FORMALISING LINEAR ELLIPTIC PDE THEORY IN LEAN 4

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Abstract

We formalise in Lean 4, on top of Mathlib, the solvability of the Dirichlet problem for second-order linear elliptic operators in divergence form. The machine-verified results, with no `sorry` in the development, include the Poincaré inequality, the existence of weak solutions by the Lax–Milgram theorem, Rellich–Kondrachov compactness, the Fredholm alternative, the spectral theorem, interior regularity estimates, and the Sobolev embedding theorem. From these results we obtain a formalisation of classical solvability for sufficiently regular coefficients and data. Our Lean library includes a self-contained theory of Sobolev spaces developed independently of existing formalisations. Throughout the paper we associate each prose statement with the named machine-checked Lean declaration that discharges it.

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Contents

1 Introduction

Mathematical analysis has recently come within reach of interactive proof assistants, but the formalisation of partial differential equations (PDE) theory is still at an early stage. [AK26] formalised the interior De Giorgi–Nash–Moser regularity theory in Lean 4 (see [MU21]) on top of Mathlib (see [The20]), building in the process a library for Sobolev spaces on bounded domains. The surrounding analytic infrastructure has grown alongside it. A non-exhaustive list of examples includes formalisations of the Gagliardo–Nirenberg–Sobolev inequality in [DM24], Schwartz functions, tempered distributions, and Sobolev spaces on Euclidean space through the Fourier transform in [Dol25], as well as the h-principle and sphere eversion in [MDN22]. We also mention that, beyond Lean, a Coq proof of the Lax–Milgram theorem was formalised in [Bol+17]. These developments suggest that a substantial part of modern analysis can now be placed on machine-verified footing.

To this end, the present work concerns formalisation, in Lean 4 on top of Mathlib, of the solvability of the Dirichlet problem for general second-order linear elliptic operators in divergence form. This theory is central to modern PDE, with its treatment via functional-analytic methods by now standard (e.g. see [GT01; Eva10; Bre11]). While we focus on solvability for the Dirichlet problem specifically, we expect that the various tools developed here bring much of the remaining standard theory for linear elliptic PDE (boundary regularity, variational methods, the Neumann problem, etc.) well within reach of formalisation within Lean.

In certifying a formal proof, a reader must confirm that the Lean code provided indeed expresses the intended statement. We therefore accompany our formalisation with a transcription from Lean to mathematical prose; each of the key results is stated in standard mathematical prose and accompanied by the Lean statement of the declaration that proves it.

1.1 Mathematical preliminaries

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set. We consider second-order linear partial differential operators L in divergence form, with coefficients $a^{ij}, b^i, c \in L^\infty(\Omega)$ for each integer $1 \le i, j \le n$, acting on functions $u \in C^2(\Omega)$ by

$$Lu = -\sum_{j=1}^{n} D_j\Big(\sum_{i=1}^{n} a^{ij}\, D_i u\Big) + \sum_{i=1}^{n} b^i\, D_i u + c\, u. \tag{1}$$

We assume that L is uniformly elliptic in the sense that the matrix of coefficients a^{ij} is uniformly positive definite and bounded. Equivalently, we assume that there exist constants $0 < \lambda \le \Lambda < \infty$ such that for each $\xi \in \mathbb{R}^n$

$$0 < \lambda|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\xi^i\xi^j \le \Lambda|\xi|^2 \tag{2}$$

for almost every $x \in \Omega$. Given $f \in L^2(\Omega)$, we address the solvability, both weakly and classically, of the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Multiplying the equation $Lu = f$ by a test function and integrating by parts yields a weak formulation of the Dirichlet problem on the Sobolev space $H^1_0(\Omega)$. Namely, we say that a function $u \in H^1_0(\Omega)$ is a weak solution of (3) if

$$B[u,v] = \langle f, v\rangle_{L^2(\Omega)} \qquad \text{for all } v \in H^1_0(\Omega), \tag{4}$$

where the bilinear form $B\colon H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ associated with L (e.g. see [Eva10, §6.1.2]) is defined by

$$B[u,v] = \sum_{i,j=1}^{n} \int_\Omega \big(a^{ij}\, D_i u\, D_j v + b^i (D_i u)\, v + c\, uv\big). \tag{5}$$

If in addition $a^{ij} \in C^1(\Omega)$ and $u \in C^2(\Omega)$, then by the fundamental lemma of the calculus of variations a weak solution of (3) satisfies $Lu = f$ almost everywhere in Ω and is in this sense a classical solution of (3), whose boundary condition is expressed by the membership $u \in H^1_0(\Omega)$.

A common approach to the classical solvability of (3) proceeds in two stages. One first obtains a weak solution by functional-analytic methods, namely the Lax–Milgram theorem and the Fredholm alternative, and then shows by interior regularity estimates and the Sobolev embedding theorem that this weak solution is a classical solution whenever the coefficients and the data are sufficiently regular. We formalise this approach, largely following the presentation of [Eva10, Chapter 6], adapting several of the proofs from the notes of [Guo26, Chapters 1–3] (based on a course on elliptic PDE taught by the second-named author).

1.2 Formalisation contributions

We provide a Lean 4 library, built on top of Mathlib, containing the formalisation of statements leading to the classical solvability of (3). The Lean 4 library that accompanies this paper is referred to as the **library** throughout the manuscript. This is a standalone Mathlib-based development containing each declaration against which the statements of Section 2 are discharged. The structure of the library is further described in Section 3 and pinned at commit `da8dd98` by our code availability statement. The library can be found here:

`https://github.com/alejandro-soto-franco/EllipticPDE`

The main contributions contained in the library are the following:

1. A self-contained weak-derivative Sobolev layer, developed independently of the one in [AK26], together with a formalised account built on top of it of the weak solvability of (3). The precise statements formalised are detailed in Subsection 2.1.

2. Higher interior regularity for weak solutions of (3) and the Sobolev embedding, leading to classical solvability. The precise statements formalised are detailed in Subsection 2.2.

3. An explicit correspondence that associates with each prose statement the named machine-checked Lean declaration proving it. Section 2 transcribes this correspondence by printing each Lean statement between the result that it proves and the sketch of its proof. The definitions used in these statements are collected in Appendix A.

1.3 Formalisation framework

Each step of a proof in Lean gives rise to an obligation to which we assign one of four statuses. An obligation is *discharged* if a machine-checked Lean proof term establishes it, so that its correctness depends on nothing beyond the kernel and the stated axioms. It is *warranted* if it is deferred to a located external source, *routine* if it relies on shared mathematical competence, and *open* if no justification for it has yet been given. The discharge of an obligation thus has two components, namely the machine check of the type of the declaration and a human audit confirming that the prose statement agrees with that type. A proof is called *formal* if every one of its steps is discharged, so that no obligation remains.

The library was written with the language models Claude Opus 4.5 and Claude Opus 5, used as agents in the Claude Code harness under the direction of the first-named author. The authors chose the target theorems, the weak formulation, the encoding of $H^1_0(\Omega)$ as a graph, the coefficient bundle, the route of each proof, and compared each Lean type with the corresponding prose statement. The agents searched Mathlib and the library for existing declarations and their exact types, decomposed each target into named auxiliary declarations, attempted the tactic proofs, and proposed improvements. No output of a model forms part of the trusted proof object, since the Lean kernel checks every proof term by the same rules irrespective of how the term was produced.

The criteria for acceptance were mechanical and can be reproduced from the pinned commit. A proposed change was admitted to the library only if the whole development built from a clean clone with no `sorry` and no warning, the environment linter passed, and `#print axioms` reported for every named declaration exactly the three standard axioms of the Lean core library (namely propositional extensionality (`propext`), the axiom of choice (`Classical.choice`), and quotient soundness (`Quot.sound`)). The library pins those reports at every named declaration, so a dependence on a further axiom fails the build.

1.4 Structure

The paper is organised as follows. In Section 2 we state the main formalised results. Each result is followed by the Lean statement of the declaration that proves it along with a sketch of proof naming the declarations on which the formal proof depends. Section 3 describes the structure of the Lean library. Section 4 concludes the paper with the discharge ratio, the part of the audit that the kernel cannot perform, the relation to prior work and directions for further research. Appendix A gives the Lean code for each definition used in the statements.

Declarations

Code availability. The Lean library is openly available at `https://github.com/alejandro-soto-franco/EllipticPDE`. An archival release with a persistent identifier will be deposited upon publication. The version described here is commit `da8dd98`, built with Lean 4 v4.31.0-rc1 and Mathlib commit `542645a`.

Author contributions. Both authors contributed to the mathematical design and to the writing of the manuscript. The first-named author carried out the Lean formalisation and its transcription into mathematical prose.

Use of language-model assistance. The Lean library and this manuscript were written with the assistance of language-model agents under the direction of the first-named author. No output of a language model forms part of the trusted proof object. The models used, the division of work and the acceptance checks are described in Subsection 1.3.

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2 Formalised statements

In this section we state the formalised results that lead to the classical solvability of (3). Subsection 2.1 gives the statements that lead to weak solvability, and Subsection 2.2 those that lead to the regularity of weak solutions. We state only those formalised results leading directly to the classical solvability of (3), and so do not survey every formalised result in the library.

Each result is stated in standard mathematical language together with a citation to a standard treatment that it formalises. The Lean statement of the declaration that proves it follows the result and is itself followed by a sketch of the proof. The proof sketches follow the cited proof and say where the formal proof departs from it. This happens when a hypothesis is replaced by one that is more convenient to state in Lean, when a constant that the cited proof leaves implicit has to be produced, or when an argument is reorganised as an induction that the proof assistant can check. The sketch also places the result in the formal development by naming in footnotes the auxiliary declarations on which its proof depends, so that the adaptations of the cited proofs can be traced to the library.

