG19] Pedro Gaspar and Marco A. M. Guaraco. The Weyl law for the phase transition spectrum and density of limit interfaces. *Geometric and Functional Analysis*, 29(2):382410, April 2019.

[GL24] Marco A. M. Guaraco and Stephen Lynch. Plateau's problem via the Allen–Cahn functional. `https://arxiv.org/abs/2305.00363`, 2024.

[GMT83] Eduardo Gonzalez, Umberto Massari, and Italo Tamanini. On the regularity of boundaries of sets minimizing perimeter with a volume constraint. *Indiana University Mathematics Journal*, 32(1):25, 1983.

[Gra04] Alfred Gray. *Tubes*. Birkhäuser, 2004.

[Gro79] Misha Gromov. Paul Levy's isoperimetric inequality. `https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/1133.pdf`, 1979.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Berlin Heidelberg, 2001.

[Gua18] Marco A. M. Guaraco. Min–max for phase transitions and the existence of embedded minimal hypersurfaces. *Journal of Differential Geometry*, 108(1), 2018.

[Had98] Jacques Hadamard. Les surfaces à courbures opposées et leurs lignes géodésiques. *Journal de Mathématiques Pures et Appliquées*, 4:27–74, 1898.

[Hie18] Fritz Hiesmayr. Spectrum and index of two-sided allencahn minimal hypersurfaces. *Communications in Partial Differential Equations*, 43(11):15411565, November 2018.

[HS79] Robert Hardt and Leon Simon. Boundary regularity and embedded solutions for the oriented plateau problem. *The Annals of Mathematics*, 110(3):439, November 1979.

[HS85] Robert Hardt and Leon Simon. Area minimizing hypersurfaces with isolated singularities. *Journal für die reine und angewandte Mathematik*, 362:102–129, 1985.

[HT00] John E. Hutchinson and Yoshihiro Tonegawa. Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. *Calculus of Variations and Partial Differential Equations*, 10(1):49–84, 2000.

[IMN18] Kei Irie, Fernando Marques, and André Neves. Density of minimal hypersurfaces for generic metrics. *Annals of Mathematics*, 187(3), May 2018.

[Kol15] Jan Kolář. Non-unique conical and non-conical tangents to rectifiable stationary varifolds in $\mathbb{R}^4$. *Calculus of Variations and Partial Differential Equations*, 54(2):18751909, May 2015.

[Kru14] Brian Krummel. Constant frequency and the higher regularity of branch sets. `https://arxiv.org/abs/1410.7339`, 2014.

[KS89] Robert V. Kohn and Peter Sternberg. Local minimisers and singular perturbations. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 111(1-2):6984, 1989.

[KW23] Brian Krummel and Neshan Wickramasekera. Analysis of singularities of area minimizing currents: planar frequency, branch points of rapid decay, and weak locally uniform approximation. `https://arxiv.org/abs/2304.10653`, 2023.

[Law91] Gary R. Lawlor. A sufficient criterion for a cone to be area-minimizing. *Memoirs of the American Mathematical Society*, 91, 1991.

[Lee18] John M. Lee. *Introduction to Riemannian Manifolds*. Springer International Publishing, 2018.

[Les23] Konstantinos Leskas. PhD thesis, University College London. *https://discovery.ucl.ac.uk/id/eprint/10176621*, 2023.

[Li23] Yangyang Li. Existence of infinitely many minimal hypersurfaces in higher-dimensional closed manifolds with generic metrics. *Journal of Differential Geometry*, 124(2), June 2023.

[Liu19] Zhenhua Liu. Stationary one-sided area-minimizing hypersurfaces with isolated singularities. https://arxiv.org/abs/1904.02289, 2019.

[LW21] Yangyang Li and Zhihan Wang. Generic regularity of minimal hypersurfaces in dimension 8. https://arxiv.org/abs/2012.05401, 2021.

[LW22] Yangyang Li and Zhihan Wang. Minimal hypersurfaces for generic metrics in dimension 8. https://arxiv.org/abs/2205.01047, 2022.

[Mag12] Francesco Maggi. Sets of finite perimeter and geometric variational problems: An introduction to geometric measure theory. *Cambridge University Press*, 2012.

[Man21] Christos Mantoulidis. AllenCahn min-max on surfaces. *Journal of Differential Geometry*, 117(1), January 2021.

[Men11] Ulrich Menne. Second order rectifiability of integral varifolds of locally bounded first variation. *Journal of Geometric Analysis*, 23(2):709763, September 2011.

[MM77] Luciano Modica and Stefano Mortola. Un esempio di Γ^--convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.

[MM02] Carlo Mantegazza and Andrea Carlo Mennucci. Hamilton–Jacobi equations and distance functions on Riemannian manifolds. *Applied Mathematics and Optimization*, 47(1):1–25, 2002.

[MN14] Fernando Marques and André Neves. Min-max theory and the Willmore conjecture. *Annals of Mathematics*, 179(2):683–782, March 2014.

[MN20] Fernando C. Marques and André Neves. *Applications of MinMax Methods to Geometry*, page 4177. Springer International Publishing, 2020.

[Moo06] John Douglas Moore. Bumpy metrics and closed parametrized minimal surfaces in Riemannian manifolds. *Transactions of the American Mathematical Society*, 358(12):5193–5257, 2006.

[Moo17] John Douglas Moore. Introduction to global analysis: Minimal surfaces in Riemannian manifolds. *American Mathematical Soc.*, 2017.