2.1 Weak solvability of the Dirichlet problem

We begin with the Poincaré inequality on bounded sets:

Theorem 1 (Poincaré inequality, [Eva10, §5.6.1, Theorem 3]). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set. Then there is a constant $C_P \geq 0$, depending only on n and Ω, such that*

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)} \qquad \text{for all } u \in H_0^1(\Omega).$$

`poincare_H01_of_bounded` The Lean theorem is stated as

```
theorem poincare_H01_of_bounded {Ω : Set (EuclideanSpace ℝ (Fin (n + 1)))}
    (hΩb : Bornology.IsBounded Ω) :
    ∃ C : ℝ, 0 ≤ C ∧ ∀ (U : H1amb Ω), U ∈ H01 Ω →
      ‖U 0‖ ^ 2 ≤ C * ∑ i : Fin (n + 1), ‖U i.succ‖ ^ 2
```

Proof sketch. Whereas [Eva10, §5.6.1, proof of Theorem 3] deduces the inequality from the Gagliardo–Nirenberg–Sobolev inequality, which requires $p < n$, the formal proof reduces it to one dimension on a box and obtains an explicit constant in every dimension (as in the statement of [Bre11, Corollary 9.19]). Let $Q = \prod_i (a_i, b_i)$ be such a box. On each line parallel to the ith axis a test function on Q is compactly supported in (a_i, b_i), so that by the one-dimensional inequality of Lemma 2 (proved in Section 3 together with the Cauchy–Schwarz inequality of Lemma 1 on which it rests), the integral of its square along that line is bounded by $\frac{(b_i - a_i)^2}{2}$ multiplied by the integral of $(\partial_i u)^2$ along it. Integrating over the remaining directions by Fubini's theorem, we obtain the corresponding bound on Q for each direction. Averaging these n bounds with $L = \max_i(b_i - a_i)$ gives the inequality on $C_c^\infty(Q)$ with $C_P = L/\sqrt{2n}$, from which the inequality on $H_0^1(Q)$ follows by approximation. To pass from a box to a bounded set Ω we enclose it in a box Q as above. A test function on Ω is then a test function on Q whose integrals over the two sets coincide. Hence the constant for Q is also admissible for Ω. The formalised statement is in the squared form $\|u\|^2_{L^2(\Omega)} \le C\|\nabla u\|^2_{L^2(\Omega)}$ with $C = C_P^2$ and asserts only the existence of C, whereas the statement for boxes gives the admissible value $C = L^2/(2n)$. Every constant quoted below is the unsquared C_P.[1] $\square$

We state the Lax–Milgram theorem next:

Theorem 2 (Lax–Milgram theorem, [Eva10, §6.2.1, Theorem 1]). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $B\colon H \times H \to \mathbb{R}$ be a bounded coercive bilinear form, with $B[u,u] \ge \beta\|u\|_H^2$ for some $\beta > 0$, and $f \in H^*$. Then, there is a unique $u \in H$ such that*

$$B[u,v] = \langle f, v\rangle \qquad \textit{for all } v \in H.$$

`lax_milgram` The Lean theorem is stated as

```
theorem lax_milgram {H : Type*} [NormedAddCommGroup H] [InnerProductSpace ℝ H]
    [CompleteSpace H] {B : H →L[ℝ] H →L[ℝ] ℝ} (hB : IsCoercive B) (f : H →L[ℝ] ℝ) :
    ∃! u : H, ∀ v : H, B u v = f v
```

Proof sketch. Since B is bounded, $B[u,\cdot\,]$ is a continuous linear functional for each $u \in H$. The Riesz representation theorem therefore yields a bounded linear operator $A\colon H \to H$ with $\langle Au, v\rangle = B[u,v]$ for every $v \in H$. By coercivity, $\beta\|u\|_H^2 \le B[u,u] = \langle Au, u\rangle \le \|Au\|_H\|u\|_H$, so A is injective with closed range. Applying the same bound on the orthogonal complement of the range, we deduce that A is surjective. If g denotes the Riesz representative of f, the solution is $u = A^{-1}g$. A second solution u' satisfies $\langle Au', w\rangle = \langle g, w\rangle$ for every w, whence $Au' = Au$ and $u' = u$. No hypothesis is imposed on B apart from boundedness and coercivity, which the Lean statement expresses respectively by the continuity built into the type of B and by Mathlib's predicate `IsCoercive`, as elsewhere in the library. The solution satisfies $\|u\|_H \le \beta^{-1}\|f\|_{H^*}$, as follows by combining coercivity with $B[u,u] = \langle f, u\rangle \le \|f\|_{H^*}\|u\|_H$. Each a priori bound below is this estimate applied with the coercivity constant of the form in question.[2] $\square$

We first apply Theorem 2 to the Dirichlet problem for the Laplacian.

Corollary 1 (Weak solvability of the Poisson equation, [Eva10, §6.2.2]). *Let $\Omega \subseteq \mathbb{R}^n$ and let $C \ge 0$ satisfy the test-function bound $\|\varphi\|^2_{L^2(\Omega)} \le C\sum_{i=1}^n \|\partial_i\varphi\|^2_{L^2(\Omega)}$ for every $\varphi \in C_c^\infty(\Omega)$. Then, for every $f \in H^{-1}(\Omega)$ there is a unique $u \in H_0^1(\Omega)$ with $\sum_{i=1}^n \langle \partial_i u, \partial_i v\rangle_{L^2(\Omega)} = f(v)$ for every $v \in H_0^1(\Omega)$.*

`poisson_weak_solution` The Lean theorem is stated as

```
theorem poisson_weak_solution (Ω : Set (EuclideanSpace ℝ (Fin d))) (C : ℝ) (hC : 0 ≤ C)
    (hbase : ∀ {φ : EuclideanSpace ℝ (Fin d) → ℝ} (h : IsTestFn Ω φ),
      ‖(h.testGraph 0 : L2D Ω)‖ ^ 2 ≤ C * ∑ i : Fin d, ‖h.testGraph i.succ‖ ^ 2)
    (f : H01 Ω →L[ℝ] ℝ) :
    ∃! u : H01 Ω, ∀ v : H01 Ω, laplaceBilin Ω u v = f v
```

[1] `EllipticPdes.Poincare.poincare_H01_of_bounded`, with the statement for boxes `poincare_H01_euclBox`.

[2] `EllipticPdes.lax_milgram`, with the estimate `EllipticPdes.norm_weak_solution_le`.

Proof sketch. We apply Theorem 2 to $B[u,v] = \sum_{i=1}^n \langle \partial_i u, \partial_i v\rangle_{L^2(\Omega)}$ with $H = H_0^1(\Omega)$. Since each of the n terms of B is bounded by the product of the norms by the Cauchy–Schwarz inequality, $|B[u,v]| \le n\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}$. Both $B[u,u] = \sum_{i=1}^n \|\partial_i u\|_{L^2(\Omega)}^2$ and the assumed test-function bound, extended to $H_0^1(\Omega)$ by density, imply that $B[u,u] \ge (C+1)^{-1}\|u\|_{H_0^1(\Omega)}^2$. We then conclude by applying Theorem 2 with $\beta = (C+1)^{-1}$. The formal statement takes the test-function bound as a hypothesis and imposes no condition on Ω, so that it also applies to unbounded sets on which such a bound holds. The hypothesis enters the proof only through the coercivity estimate.[3] □

To obtain coercivity for more general operators of the form (1) we have:

Theorem 3 (Gårding inequality, [Eva10, §6.2.2, Theorem 2])**.** *Let $\Omega \subseteq \mathbb{R}^n$. Let L be the operator* (1) *with ellipticity constant λ as in* (2)*. If $\gamma = \lambda/2 + \|c\|_{L^\infty(\Omega)} + n\,(\max_i \|b^i\|_{L^\infty(\Omega)})^2/(2\lambda)$, then*

$$\tfrac{\lambda}{2}\|u\|_{H_0^1(\Omega)}^2 \le B[u,u] + \gamma\|u\|_{L^2(\Omega)}^2 \qquad \textit{for all } u \in H_0^1(\Omega).$$

`garding` The Lean theorem is stated as

```
theorem garding (Ω : Set (EuclideanSpace ℝ (Fin d))) (U : H01 Ω) :
    Op.lam / 2 * ‖U‖ ^ 2
      ≤ Op.fullBilin Ω U U + Op.gardingγ * ‖(U : H1amb Ω) 0‖ ^ 2
```

Proof sketch. One first shows that B is bounded, which follows by the pointwise bounds on the coefficients[4]. By ellipticity, for the principal part we see that $B_A[u,u] = \sum_{i,j}\langle a^{ij}\,\partial_i u, \partial_j u\rangle \ge \lambda\|\nabla u\|_{L^2(\Omega)}^2$[5]. By the Peter–Paul inequality, the first order terms of B are absorbed, and the zeroth-order terms are controlled pointwise by $\|c\|_{L^\infty(\Omega)}$. The desired inequality follows by using these facts on the difference between B_A and B. Whereas the cited statement leaves γ unspecified, the formal statement gives its value.[6] □

It follows from the proof above that for every $\mu \ge \gamma$ the bilinear form $B_\mu = B + \mu\langle\cdot,\cdot\rangle_{L^2}$ is coercive, with Theorem 2 then providing a unique weak solution $u \in H_0^1(\Omega)$ of the equation $Lu + \mu u = f$. If $b = 0$ and $c \ge 0$ on a bounded set Ω then B is coercive and thus:

Theorem 4 (Existence I, [Eva10, §6.2.2, Theorem 3 and the Examples following it])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set. Let L be an operator of the form* (1) *which is uniformly elliptic with constants λ, Λ in the sense of* (2)*, with $b = 0$ almost everywhere, and with $c \ge 0$ almost everywhere. Let C_P be the Poincaré constant of Theorem 1 and set $\alpha = \lambda/(1 + C_P^2)$. Then for every $f \in L^2(\Omega)$ the problem* (3) *has a unique weak solution $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} \le \alpha^{-1}\|f\|_{L^2(\Omega)}$.*

`weak_solution_L2_of_nonneg_zeroth_of_bounded` The Lean theorem is stated as

```
theorem weak_solution_L2_of_nonneg_zeroth_of_bounded {n : ℕ}
    (Op : FullEllipticOp (n + 1)) {Ω : Set (EuclideanSpace ℝ (Fin (n + 1)))}
    (hΩb : Bornology.IsBounded Ω)
    (hb : ∀ i, ∀ᵐ x ∂(volume.restrict Ω), Op.b x i = 0)
    (hc : ∀ᵐ x ∂(volume.restrict Ω), 0 ≤ Op.c x) :
    ∃ CP : ℝ, 0 ≤ CP ∧ ∀ f : L2D Ω,
```

[3] `EllipticPdes.poisson_weak_solution`, with coercivity proved in `laplaceBilin_coercive`.
[4] `EllipticPdes.Sobolev.EllipticCoeff.bilin` whose bound is part of its construction.
[5] `EllipticPdes.Sobolev.EllipticCoeff.bilin_coercive`.
[6] `garding` which admits any almost-everywhere bounds on b and c in place of their essential suprema.

```
((∃! u : H01 Ω, ∀ v : H01 Ω,
  Op.fullBilin Ω u v = ∫ x in Ω, (f x : ℝ) * ((v : H1amb Ω) 0 x : ℝ))
∧ ∀ u : H01 Ω,
    (∀ v : H01 Ω,
      Op.fullBilin Ω u v = ∫ x in Ω, (f x : ℝ) * ((v : H1amb Ω) 0 x : ℝ)) →
      ‖u‖ ≤ (CP + 1) / Op.lam * ‖f‖)
```

Proof sketch. The Lean statement names the constant of the squared Poincaré inequality as C_P^2 and therefore writes α^{-1} as $(C_P^2+1)/\lambda$. The coercivity argument requires the inequality on all of $H_0^1(\Omega)$. Let $\Lambda_f \in H_0^1(\Omega)^*$ be the functional $v \mapsto \int_\Omega fv$. We apply Theorem 2 to B on $H_0^1(\Omega)$. Boundedness follows from the upper bound Λ on A and the bounds on the suprema of b and c. Since $b=0$ and $c \geq 0$ almost everywhere, coercivity of B with constant α follows from the lower bound λ together with the Poincaré inequality. Theorem 2 then yields a unique u with $B[u,v] = \Lambda_f(v)$ for every $v \in H_0^1(\Omega)$, which is the integral form of the statement. Every solution then satisfies $\|u\| \leq \alpha^{-1}\|\Lambda_f\|$. Since the Cauchy–Schwarz inequality gives $\|\Lambda_f\| \leq \|f\|_{L^2(\Omega)}$, the estimate follows. A sharper constant obtained from the geometry of a particular Ω, such as the Friedrichs constant $C_P \leq d/\pi$ of a convex domain of diameter d, may be used in the abstract statement without change.[7] □

For solvability in the absence of coercivity we have:

Theorem 5 (Existence II, [Eva10, §6.2.3, Theorem 4(i)]). *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and measurable and let L be the operator* (1), *uniformly elliptic in the sense of* (2). *Then either the homogeneous Dirichlet problem ($f=0$) has a non-zero weak solution $u \in H_0^1(\Omega)$, or for every $f \in H^{-1}(\Omega)$ the problem $Lu = f$ has a unique weak solution $u \in H_0^1(\Omega)$.*

`FullEllipticOp.fredholm_alternative_of_bounded` The Lean theorem is stated as

```
theorem FullEllipticOp.fredholm_alternative_of_bounded (Op : FullEllipticOp d)
    (Ω : Set (EuclideanSpace ℝ (Fin d))) (hΩm : MeasurableSet Ω)
    (hΩb : Bornology.IsBounded Ω) :
    (∃ u : H01 Ω, u ≠ 0 ∧ ∀ v : H01 Ω, Op.fullBilin Ω u v = 0)
      ∨ (∀ f : H01 Ω →L[ℝ] ℝ, ∃! u : H01 Ω, ∀ v : H01 Ω, Op.fullBilin Ω u v = f v)
```

Proof sketch. The library proves the Rellich–Kondrachov theorem, that is, the compactness of the embedding $\iota\colon H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, on every $\Omega \subseteq \mathbb{R}^n$ which is bounded[8]. We use three bounded operators on $H_0^1(\Omega)$: write $\mathcal{A}$ for the Riesz representative of B, E for the Riesz representative of $B_\gamma = B + \gamma\langle\cdot,\cdot\rangle_{L^2(\Omega)}$, and $T = \iota^*\iota$. Since B_γ is coercive, E is invertible by Theorem 2. Since $\mathcal{A} = E - \gamma T$ we write

$$\mathcal{A} = E\,(1-K), \qquad K = \gamma\, E^{-1} T, \tag{6}$$

where K is compact since $T = \iota^*\iota$.[9]

By (6), weak solvability of (3) reduces to solving $(1-K)u = h$ with $h = E^{-1}g$ and g the Riesz representative of f. Since K is compact, 1 is either an eigenvalue of K or a point of its resolvent set. If $Kx = x$ for some $x \neq 0$, then $\mathcal{A}x = E(1-K)x = 0$, so $B[x,v] = \langle \mathcal{A}x, v\rangle = 0$ for every $v \in H_0^1(\Omega)$ and x is a non-zero solution of the homogeneous problem. On the other hand, if 1 lies in the resolvent set of K, then $1-K$ is bijective, so that $\mathcal{A} = E(1-K)$ is bijective as the composition

[7] `EllipticPdes.Sobolev.FullEllipticOp.weak_solution_L2_of_nonneg_zeroth_of_bounded`, based on `weak_solution_of_nonneg_zeroth` and `weak_solution_of_nonneg_zeroth_bound`.

[8] `EllipticPdes.Sobolev.embW12_isCompact`

[9] `opA`, `opT`, `opE`, `opK`, `opA_factor`.

of two bijections. We then see that $\mathcal{A}u = g$ holds exactly when $B[u, v] = f(v)$ for every $v \in H_0^1(\Omega)$, so the inhomogeneous problem has a unique weak solution for every $f \in H^{-1}$. Evans takes $f \in L^2(\Omega)$, whereas the formal statement admits every element of the dual space and asserts the disjunction of the two alternatives without their mutual exclusion.[10]

The library also formalises a characterisation of H^{-1} functions so that the data in the solvability statements (here and the one below) may be presented as a tuple of L^2 functions: if $\Omega \subseteq \mathbb{R}^n$ is open, every $f \in H^{-1}(\Omega)$ is represented as $\langle f, v\rangle = \int_\Omega f_0 v - \sum_{i=1}^n \int_\Omega f_i D_i v$ by functions $f_0, \ldots, f_n \in L^2(\Omega)$ with $\|f\|_{H^{-1}(\Omega)} = (\int_\Omega \sum_{i=0}^n |f_i|^2)^{1/2}$.[11] □

If we write $N = \{u \in H_0^1(\Omega) : B[u, v] = 0 \text{ for all } v\}$ and $N^* = \{u \in H_0^1(\Omega) : B[v, u] = 0 \text{ for all } v\}$ for the spaces of homogeneous solutions to the Dirichlet problem for L and its formal adjoint L^*:

Theorem 6 (Existence II, [Eva10, §6.2.3, Theorem 4(ii)+(iii)]). *With the same hypotheses as Theorem 5:*

1. *The space N is finite-dimensional;*
2. $\dim N^* = \dim N$*;*
3. *For every $f \in H^{-1}(\Omega)$, the problem $Lu = f$ has a weak solution if and only if $\langle f, w\rangle_{L^2(\Omega)} = 0$ for every $w \in N^*$.*

`solvable_iff_orthogonal_solSpaceStar` The Lean theorem is stated as

```
theorem solvable_iff_orthogonal_solSpaceStar (hK : IsCompactOperator (Op.opK Ω))
    (f : H01 Ω →L[ℝ] ℝ) :
    (∃ u : H01 Ω, ∀ v : H01 Ω, Op.fullBilin Ω u v = f v)
      ↔ ∀ w ∈ Op.solSpaceStar Ω, f w = 0
```

Proof sketch. All three clauses follow from the Riesz theory of compact perturbations of the identity applied to the relation (6). The space N is the eigenspace of K at the eigenvalue 1, the identity of dimensions is the Fredholm index theorem in the Hilbert space setting and the solvability criterion is the closed-range property of $1 - K$. Recall that L^* denotes the formal adjoint defined by

$$L^* v = -D_i\big(a^{ij} D_j v\big) - b^i D_i v + \big(c - D_i b^i\big) v,$$

whose bilinear form B^* satisfies $B[u, v] = B^*[v, u]$. The formal statement places the compactness hypothesis on K, which follows from the compactness of $T = \iota^* \iota$.[12] □

The results stated above hold for arbitrary second order coefficients with no assumption of symmetry; only the spectral results below require these coefficients to be symmetric. We have the variational characterisation of the first eigenvalue via the Rayleigh quotient:

---

[10]`EllipticPdes.Sobolev.FullEllipticOp.fredholm_alternative_of_bounded`, based on `fredholm_alternative` with the compactness of K as a hypothesis and on `embL2_isCompact`.

[11]`EllipticPdes.hneg_characterization`.

[12]`EllipticPdes.Sobolev.FullEllipticOp.finiteDimensional_solSpace`, `EllipticPdes.Sobolev.FullEllipticOp.finrank_solSpaceStar_eq_finrank_solSpace` and `EllipticPdes.Sobolev.FullEllipticOp.solvable_iff_orthogonal_solSpaceStar`.

Theorem 7 (Variational principle for the principal eigenvalue, [Eva10, §6.5.1, Theorem 2])**.** *Let B be symmetric and coercive on $H^1_0(\Omega)$ with the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ compact. Set*

$$\lambda_1 = \inf\big\{B[u,u] \ : \ u \in H^1_0(\Omega),\ \|u\|_{L^2(\Omega)} = 1\big\}.$$

Then, $\lambda_1 > 0$, and the infimum is attained at some u with $\|u\|_{L^2(\Omega)} = 1$ satisfying

$$B[u,v] = \lambda_1 \langle u, v\rangle_{L^2(\Omega)} \qquad \text{for all } v \in H^1_0(\Omega).$$

`exists_principal_eigenpair` The Lean theorem is stated as

```
theorem exists_principal_eigenpair (hco : IsCoercive B) (hsymm : ∀ U V : H01 Ω, B U V = B V U)
    (hRellich : IsCompactOperator (embL2 Ω)) (hne : ∃ V : H01 Ω, embL2 Ω V ≠ 0) :
    ∃ U : H01 Ω, ‖embL2 Ω U‖ = 1 ∧ B U U = principalEigenvalue B ∧
      ∀ V : H01 Ω, B U V = principalEigenvalue B * ⟪embL2 Ω U, embL2 Ω V⟫
```

Proof sketch. The proof uses the direct method of the calculus of variations. By coercivity a minimising sequence is bounded, and by weak sequential compactness has a weak limit satisfying the constraint by the compact embedding. The weak lower semicontinuity of B follows by expanding $0 \le B[u_k - w, u_k - w]$ and using $B[u_k, w] \to B[w,w]$. Whereas Evans states the principle for the symmetric operator with no lower-order terms on a bounded connected open set and also proves that the minimiser is positive and simple, the formal statement concerns an abstract symmetric coercive form and takes the compactness of the embedding as a hypothesis.[13]

For the Laplacian, $B[u,v] = \int_\Omega Du \cdot Dv$ and λ_1 is the largest constant for which $\lambda_1 \|u\|^2_{L^2(\Omega)} \le \int_\Omega |Du|^2$ holds on all of $H^1_0(\Omega)$. Since equality is attained by some u of unit L^2 norm, it is the best such constant, whereas Theorem 1 gives a constant only in terms of a box containing Ω.[14] □

The same compactness yields a complete family of eigenfunctions:

Corollary 2 (Completeness of the eigenspaces, [Eva10, §6.5.1, Theorem 1])**.** *Suppose that B is symmetric and coercive, and the embedding $\iota : H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Let $G = \iota \circ (B^\sharp)^{-1} \circ \iota^*$ be the solution operator on $L^2(\Omega)$, where $B^\sharp$ is the Riesz representative of B on $H^1_0(\Omega)$ and ι^* is the adjoint of the embedding. Then the eigenspaces of G span a dense subspace of $L^2(\Omega)$. Each eigenpair $G\varphi = \mu\varphi$ with $\mu \neq 0$ lifts to a weak eigenfunction $u \in H^1_0(\Omega)$ of L at the elliptic eigenvalue μ^{-1}.*

`solOp_spectral` The Lean theorem is stated as

```
theorem solOp_spectral (hco : IsCoercive B) (hsymm : ∀ U V, B U V = B V U)
    (hRellich : IsCompactOperator (embL2 Ω)) :
    (⨆ μ : ℝ, Module.End.eigenspace (solOp B hco : Module.End ℝ (L2D Ω)) μ)ᗮ = ⊥
```

[13] `EllipticPdes.Sobolev.exists_principal_eigenpair`, with the minimality `EllipticPdes.Sobolev.principalEigenvalue_le_of_weak_eigen` and the positivity `EllipticPdes.Sobolev.principalEigenvalue_pos`. For the form of the Laplacian on a bounded measurable domain, coercivity follows from the Poincaré inequality and compactness of the embedding from the Rellich theorem (`EllipticPdes.Sobolev.dirichlet_principal_eigenpair_of_bounded`). On the unit ball of $\mathbb{R}^n$ with $n > 2$ the remaining hypothesis is verified by a renormalised bump function (`EllipticPdes.Sobolev.dirichlet_principal_eigenpair_ball`).

[14] `EllipticPdes.Sobolev.dirichlet_poincare_sharp` with the equality case `EllipticPdes.Sobolev.dirichlet_poincare_attained`.