[Mor16] Frank Morgan. *Geometric Measure Theory*. Elsevier, 2016.

[NV20] Aaron Naber and Daniele Valtorta. The singular structure and regularity of stationary varifolds. *Journal of the European Mathematical Society*, 22(10):33053382, July 2020.

[Oss70] Robert Osserman. A proof of the regularity everywhere of the classical solution to plateaus problem. *The Annals of Mathematics*, 91(3):550, May 1970.

[Per08] G. Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. *Topologica*, 1(1):005, 2008.

[Pit81] Jon T. Pitts. *Existence and Regularity of Minimal Surfaces on Riemannian Manifolds. (MN-27)*. Princeton University Press, December 1981.

[PS20] Alessandro Pigati and Daniel Stern. Minimal submanifolds from the abelian higgs model. *Inventiones mathematicae*, 223(3):10271095, September 2020.

[Rad30] Tibor Radó. On plateaus problem. *Annals of Mathematics*, 31(3):457, July 1930.

[Rei60] E. R. Reifenberg. Solution of the Plateau problem for m-dimensional surfaces of varying topological type. *Acta Mathematica*, 104(12):192, 1960.

[RT07] Matthias Röger and Yoshihiro Tonegawa. Convergence of phase-field approximations to the Gibbs–Thomson law. *Calculus of Variations and Partial Differential Equations*, 32(1):111–136, 2007.

[Sak96] Takashi Sakai. *Riemannian Geometry*. American Mathematical Society, 1996.

[Sav10] O. Savin. Phase transitions, minimal surfaces and a conjecture of de Giorgi. In *Current developments in mathematics, 2009*, pages 59–113. Somerville, MA: International Press, 2010.

[Sim68] James Simons. Minimal varieties in Riemannian manifolds. *Annals of Mathematics*, 88:62, 1968.

[Sim83] Leon Simon. Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. *Annals of Mathematics*, 118(3):525, 1983.

[Sim84] Leon Simon. Lectures on geometric measure theory. *Proceedings of the Centre for Mathematical Analysis 3, Canberra, VII+272*, 1984.

[Sim93] Leon Simon. Cylindrical tangent cones and the singular set of minimal submanifolds. *Journal of Differential Geometry*, 38(3), January 1993.

[Sim23] Leon Simon. Stable minimal hypersurfaces in $\mathbb{R}^{N+1+\ell}$ with singular set an arbitrary closed $K \subset \{0\} \times \mathbb{R}^{\ell}$. *Annals of Mathematics*, 197(3), May 2023.

[Sma93] Nathan Smale. Generic regularity of homologically area minimizing hypersurfaces in eight dimensional manifolds. *Communications in Analysis and Geometry*, 1:217–228, 1993.

[Son23] Antoine Song. Existence of infinitely many minimal hypersurfaces in closed manifolds. *Annals of Mathematics*, 197(3), May 2023.

[SS81] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. *Communications on Pure and Applied Mathematics*, 34(6):741–797, November 1981.

[SSY75] R. Schoen, L. Simon, and S. T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Mathematica*, 134(0):275288, 1975.

[ST20] Salvatore Stuvard and Yoshihiro Tonegawa. Dynamical instability of minimal surfaces at flat singular points. `https://arxiv.org/abs/2008.13728`, 2020.

[SY79a] Richard Schoen and Shing-Tung Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45–76, February 1979.

[SY79b] Richard Schoen and Shing-Tung Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta mathematica*, 28:159–184, 1979.

[Szé21] Gábor Székelyhidi. Minimal hypersurfaces with cylindrical tangent cones. `https://arxiv.org/abs/2107.14786`, 2021.

[Ton05] Yoshihiro Tonegawa. On stable critical points for a singular perturbation problem. *Communications in Analysis and Geometry*, 13(2):439459, 2005.

[TW12] Yoshihiro Tonegawa and Neshan Wickramasekera. Stable phase interfaces in the van der Waals–Cahn–Hilliard theory. *Journal für die reine und angewandte Mathematik*, 2012(668), January 2012.

[Whi85] Brian White. Generic regularity of unoriented two-dimensional area minimizing surfaces. *Annals of Mathematics*, 121(3):595, 1985.

[Whi91] Brian White. The space of minimal submanifolds for varying Riemannian metrics. *Indiana University Mathematics Journal*, 40:161–200, 1991.

[Whi15] Brian White. On the bumpy metrics theorem for minimal submanifolds. *American Journal of Mathematics*, 139(4):1149 – 1155, 2015.

[Whi19] Brian White. Generic transversality of minimal submanifolds and generic regularity of two-dimensional area-minimizing integral currents. `https://arxiv.org/abs/1901.05148`, 2019.

[Wic14a] Neshan Wickramasekera. A general regularity theory for stable codimension 1 integral varifolds. *Annals of Mathematics*, 179(3):843–1007, 2014.

[Wic14b] Neshan Wickramasekera. Regularity of stable minimal hypersurfaces: Recent advances in the theory and applications. *Surveys in differential geometry*, 19:231–301, 2014.

[WW19] Kelei Wang and Juncheng Wei. Finite morse index implies finite ends. *Communications on Pure and Applied Mathematics*, 72(5):1044–1119, 2019.

[Zho17] Xin Zhou. Minmax hypersurface in manifold of positive ricci curvature. *Journal of Differential Geometry*, 105(2), February 2017.