Proof sketch. Minimising over unit-norm functions orthogonal to the eigenfunction for λ_1 above yields the next eigenvalue. Repeating this, for each $n \geq 1$ we obtain functions $w_1, \ldots, w_n \in H_0^1(\Omega)$, orthonormal in $L^2(\Omega)$, and $0 < \lambda_1 \leq \cdots \leq \lambda_n$ with $B[w_k, v] = \lambda_k \langle w_k, v \rangle_{L^2(\Omega)}$ for every $v \in H_0^1(\Omega)$, where each λ_k is the infimum of the Rayleigh quotient over the elements of unit L^2 norm orthogonal to $w_1, \ldots, w_{k-1}$.[15]

Since the operator G is compact, self-adjoint, positive and injective, the spectral theorem for compact self-adjoint operators applies [Eva10, App. D.6, Theorem 7; Bre11, §6.4]. The elliptic eigenvalues are moreover positive and of finite multiplicity. Both properties follow from results already established because every weak eigenvalue is bounded below by the positive principal eigenvalue of Theorem 7 and the eigenspace of a compact operator at a nonzero eigenvalue is finite dimensional. Evans obtains a countable orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions with increasing eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ diverging to $+\infty$, whereas the library provides the density statement above together with the orthonormal families of every finite length constructed before it. It does not construct the basis as a single object and does not state that the eigenvalues diverge. Since only finitely many spectral points of a compact operator satisfy $|\mu| \geq \delta$ by Theorem 8 below, the divergence of the eigenvalues would follow from results already formalised by an argument that we have not formalised.[16] □

The direct method also applies under an L^q constraint with $q \neq 2$, in which case the minimiser satisfies a semi-linear equation. Let $n > 2$, let $B_1 \subseteq \mathbb{R}^n$ be the unit ball and let $2 \leq q < 2^\star$. Minimising the Dirichlet energy over $\{\|u\|_{L^q(B_1)} = 1\}$ gives $u \in H_0^1(B_1)$ and $\lambda = \int_{B_1} |Du|^2 > 0$ with

$$\int_{B_1} Du \cdot Dv = \lambda \int_{B_1} |u|^{q-2} u\, v \qquad \big(v \in H_0^1(B_1)\big),$$

so that u is a weak solution of $-\Delta u = \lambda |u|^{q-2} u$ with zero boundary data. Existence again follows from the direct method, in which the Rellich–Kondrachov theorem below the critical exponent gives the strong L^q convergence needed to pass the constraint to the weak limit. This step fails at $q = 2^\star$, in accordance with the restriction on the exponent $p + 1 < 2^\star$ for $-\Delta u = u^p$.[17]

The third existence theorem requires no coercivity, but its proof uses the following result on the spectrum of a compact operator:

Theorem 8 (Spectrum of a compact operator, [Eva10, App. D.5, Theorem 6])**.** *Let K be a compact operator on an infinite-dimensional real Hilbert space. Then zero lies in the spectrum of K whose nonzero part consists of countably many eigenvalues with only finitely many spectral points satisfying $|\mu| \geq \delta$ for each $\delta > 0$.*

---

[15] `EllipticPdes.Sobolev.exists_eigen_family`, with the step `EllipticPdes.Sobolev.exists_higher_eigenpair`. At each stage the recursion requires an element of nonzero L^2 class orthogonal to the family already constructed. The existence of such elements at every stage is precisely the infinite dimensionality of $H_0^1(\Omega)$. On the unit ball of $\mathbb{R}^n$ with $n > 2$ it is verified by m bump functions centred at the points $((2k+1)/(2m) - 1/2)e_1$ whose supports are disjoint because the centres are $1/m$ apart and the radii equal $1/(2m)$. The resulting statement `EllipticPdes.Sobolev.dirichlet_eigen_family_ball` assumes $n > 2$.

[16] `solOp_spectral` with the lifting `solOp_weak_eigen` and the divergence-form specialisation `symmetric_fullElliptic_spectral` whose form for bounded sets is `symmetric_fullElliptic_spectral_of_bounded`. The specialisation deduces the symmetry of B from the coefficients under the assumptions $a^{ij} = a^{ji}$ and $b \equiv 0$ almost everywhere on Ω, with coercivity following from $c \geq 0$. The positivity and the finite multiplicity are `EllipticPdes.Sobolev.weak_eigenvalue_pos` and `EllipticPdes.Sobolev.solOp_finiteDimensional_eigenspace`, with the instance at $-\Delta$ on the unit ball `EllipticPdes.Sobolev.dirichlet_eigenvalue_pos_ball`.

[17] `EllipticPdes.Embedding.exists_weakSolution_dirichlet_of_lt`, with the minimiser `EllipticPdes.Embedding.exists_dirichlet_minimiser_of_lt` and the abstract equation `EllipticPdes.Analysis.euler_lagrange_of_quadratic_min`. Minimising the H_0^1 norm in place of the Dirichlet energy gives $-\Delta u + u = \lambda |u|^{q-2} u$ (`EllipticPdes.Embedding.exists_weakSolution_semilinear_of_lt`).

`spectrum_compact_operator` The Lean theorem is stated as

```
theorem spectrum_compact_operator (hK : IsCompactOperator K)
    (hinf : ¬ FiniteDimensional ℝ E) :
    (0 : ℝ) ∈ spectrum ℝ K
    ∧ spectrum ℝ K \ {0}
        = {μ : ℝ | Module.End.HasEigenvalue (K.toLinearMap) μ} \ {0}
    ∧ (spectrum ℝ K \ {0}).Countable
    ∧ ∀ δ : ℝ, 0 < δ → {μ ∈ spectrum ℝ K | δ ≤ |μ|}.Finite
```

Proof sketch. Away from zero the spectrum consists of eigenvalues by the Fredholm alternative for $\mu - K$. The statement on accumulation follows from the same argument as in [Eva10, App. D.5, Theorem 6], and consequently the non-zero spectrum converges to 0. Whereas Evans states that the non-zero eigenvalues form a finite set or a sequence tending to 0, the formal statement asserts the equivalent finiteness of the spectrum outside every neighbourhood of 0.[18] $\square$

We return to the general operator given in (1), with no assumption of symmetry:

Theorem 9 (Existence III, [Eva10, §6.2.3, Theorem 5])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and measurable, and L be an operator of the form* (1) *which is uniformly elliptic as in* (2)*. Then, there is a countable set $\Sigma \subseteq \mathbb{R}$ with $\Sigma \cap (-\infty, C]$ finite for every $C \in \mathbb{R}$. The problem $Lu = su + f$ has a unique weak solution $u \in H_0^1(\Omega)$ for every $f \in L^2(\Omega)$ exactly when $s \notin \Sigma$. Moreover, for $s \notin \Sigma$ there is a constant $C > 0$ depending on s, Ω and L with $\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ for every such weak solution.*

`existence_three_of_bounded` The Lean theorem is stated as

```
theorem existence_three_of_bounded (hΩm : MeasurableSet Ω)
    (hΩb : Bornology.IsBounded Ω) :
    ∃ S : Set ℝ, S.Countable ∧ (∀ C : ℝ, (S ∩ Set.Iic C).Finite) ∧
      ∀ lam : ℝ, lam ∉ S ↔ ∀ f : L2D Ω, ∃! u : H01 Ω, ∀ v : H01 Ω,
        Op.fullBilin Ω u v
          = lam * ⟪(u : H1amb Ω) 0, ((v : H1amb Ω) 0)⟫
            + ∫ x in Ω, (f x : ℝ) * ((v : H1amb Ω) 0 x : ℝ)
```

Proof sketch. The set Σ is determined by $\mathcal{A}_s = E(1 - \frac{\gamma+s}{\gamma}K)$ as in (6) since the problem at s is uniquely solvable exactly when $1 - \frac{\gamma+s}{\gamma}K$ is injective. By Theorem 8 the eigenvalues of K that obstruct injectivity form a countable set whose only accumulation point is 0. Since the eigenvalues of K are positive, Σ is contained in $(-\gamma, \infty)$ and each set $\Sigma \cap (-\infty, C]$ is finite, it follows that any infinite Σ is a sequence diverging to $+\infty$. The resolvent bound, which Evans states as a separate theorem, is included in the formal statement.[19] $\square$

2.2 Regularity of weak solutions

Once one has produced a weak solution to (3), the Sobolev embedding ensures weak derivatives can be traded for higher integrability via repeated application of the Gagliardo–Nirenberg–Sobolev and Morrey inequalities:

[18] `EllipticPdes.Sobolev.spectrum_compact_operator`.

[19] `EllipticPdes.Sobolev.FullEllipticOp.existence_three_of_bounded` and `EllipticPdes.Sobolev.FullEllipticOp.resolvent_bound_of_bounded`.

Theorem 10 (Sobolev embedding, [Eva10, §5.6.3, Theorem 6])**.** *Let $n \geq 2$, $\Omega \subseteq \mathbb{R}^n$ be open and bounded with C^1 boundary, $p \in [1, \infty)$, $k \in \mathbb{N}$, and $u \in W^{k,p}(\Omega)$. Then:*

1. *If $k < n/p$, then $u \in L^q(\Omega)$ with $1/q = 1/p - k/n$, and $\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}$ for a constant $C > 0$ depending on k, p, n and Ω.*
2. *If $k > n/p$, then $u \in C^{k-1-\lfloor n/p \rfloor, \gamma}(\overline{\Omega})$ with*

$$\gamma = \begin{cases} \text{any value in } (0,1), & n/p \in \mathbb{N}, \\ \lfloor n/p \rfloor - n/p + 1, & n/p \notin \mathbb{N}, \end{cases}$$

and $\|u\|_{C^{k-1-\lfloor n/p \rfloor, \gamma}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)}$ for a $C > 0$ depending on k, n, p, γ and Ω.

`exists_const_memLp_of_gradClosed_domain_ideal` The Lean theorem is stated as

```
theorem exists_const_memLp_of_gradClosed_domain_ideal (hd : 1 < d)
    {Ω : Set (EuclideanSpace ℝ (Fin d))} (hΩopen : IsOpen Ω) (hΩb : Bornology.IsBounded Ω)
    (hC1 : HasC1Boundary Ω) {p₀ : ℝ≥0} (hp₀ : 1 ≤ p₀) (ι : Type*) (s : ℕ)
    (hsd : (p₀ : ℝ) * s < (d : ℝ)) :
    ∃ K : ℝ≥0, ∀ {F : ι → EuclideanSpace ℝ (Fin d) → ℝ} {nxt : ι → Fin d → ι}
      {dep : ι → ℕ} {m : ℕ}, (∀ i k, dep (nxt i k) ≤ dep i + 1) →
      (∀ i, dep i < m → HasWeakGradOn Ω (F i) (fun k => F (nxt i k))) →
      (∀ i, dep i ≤ m → MemLp (F i) p₀ (volume.restrict Ω)) →
      ∀ M : ℝ≥0∞, (∀ j, dep j ≤ m → eLpNorm (F j) p₀ (volume.restrict Ω) ≤ M) →
      ∀ i, dep i + s ≤ m →
        MemLp (F i) (Real.toNNReal ((p₀ : ℝ)⁻¹ - (s : ℝ) * (d : ℝ)⁻¹)⁻¹)
            (volume.restrict Ω) ∧
          eLpNorm (F i) (Real.toNNReal ((p₀ : ℝ)⁻¹ - (s : ℝ) * (d : ℝ)⁻¹)⁻¹)
            (volume.restrict Ω) ≤ (K : ℝ≥0∞) * M
```

`exists_const_contDiffOn_holderOnWith_domain_ideal` The Lean theorem is stated as

```
theorem exists_const_contDiffOn_holderOnWith_domain_ideal (hd : 1 < d)
    {Ω : Set (EuclideanSpace ℝ (Fin d))} (hΩopen : IsOpen Ω) (hΩb : Bornology.IsBounded Ω)
    (hC1 : HasC1Boundary Ω) {p₀ : ℝ≥0} (hp₀ : 1 ≤ p₀) (ι : Type*) (s : ℕ)
    (hsd : (p₀ : ℝ) * s < (d : ℝ)) (hlt : (d : ℝ) < (p₀ : ℝ) * ((s : ℝ) + 1)) :
    ∃ C : ℝ≥0, ∀ {F : ι → EuclideanSpace ℝ (Fin d) → ℝ} {nxt : ι → Fin d → ι}
      {dep : ι → ℕ} {m : ℕ}, (∀ i k, dep (nxt i k) ≤ dep i + 1) →
      (∀ i, dep i < m → HasWeakGradOn Ω (F i) (fun k => F (nxt i k))) →
      (∀ i, dep i ≤ m → MemLp (F i) p₀ (volume.restrict Ω)) →
      ∀ M : ℝ≥0, (∀ j, dep j ≤ m → eLpNorm (F j) p₀ (volume.restrict Ω) ≤ (M : ℝ≥0∞)) →
      ∃ v : ι → EuclideanSpace ℝ (Fin d) → ℝ,
        (∀ i, dep i + 1 + s ≤ m → v i =ᵐ[volume.restrict Ω] F i) ∧
        (∀ i, dep i + 1 + s ≤ m → ∀ y ∈ closure Ω, ‖v i y‖ ≤ ((C * M : ℝ≥0) : ℝ)) ∧
        (∀ i, dep i + 1 + s ≤ m →
          HolderOnWith (C * M) (Real.toNNReal ((s : ℝ) + 1 - (d : ℝ) / (p₀ : ℝ)))
            (v i) (closure Ω)) ∧
        (∀ (n : ℕ) (i : ι), dep i + n + 1 + s ≤ m → ContDiffOn ℝ (n : ℕ) (v i) Ω) ∧
        (∀ i, dep i + 2 + s ≤ m → ∀ y ∈ Ω,
          HasFDerivAt (v i) (gradCLM (fun k => v (nxt i k)) y) y)
```

Proof sketch. Cases (i) and (ii) refer to the space $W^{k,p}(\Omega)$, for which the library interprets each element as a family closed under weak differentiation all of whose members lie in $L^p(\Omega)$. Such a family consists of an index type, a function for each index, a successor map that assigns to an index the weak derivatives of its function and a depth function that bounds the number of available derivatives. For $u \in W^{k,p}(\Omega)$ every member of the family $\{D^\alpha u : |\alpha| \le k\}$ is bounded by $\|u\|_{W^{k,p}(\Omega)}$, and hence the conclusion stated for the member of depth zero gives the estimates of cases (i) and (ii). This formulation is the principal departure from the cited proof, which iterates on $W^{k,p}(\Omega)$ and writes $D^\alpha u$ as a single symbol throughout. In a proof assistant one has to specify which function the symbol denotes at each order, and closure under weak differentiation allows a single induction with no separate induction on the order of differentiation. Theorems 11 and 12 below use the same interpretation.

We begin with a local form of the inequality, which requires no assumption on $\partial\Omega$. Let $x_0 \in \mathbb{R}^n$, let $0 < r < R$, and let p' satisfy $1/p' = 1/p - 1/n$. Then there is a constant C, depending only on n, p, x_0, r and R, such that every $v \in L^p(B_R(x_0))$ with weak gradient $g \in L^p(B_R(x_0))^n$ lies in $L^{p'}(B_r(x_0))$, with

$$\|v\|_{L^{p'}(B_r(x_0))} \le C\Big(\|v\|_{L^p(B_R(x_0))} + \sum_\ell \|g_\ell\|_{L^p(B_R(x_0))}\Big).$$

Following [Eva10, §5.6.1], for a compactly supported C^1 function the inequality above is the Gagliardo–Nirenberg–Sobolev estimate, which is available in Mathlib. We extend it to a function with a weak gradient on a ball by means of a cutoff and mollification. Let η be a test function with $0 \le \eta \le 1$ that is equal to 1 on $\overline{B_r(x_0)}$ and supported in $B_R(x_0)$, and let M bound the derivatives of η. By the product rule for weak gradients the compactly supported function $w = \eta v$ has the weak gradient $\eta g + v\nabla\eta$ on all of $\mathbb{R}^n$. Both functions lie in L^p with a bound that introduces the factor $\max(1, nM)$ into the constant. The mollifications of w are smooth with compact support and have as classical partial derivatives the mollifications of the weak gradient, whose L^p norms are bounded uniformly in the inner radius by Young's inequality. Applying the C^1 case to each mollification, we obtain a uniform bound on the $L^{p'}$ norm that is independent of the radius. By Fatou's lemma this bound passes to the almost-everywhere limit, which agrees with v on $B_r(x_0)$.[20]

Case (i) is obtained by iterating the above, following [Guo26, Proof of Theorem IV.2.3]. On $\mathbb{R}^n$ the iteration runs over a family of compact support with no cutoff, thus giving a bound in terms of a constant and the gradient. We argue by induction on the number s of steps, for all indices simultaneously, and with base exponent p we write $t = 1/p - s/n$. The condition $ps < n$ gives $t > 0$, so that $1/t$ is a finite exponent with $p \le 1/t$. If $s = 0$, the member lies in L^p and hence in every L^q with $q \le p$ since Ω is bounded. For the inductive step, the inductive hypothesis applies to the index and each of its derivatives, because the successor increases the depth function mentioned above by at most one.[21] The statement is transferred to Ω itself by the extension operator of [Eva10, §5.4, Theorem 1], which extends $W^{1,p}(\Omega)$ functions on a bounded C^1 domain to a compactly supported $W^{1,p}(\mathbb{R}^n)$ function with norm bounded, up to a constant, by the original norm.[22] Applying the operator to one member of the family at a time, we transfer the above step from $\mathbb{R}^n$ to Ω.[23]

For case (ii) the same idea is run for $s = \lfloor n/p \rfloor$ steps to an exponent above n, after which Morrey's inequality is applied on a ball containing $\overline{\Omega}$ giving Hölder exponent $s+1-n/p$, which is the value in the statement. If n/p is an integer, the Hölder exponent may be chosen freely in $(0, 1)$. A step whose target reciprocal is 0 requires an additional device. We allow a step to take its hypothesis in L^q for every

[20] `EllipticPdes.Embedding.exists_eLpNorm_sobolevConj_le`.

[21] `EllipticPdes.Embedding.memLp_of_gradClosed_compactSupport_ideal`, based on the step `EllipticPdes.Embedding.exists_eLpNorm_sobolevConj_le_compactSupport`.

[22] `EllipticPdes.Extension.exists_extension_subset_bound`, based on `EllipticPdes.Extension.exists_localExtension_bound`.

[23] `EllipticPdes.Embedding.exists_eLpNorm_sobolevConj_le_domain`, iterated by `EllipticPdes.Embedding.exists_const_memLp_of_gradClosed_domain`.

$q \geq p$ while its conclusion is stated at the conjugate exponent of p.[24] Morrey's inequality produces the continuous representatives and bounds their supremum and their Hölder seminorm together, which gives the estimate for the full norm. The representatives are chosen once and for all so that they are the classical partial derivatives of one another on Ω, as membership of $C^{k-1-\lfloor n/p \rfloor, \gamma}(\overline{\Omega})$ requires, with an induction on the remaining order giving inclusion in all relevant $C^m(\Omega)$.[25] $\square$

The interior theory is stated for a weak solution that is subject to no boundary condition. Throughout the following three theorems $\Omega \subseteq \mathbb{R}^n$ is bounded and open, L is an operator of the form (1) with second-order coefficients satisfying $a^{ij} = a^{ji}$ and the ellipticity condition (2) with some $\theta > 0$ in place of λ. Throughout we write ℓ and ℓ' for directions of differentiation and reserve i and j for the coefficient indices and m for the order in Theorem 12. A function $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω if

$$\sum_{i,j=1}^{n} \int_\Omega a^{ij}\, D_i u\, D_j \varphi + \sum_{i=1}^{n} \int_\Omega b^i\, D_i u\, \varphi + \int_\Omega c\, u\, \varphi = \int_\Omega f\, \varphi \qquad \text{for every } \varphi \in C_c^\infty(\Omega). \tag{7}$$

The first gain of regularity transfers the regularity of the second-order coefficients onto the weak solution, giving one derivative more in the interior under a hypothesis on those coefficients alone.

Theorem 11 (Interior H^2 regularity, [Eva10, §6.3.1, Theorem 1])**.** *Assume $a^{ij} \in C^1(\Omega)$, $b^i, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω. Then $u \in H^2_{\mathrm{loc}}(\Omega)$ and for each open $V \Subset \Omega$*

$$\|u\|_{H^2(V)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big),$$

where the constant C depends only on V, Ω and the coefficients of L.

`interior_H2_regularity_evans` The Lean theorem is stated as

```
theorem interior_H2_regularity_evans {U : Set (EuclideanSpace ℝ (Fin d))} (hU : IsOpen U)
    (_hUb : Bornology.IsBounded U)
    {a : EuclideanSpace ℝ (Fin d) → Fin d → Fin d → ℝ} {b : EuclideanSpace ℝ (Fin d) → Fin d → ℝ}
    {c : EuclideanSpace ℝ (Fin d) → ℝ}
    (ha : ∀ i j, ContDiffOn ℝ 1 (fun x => a x i j) U)
    (hb : ∀ i, MemLp (fun x => b x i) ⊤ (volume.restrict U))
    (hc : MemLp c ⊤ (volume.restrict U)) {θ : ℝ} (hθ : 0 < θ)
    (hell : ∀ᵐ x ∂(volume.restrict U), ∀ ξ : Fin d → ℝ,
      θ * ∑ i, ξ i ^ 2 ≤ ∑ i, ∑ j, a x i j * ξ i * ξ j)
    (_hsymm : ∀ x ∈ U, ∀ i j, a x i j = a x j i)
    {V : Set (EuclideanSpace ℝ (Fin d))} (hVo : IsOpen V) (hVc : IsCompact (closure V))
    (hVU : closure V ⊆ U) :
    ∃ C : ℝ, 0 ≤ C ∧ ∀ (f u : EuclideanSpace ℝ (Fin d) → ℝ)
      (G : Fin d → EuclideanSpace ℝ (Fin d) → ℝ)
      (hf : MemLp f 2 (volume.restrict U)) (hu : MemLp u 2 (volume.restrict U)),
      (∀ i, MemLp (G i) 2 (volume.restrict U)) → HasWeakGradOn U u G →
      LocalWeakSol U a b c f u G →
      ∃ H : HasIteratedWeakDerivOn V 2
          ((hu.mono_measure (Measure.restrict_mono (subset_closure.trans hVU) le_rfl)).toLp u),
        iteratedNorm H ≤ C * (‖hf.toLp f‖ + ‖hu.toLp u‖)
```

[24] `EllipticPdes.Embedding.exists_eLpNorm_sobolevConj_le_of_le`.

[25] `EllipticPdes.Embedding.exists_const_contDiffOn_holderOnWith_domain_ideal` and `EllipticPdes.Embedding.exists_const_contDiffOn_holderOnWith_domain_free`, based on `EllipticPdes.Embedding.exists_const_contDiffOn_holderOnWith_of_gradClosed_domain` and `EllipticPdes.Embedding.hasFDerivAt_of_continuousOn_hasWeakGradOn`.

Proof sketch. The proof reduces the statement to the corresponding estimate for a solution in $H^1_0(\Omega)$ with globally defined coefficients, which is proved by the method of difference quotients. After the weak formulation is localised by a cutoff, discrete integration by parts and uniform ellipticity bound the difference quotients of ∇u in L^2 uniformly in the step. By the Peter–Paul inequality, which Evans calls Cauchy's inequality with ε, the transport term is absorbed into the lower bound given by ellipticity, so that no sign condition on c is needed and the constant depends on the ellipticity constant and on the suprema of $|b|$ and $|c|$. Passing to the limit, we obtain $\nabla u \in H^1_{\mathrm{loc}}(\Omega)$ [Eva10, §6.3.1, Theorem 1; GT01, Theorem 8.8]. That estimate bounds its lower-order terms by testing the equation with u itself and by extending u by zero.[26] Of the principal coefficients the estimate asks the Lipschitz bound $|a^{ij}(x) - a^{ij}(y)| \le A_1|x-y|$ alone, which is $W^{1,\infty}$ in pointwise form and which bounds the difference quotients of the coefficients in the commutator term. The hypothesis $a^{ij} \in C^1(\Omega)$ of Theorem 11 is the one Evans states, and it reaches the formal proof through the mean value inequality.[27]

Two reductions remove the boundary condition and the global coefficients. Let $\eta \in C^\infty_c(\Omega)$ be equal to 1 near $\overline{V}$. For $u \in H^1(\Omega)$ the product ηu lies in $H^1_0(\Omega)$ and is a weak solution of $L(\eta u) = F$ with

$$F = \eta f - a^{ij} D_i u\, D_j \eta - D_j\big(a^{ij} D_i \eta\, u\big) + b^i D_i \eta\, u,$$

as a direct computation from (7) shows. Setting $\varphi = \eta^2 u$ in (7), which is admissible by density since $\eta^2 u \in H^1_0(\Omega)$ has compact support in Ω, and performing elementary calculations, we discover

$$\int_\Omega \eta^2 |\nabla u|^2 \le C \int_\Omega f^2 + u^2,$$

whence the norm of F is bounded by a multiple of $\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$ and the estimate for ηu gives the estimate of Theorem 11.[28] The coefficients are required only on the support of η. With a second cutoff χ equal to 1 there we replace a by $\theta I + \chi(a - \theta I)$ and b, c by χb, χc, which gives coefficients on $\mathbb{R}^n$ with the ellipticity constant θ, bounded derivatives and the original values near $\overline{V}$. For $b^i, c \in L^\infty(\Omega)$ the formal statement first passes to measurable representatives equal to them almost everywhere.[29]

The formal statement expresses $H^2_{\mathrm{loc}}(\Omega)$ by a family of weak derivatives on V and measures $\|u\|_{H^2(V)}$ by a sum over lists of directions, in which a mixed derivative is counted once for each ordering, so that the norm is equivalent to Evans's with constants depending only on n. The estimate is proved one pair of directions ℓ, ℓ' at a time, bounding $\|\partial_\ell \partial_{\ell'} u\|_{L^2(V)} + \|\partial_{\ell'} u\|_{L^2(V)} + \|u\|_{L^2(V)}$ by the right-hand side of the estimate, and the sum over the pairs is where the factor depending only on n enters. The constant is quantified before f and u and therefore depends on neither. The weak formulation is tested against $C^\infty_c(\Omega)$ in place of $H^1_0(\Omega)$. This is a weaker hypothesis, since for coefficients that are only of class C^1 in Ω the form $B[u, v]$ need not be defined for every $v \in H^1_0(\Omega)$. The boundedness of Ω and the symmetry of the coefficients a^{ij} are assumed only because Evans assumes them, and the proof uses neither. $\square$

[26] `EllipticPdes.Regularity.interior_H2_estimate`.

[27] `EllipticPdes.Regularity.IsLipCoeff` and `EllipticPdes.Regularity.IsC1Coeff.toIsLipCoeff`. Guo [Guo26, Theorem VIII.3.2] states the interior theory under the $W^{1,\infty}$ hypothesis.

[28] `EllipticPdes.Regularity.cutoffMul_mem_H01_of_mem_W12`, `EllipticPdes.Regularity.reduction_weakForm` and `EllipticPdes.Regularity.caccioppoli_W12`. The inequality above is the Caccioppoli inequality for L on the support of η, and [Eva10, p. 331] states it in these terms without naming it, in order to replace $\|u\|_{H^1}$ by $\|u\|_{L^2}$ on the right of the estimate. Here it bounds instead the gradient terms of the datum F, which the reduction to $H^1_0(\Omega)$ introduces and which the argument of the source never forms.

[29] `EllipticPdes.Regularity.exists_localOp_C1` and `EllipticPdes.Regularity.exists_family_of_localWeakSol`.

With higher regularity assumptions on the coefficients and data, the interior estimate can be iterated:

Theorem 12 (Higher interior regularity, [Eva10, §6.3.1, Theorem 2]). *Let m be a nonnegative integer and assume $a^{ij}, b^i, c \in C^{m+1}(\Omega)$ and $f \in H^m(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω. Then $u \in H^{m+2}_{\mathrm{loc}}(\Omega)$ and for each open $V \Subset \Omega$*

$$\|u\|_{H^{m+2}(V)} \le C\big(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\big),$$

where the constant C depends only on m, Ω, V and the coefficients of L.

higher_interior_regularity_evans The Lean theorem is stated as

```
theorem higher_interior_regularity_evans {U : Set (EuclideanSpace ℝ (Fin d))} (hU : IsOpen U)
    (_hUb : Bornology.IsBounded U) (m : ℕ)
    {a : EuclideanSpace ℝ (Fin d) → Fin d → Fin d → ℝ} {b : EuclideanSpace ℝ (Fin d) → Fin d → ℝ}
    {c : EuclideanSpace ℝ (Fin d) → ℝ}
    (ha : ∀ i j, ContDiffOn ℝ (m + 1 : ℕ) (fun x => a x i j) U)
    (hb : ∀ i, ContDiffOn ℝ (m + 1 : ℕ) (fun x => b x i) U)
    (hc : ContDiffOn ℝ (m + 1 : ℕ) c U) {θ : ℝ} (hθ : 0 < θ)
    (hell : ∀ᵐ x ∂(volume.restrict U), ∀ ξ : Fin d → ℝ,
      θ * ∑ i, ξ i ^ 2 ≤ ∑ i, ∑ j, a x i j * ξ i * ξ j)
    (_hsymm : ∀ x ∈ U, ∀ i j, a x i j = a x j i)
    {V : Set (EuclideanSpace ℝ (Fin d))} (hVo : IsOpen V) (hVc : IsCompact (closure V))
    (hVU : closure V ⊆ U) :
    ∃ C : ℝ, 0 ≤ C ∧ ∀ (f u : EuclideanSpace ℝ (Fin d) → ℝ)
      (G : Fin d → EuclideanSpace ℝ (Fin d) → ℝ)
      (hf : MemLp f 2 (volume.restrict U)) (hu : MemLp u 2 (volume.restrict U)),
      (∀ i, MemLp (G i) 2 (volume.restrict U)) →
      ∀ Hf : HasIteratedWeakDerivOn U m (hf.toLp f), HasWeakGradOn U u G →
      LocalWeakSol U a b c f u G →
      ∃ H : HasIteratedWeakDerivOn V (m + 2)
          ((hu.mono_measure (Measure.restrict_mono (subset_closure.trans hVU) le_rfl)).toLp u),
        iteratedNorm H ≤ C * (iteratedNorm Hf + ‖hu.toLp u‖)
```

Proof sketch. The proof proceeds by induction on m whose base case is Theorem 11. For a solution in $H^1_0(\Omega)$ with globally defined coefficients we differentiate the weak formulation in one direction ℓ at a time. A weak solution of $Lu = f$ gives a weak solution of $L(\partial_\ell u) = \partial_\ell f + R_\ell$, where R_ℓ collects the terms in which a derivative falls on a coefficient and lies in $H^{m-1}(\Omega)$ once the conclusion at order $m-1$ is known, each of them pairing a derivative of a coefficient with a derivative of u of order at most two. Applying the inductive hypothesis to $\partial_\ell u$ on an intermediate set $\overline{V} \subseteq W \Subset \Omega$ and recombining the directions, we obtain $u \in H^{m+2}(V)$ [Eva10, §6.3.1, Theorem 2]. This argument differs from the argument of [Guo26], which differentiates by a multi-index α with $|\alpha| = m+1$ in a single step and tests the weak formulation against $(-1)^{|\alpha|} D^\alpha \varphi$, because differentiating in one direction at a time allows each step to apply the statement already proved. In that statement the coefficients are of the $W^{k,\infty}$ type used by [Guo26, Theorem VIII.3.2], with $a^{ij} \in W^{m+1,\infty}(\Omega)$ and $b^i, c \in W^{m,\infty}(\Omega)$, since no step of the argument requires a coefficient of the differentiated equation to be continuous. At $m = 0$ the bundle of order one is passed so that the order is stated uniformly in m, and it is never read, since the base case is Theorem 11 with its $W^{1,\infty}$ hypothesis.[30]

For a solution in $H^1(\Omega)$ the same cutoff reduction as in the proof of Theorem 11 applies at each order. The datum F of $L(\eta u) = F$ lies in H^m with a bound by $\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}$ once the conclusion at

[30] `EllipticPdes.Regularity.higher_interior_regularity`, whose step is `EllipticPdes.Regularity.interiorRegularityAt_succ`. The implication `IsCkCoeff.toIsWkInftyCoeff` shows that the classical hypothesis is the stronger one.

order $m-1$ is known near the support of η, and the coefficients of class $C^{m+1}(\Omega)$ are localised by the cutoff blend of that proof, which preserves their order.[31] The formal statement expresses $f \in H^m(\Omega)$ and the conclusion by families of weak derivatives measured by the norm of Theorem 11, and its remaining hypotheses are those of that theorem. $\square$

Applying the estimate at every order, we obtain a smooth representative:

Theorem 13 (Infinite differentiability in the interior, [Eva10, §6.3.1, Theorem 3])**.** *Assume $a^{ij}, b^i, c \in C^\infty(\Omega)$ and $f \in C^\infty(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω. Then $u \in C^\infty(\Omega)$.*

`exists_contDiffOn_of_weakSolution_evans` The Lean theorem is stated as

```
theorem exists_contDiffOn_of_weakSolution_evans {d : ℕ} {U : Set (EuclideanSpace ℝ (Fin d))}
    (hU : IsOpen U) (_hUb : Bornology.IsBounded U)
    {a : EuclideanSpace ℝ (Fin d) → Fin d → Fin d → ℝ} {b : EuclideanSpace ℝ (Fin d) → Fin d → ℝ}
    {c f u : EuclideanSpace ℝ (Fin d) → ℝ} {G : Fin d → EuclideanSpace ℝ (Fin d) → ℝ}
    (ha : ∀ i j, ContDiffOn ℝ (⊤ : ℕ∞) (fun x => a x i j) U)
    (hb : ∀ i, ContDiffOn ℝ (⊤ : ℕ∞) (fun x => b x i) U)
    (hc : ContDiffOn ℝ (⊤ : ℕ∞) c U) (hf : ContDiffOn ℝ (⊤ : ℕ∞) f U)
    {θ : ℝ} (hθ : 0 < θ)
    (hell : ∀ᵐ x ∂(volume.restrict U), ∀ ξ : Fin d → ℝ,
      θ * ∑ i, ξ i ^ 2 ≤ ∑ i, ∑ j, a x i j * ξ i * ξ j)
    (_hsymm : ∀ x ∈ U, ∀ i j, a x i j = a x j i)
    (hu : MemLp u 2 (volume.restrict U)) (hG : ∀ i, MemLp (G i) 2 (volume.restrict U))
    (hgrad : HasWeakGradOn U u G) (hsol : LocalWeakSol U a b c f u G) :
    ∃ u' : EuclideanSpace ℝ (Fin d) → ℝ,
      ContDiffOn ℝ (⊤ : ℕ∞) u' U ∧ u' =ᵐ[volume.restrict U] u
```

Proof sketch. Let B be a closed ball contained in Ω. The cutoff in the proof of Theorem 11 gives coefficients on $\mathbb{R}^n$ with bounded derivatives of every order that agree with the a^{ij}, b^i and c near B, and the datum χf is smooth with compact support. By Theorem 12 at every order the solution has weak derivatives of every order in L^2 near B. To pass from weak derivatives of every order to a classical smooth representative, we apply the Gagliardo–Nirenberg–Sobolev inequality along a family of functions closed under differentiation in steps of $1/(2n)$. Morrey's inequality gives a continuous representative of each derivative that a mollification argument identifies with a classical derivative [Eva10, §6.3.1, Theorem 3].[32] Two continuous functions that agree almost everywhere on an open set agree on it, so the representatives obtained on the balls agree on their overlaps and define a single function of class $C^\infty(\Omega)$ that is equal to u almost everywhere.[33] The formal statement reads $u \in C^\infty(\Omega)$ as the existence of such a representative, tests the weak formulation against $C_c^\infty(\Omega)$ as in Theorem 11, and uses neither the boundedness of Ω nor the symmetry of the a^{ij}. The declaration on which it rests differs from the cited statement in three respects. Evans assumes $a^{ij}, b^i, c, f \in C^\infty(\Omega)$ on a bounded open set, where the declaration takes coefficients with bounded weak derivatives of every order and a datum with weak derivatives of every order in $L^2(\Omega)$. Evans assumes $u \in H^1(\Omega)$, where the declaration takes $u \in H^1_0(\Omega)$ and the cutoff reduction of Theorem 11 recovers the weaker

[31] `EllipticPdes.Regularity.higher_interior_regularity_W12`, `EllipticPdes.Regularity.exists_reductionDatum` and `EllipticPdes.Regularity.exists_localOp_Ck`.

[32] `EllipticPdes.Regularity.interior_smooth` and `EllipticPdes.Regularity.interior_smooth_W12`, based on `EllipticPdes.Embedding.contDiffOn_of_gradClosed`.

[33] `EllipticPdes.Regularity.exists_contDiffOn_of_closedBall_ae`, based on `EllipticPdes.Regularity.exists_contDiffOn_of_locally_ae`, and the statement for a local weak solution with no boundedness or symmetry assumption `EllipticPdes.Regularity.exists_contDiffOn_of_localWeakSol`.

hypothesis. Evans concludes $u \in C^\infty(\Omega)$ for one representative, where the declaration gives one on the interior of each $V \Subset \Omega$ and the gluing above assembles them into a single representative on Ω. Under the $W^{k,\infty}$ hypotheses the statement asks of a^{ij} one thing beyond the bundles, the $W^{1,\infty}$ hypothesis of Theorem 11, which is read by the base case. Its redundancy is the statement that a $W^{1,\infty}$ function has a Lipschitz representative, which Mathlib does not prove, so the statement asks for the hypothesis and the bundle of order one separately. □

Each of the three statements above is proved in the library in the form that Guo and that Gilbarg and Trudinger state, from which Theorems 11, 12 and 13 follow by the cutoff reduction. For a weak solution $u \in H_0^1(\Omega)$ of (3) with a^{ij} Lipschitz on $\mathbb{R}^n$ and b^i, c bounded, the second weak derivatives exist in $L^2(V)$ on every compact $V \subseteq \Omega$ and are bounded one pair of directions at a time, with the constant quantified before u and f [Guo26, Theorem VIII.2.2; GT01, Theorem 8.8].[34] With $a^{ij} \in W^{m+1,\infty}(\Omega)$ and $b^i, c \in W^{m,\infty}(\Omega)$ with $f \in H^m(\Omega)$, the solution has weak derivatives to order $m+2$ on every such $V \Subset \Omega$, each bounded by $C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$ [Guo26, Theorem VIII.3.2].[35] With those hypotheses at every order the solution has a representative smooth on the interior of each such V [Guo26, Theorem VIII.3.3; GT01, Corollary 8.11].[36] In all three statements above the hypothesis on the principal coefficients is $W^{1,\infty}$, which is what the difference-quotient proof reads and what [GT01, Theorem 8.8] asks.

We thus obtain a classical solvability statement:

Corollary 3 (Classical solvability in the interior, [Eva10, §6.2.2, Theorem 3; Eva10, §6.3.1, Theorem 3])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open, L be the operator* (1)*, uniformly elliptic in the sense of* (2) *with $a^{ij} \in C^1(\Omega)$, $a^{ij}, b^i, c \in W^{k,\infty}(\Omega)$, and $f \in H^k(\Omega)$ for every $k \in \mathbb{N}$, and suppose that $b \equiv 0$ and $c \geq 0$ almost everywhere. Then,* (3) *has a unique weak solution $u \in H_0^1(\Omega)$ whose class has, on the interior of each compact $V \subseteq \Omega$, a representative that is smooth there and satisfies $Lu = f$ pointwise almost everywhere there.*

`exists_weakSolution_interior_classical` The Lean theorem is stated as

```
theorem exists_weakSolution_interior_classical {n : ℕ}
    (Op : FullEllipticOp (n + 1)) {Ω : Set (EuclideanSpace ℝ (Fin (n + 1)))}
    (hΩm : MeasurableSet Ω) (hΩo : IsOpen Ω) (hΩb : Bornology.IsBounded Ω)
    (hb : ∀ i, ∀ᵐ x ∂(volume.restrict Ω), Op.b x i = 0)
    (hc : ∀ᵐ x ∂(volume.restrict Ω), 0 ≤ Op.c x)
    (hA1 : IsC1Coeff Op.toEllipticCoeff)
    (hA : ∀ k : ℕ, IsWkInftyCoeff Op.toEllipticCoeff k)
    (hbc : ∀ k : ℕ, IsWkInftyLower Op k)
    (f : L2D Ω)
    (hf : ∀ k : ℕ, ∃ hfk : HasIteratedWeakDerivOn Ω k f, ∃ M : ℝ, IteratedL2Bound hfk M)
    {V : Set (EuclideanSpace ℝ (Fin (n + 1)))} (hVc : IsCompact V) (hVΩ : V ⊆ Ω) :
    ∃ u : H01 Ω,
      (∀ v : H01 Ω, Op.fullBilin Ω u v = ∫ x in Ω, (f x : ℝ) * ((v : H1amb Ω) 0 x : ℝ)) ∧
      ∃ u' : EuclideanSpace ℝ (Fin (n + 1)) → ℝ,
        u' =ᵐ[volume.restrict (interior V)]
            (extendL2 hΩm ((u : H1amb Ω) 0) : EuclideanSpace ℝ (Fin (n + 1)) → ℝ) ∧
          ContDiffOn ℝ (⊤ : ℕ∞) u' (interior V) ∧
          ∀ᵐ x ∂volume, x ∈ interior V →
            -(∑ i, ∑ j, partialD j (fun y => Op.a y i j * partialD i u' y) x)
              + ∑ i, Op.b x i * partialD i u' x + Op.c x * u' x = f x
```

[34] `EllipticPdes.Regularity.interior_H2_estimate`.
[35] `EllipticPdes.Regularity.higher_interior_regularity`.
[36] `EllipticPdes.Regularity.interior_smooth`.

Proof sketch. Theorem 4 provides the unique weak solution and Theorem 13 provides its smooth representatives on each compactly contained subset of Ω. The fundamental lemma of the calculus of variations shows that the weak formulation implies $Lu = f$ pointwise on the interior. The pointwise equation differentiates $a^{ij} D_i u$ classically, and the formal statement therefore also takes $a^{ij} \in C^1(\Omega)$. The hypotheses $b \equiv 0$ and $c \geq 0$ make the associated bilinear form coercive with $\gamma = 0$ in Theorem 3; these hypotheses enter the proof only through Theorem 4 and are not used by Theorem 13.[37] □

3 Structure of the Lean library

In this section we describe the Lean library in which the proofs and statements of the results in Section 2 are formalised. The library consists of three layers: Sobolev spaces, the Poincaré reduction, and the Sobolev embedding. The solvability results of Section 2 are proved in the first two layers and the regularity results are proved in the third. We also include a module graph in which every result is placed above the results it uses.

3.1 Foundational layers

Weak-derivative Sobolev layer

The Sobolev spaces are realised as weak-derivative Hilbert spaces so that a member is an L^2 function together with its L^2 gradients; the ambient space is $L^2(\Omega) \times (L^2(\Omega))^n$, encoded as `PiLp` 2 over Fin $(n+1)$, with coordinate 0 the function and coordinate $i+1$ the ith weak partial derivative. The weak-gradient relation is expressed as orthogonality to a fixed family of constraint vectors, so that $W^{1,2}(\Omega)$ is an orthogonal complement and is therefore closed, complete and a real Hilbert space. The space $H^1_0(\Omega)$ is the topological closure of the test-function graphs and is contained in $W^{1,2}(\Omega)$ by an integration by parts without boundary term.[38]

On this layer the bilinear form $B[u,v] = \sum_{i=1}^{n} \langle \partial_i u, \partial_i v \rangle$ of the Laplacian is bounded and is coercive by the Poincaré inequality obtained by density. The Lax–Milgram theorem in Mathlib then yields a unique weak solution of the Dirichlet problem for the Poisson equation for every $f \in H^{-1}(\Omega)$. Data $f \in L^2(\Omega)$ enters through the functional $v \mapsto \int_\Omega f\, v_0$, which is a contraction $L^2(\Omega) \to H^{-1}(\Omega)$.[39] The quantitative part of the Lax–Milgram theorem gives the a priori bound of Theorem 2.[40] This form is the special case $a^{ij} = \delta^{ij}$, $b = 0$, $c = 0$ in (5). The remainder of the library develops the theory of the full operator on this layer along the lines of the classical divergence-form theory as in [Eva10]. The coefficient bundle consists of measurable representatives on $\mathbb{R}^n$ that satisfy the ellipticity and entrywise bounds almost everywhere.[41] The formalised bilinear form is bounded with continuity constant $n^2\Lambda$.

[37]`EllipticPdes.Regularity.exists_weakSolution_interior_classical` which combines `EllipticPdes.Sobolev.FullEllipticOp.weak_solution_L2_of_nonneg_zeroth_of_bounded` with `EllipticPdes.Regularity.interior_smooth` and the pointwise step. The smoothness statement alone is `EllipticPdes.Regularity.exists_weakSolution_interior_smooth`.

[38]`EllipticPdes.Sobolev.W12`, `EllipticPdes.Sobolev.H01` and `EllipticPdes.Sobolev.testGraph_mem_W12`.

[39]`EllipticPdes.l2Functional`.

[40]`EllipticPdes.lax_milgram`, `EllipticPdes.poisson_weak_solution` and `EllipticPdes.norm_weak_solution_le`.

[41]`EllipticPdes.Sobolev.EllipticCoeff` and `EllipticPdes.Sobolev.FullEllipticOp`.

The Poincaré constant is derived from the geometry of the domain. We transfer the one-dimensional slice bound along the measure-preserving identification with `EuclideanSpace` and thereby obtain unconditional coercivity of the form of the Laplacian on every open coordinate box together with the Poincaré inequality on a box.[42] Since a test function on a bounded Ω and its integrals over a box can be transferred to a box containing Ω, combining these results yields the inequality on a bounded set.[43] Because the compactness of the embedding is itself proved for the weak-derivative space by means of the Fréchet–Kolmogorov criterion and the estimate of L^2 translations by the gradient, the Fredholm alternative and the spectral decomposition hold on every bounded set with no further hypothesis.[44]

Poincaré reduction layer

Coercivity of (5) follows from Theorem 1, whose proof is sketched in Section 2. The analytic content of that proof lies in the following two one-dimensional lemmas, from which the estimate is built up through intermediate statements that are formalised separately and may be used independently.

Lemma 1 (One-dimensional Cauchy–Schwarz inequality, [Eva10, App. B.2])**.** *Let f, g be continuous on $[a, b]$. Then*

$$\left(\int_a^b f(t)\, g(t)\, \mathrm{d}t\right)^2 \le \left(\int_a^b f(t)^2\, \mathrm{d}t\right)\left(\int_a^b g(t)^2\, \mathrm{d}t\right).$$

Lemma 2 (One-dimensional Poincaré inequality)**.** *Let u be continuously differentiable on $[a, b]$ with $u(a) = 0$. Then*

$$\int_a^b u(t)^2\, \mathrm{d}t \le \frac{(b-a)^2}{2} \int_a^b u'(t)^2\, \mathrm{d}t.$$

Lemma 1 is the case $p = q = 2$ of Hölder's inequality, which the formal proof obtains by the discriminant argument. Lemma 2 then follows from the representation $u(x) = \int_a^x u'$ by the fundamental theorem of calculus combined with Lemma 1.[45] We state the Cauchy–Schwarz inequality as a separate lemma because the continuity bound on the bilinear form also uses it. Both lemmas are also contained, in a form prepared for submission upstream, in the first-named author's Mathlib fork alongside the Gagliardo–Nirenberg–Sobolev inequality. Whereas the sharp Friedrichs constant would require eigenvalue theory, the one-dimensional argument yields the explicit constant $(b-a)^2/2$ that the reduction to boxes preserves.

The formalisation devotes one module to each step of the proof of Theorem 1, from Lemma 1 to the inequality on a bounded set. The reduction is carried out on a box because Fubini's theorem and the product measure are already available for boxes in Mathlib. This choice entails no loss of generality, since enclosing a bounded set in a box transfers the constant to that set.

Sobolev embedding layer

The Sobolev embedding layer, consisting of three levels, provides the passage from equivalence classes of functions (differing on sets of measure zero) to functions defined everywhere pointwise.

[42] `laplaceBilin_coercive_euclBox` and `poincare_H01_euclBox`, in the module `Poincare/BoxSlice`.
[43] `EllipticPdes.Poincare.poincare_H01_of_bounded`.
[44] `embL2_isCompact`.
[45] `EllipticPdes.Poincare.intervalIntegral_mul_sq_le` and `EllipticPdes.Poincare.poincare_oneDim`.

The lowest level consists of a single step. The Gagliardo–Nirenberg–Sobolev inequality of Mathlib is stated for compactly supported functions on the whole space. We extend it to weak-derivative classes on a ball contained in a larger ball and thereby obtain the local form of Theorem 10 that is used in the cutoff argument, in which the cutoff is identically one on the smaller ball.[46] Morrey's inequality is extended in the same way and yields a Hölder continuous representative whenever the exponent exceeds the dimension.[47] The radii in the bootstrap decrease only within the interval between its two given radii. Since the derivative of a representative is the representative of the derivative, the differentiability argument is carried out on the inner ball itself.

The intermediate level is the bootstrap over a family closed under weak differentiation, whose design and step condition are described in the proof of Theorem 10. The bootstrap with the full step $1/n$ applies when a depth function bounds the supply of derivatives and that with the half step $1/(2n)$ when derivatives of every order are available.[48] The exponent-lowering form of the step is the formal counterpart of the passage through $W^{k-\ell,s}$ for $s \in [1, n)$, a step that the classical proof treats separately and in which it uses the boundedness of Ω.[49]

At the highest level weak derivatives are converted into classical ones. A weak gradient is unique almost everywhere and, if it has a continuous representative, this representative is a classical Fréchet derivative, as we prove by mollification and uniform convergence on a compact enlargement.[50] Combining these two results, we show that the members of a closed family are smooth, as the statement of Theorem 13 requires, or, at finite depth, Hölder regular, as Theorem 10 requires.[51] The closed family is assembled from the interior estimates.[52] No separate uniqueness argument is required there, since a single iterated weak-derivative hypothesis already assigns a class to every index list and the successor is therefore the closure.

The library also contains a shorter argument that provides an alternative to this layer in low dimension. Since H^2 data alone provides only two derivatives, the argument consists of a single step that improves the integrability of the gradient from L^2 to L^6, after which Morrey's inequality applies for $n \leq 3$.[53]

3.2 Module architecture

The library is organised so that the logical dependencies between results are reflected in the module graph and every import points strictly upward. Consequently, each result can be reused without the results above it. Figure 1 shows the modules and the imports between them, with the Sobolev and Poincaré layers in its upper part and the modules of solvability and regularity built on them in its lower part. It is extracted from the import headers of the modules.

The Poincaré reduction is formalised with one module for each step of the proof of Theorem 1: the interval Cauchy–Schwarz and one-dimensional inequalities of Lemmas 1 and 2 (`Poincare/OneDim`), the per-direction bound on a box (`Fubini`), the average over the coordinate directions (`Domain`), the extension from test functions to $H^1_0(\Omega)$ (`Density`), the bridge from a slice bound to the graph-coordinate

[46] The module `Embedding/GagliardoNirenberg`.

[47] The modules `Embedding/Morrey`, `Embedding/MorreyOneDim` and `Embedding/RayIntegral`.

[48] The modules `Embedding/SobolevLadderFullStep` and `Embedding/SobolevLadder`.

[49] `EllipticPdes.Embedding.exists_eLpNorm_sobolevConj_le_of_le`.

[50] The modules `Embedding/WeakGradUnique` and `Embedding/ClassicalDeriv`.

[51] The modules `Embedding/SmoothOfGradClosed`, `Embedding/HolderOfGradClosed` and `Regularity/InteriorHolderFinite`.

[52] The module `Regularity/IteratedFamily`.

[53] The module `Embedding/InteriorHolder`.

hypothesis (`Geometry`), the transport to Euclidean space and the statement on a box (`BoxSlice`), and the enclosure of a bounded set in a box (`BoundedDomain`). These modules depend on the weak-derivative Hilbert spaces and the action of the coefficients (`Sobolev/Basic`, `Sobolev/Coefficients`). The existence and spectral theory, including its unconditional forms on bounded domains, is developed in the modules above them.

The regularity theory begins with difference quotients along with their control by the translation estimate, and by a bound uniform in the step (`Regularity/DifferenceQuotient`, `DiffQuotientBound`). These estimates are localised by the Caccioppoli inequality (`Caccioppoli`). The test functions required in the interior argument are constructed from the cutoff tower and the calculus of C^1 coefficients (`CutoffTower`, `CoeffC1`). This part of the library ends with the interior H^2 estimate of Theorem 11 (`Regularity/Interior`). On the basis of this estimate the Sobolev bootstrap improves the integrability of the gradient to an exponent at which Morrey's inequality applies and yields interior Hölder continuity with exponent $1/2$ (`Embedding/GagliardoNirenberg`, `Morrey`, `InteriorHolder`). Since the bootstrap uses the ray-integral estimates on which the proof of Morrey's inequality is based, the bootstrap is placed above Morrey's inequality in the import graph.

The regularity theory is extended beyond the H^2 estimate to every order by induction. In the differentiated weak formulation every derivative of the coefficients is taken from a $W^{k,\infty}$ bundle (`Regularity/DifferentiatedWkInfty`, `LeibnizWkInfty`, `CoeffWkInfty`). The induction, for which the cutoff calculus provides the datum at each step (`CutoffDeriv`, `CutoffDatum`, `DatumPiece`), concludes with Theorem 12 (`HigherInterior`). The output of this theorem, a family of weak derivatives of every order up to $m + 2$, is assembled in `Regularity/IteratedFamily` into the closed family to which the embedding layer of Section 3.1 applies. This closed family is used in two modules, namely `Regularity/InteriorSmooth` at every order and `Regularity/InteriorHolderFinite` at finite depth.

Two directories are not used in the proofs of the results above. Campanato's characterisation of Hölder continuity by the decay of the mean oscillation is proved in both directions (`Campanato`). No module outside the directory uses it, since in the library Hölder continuity is derived from Morrey's inequality. Half-ball geometry, tangential difference quotients, the admissibility of a cutoff tested against it and the division of a C^1 weight are steps of the boundary H^2 estimate in [Eva10, §6.3.2] and are proved (`Regularity/Boundary`). No boundary regularity estimate has yet been derived from these steps.

The library distinguishes an elementary core suitable for inclusion in Mathlib, namely the Cauchy–Schwarz inequality on an interval and the one-dimensional Poincaré inequality added to the first-named author's fork of Mathlib, from the remaining development built on existing Mathlib results. The division between reused and new infrastructure was determined mechanically by a query over the combined declaration graph of both Lean environments. The query locates each premise together with its home module and therefore allows the division to be rechecked against the libraries as they evolve.

4 Discussion

Every result claimed in Section 2 is proved in Lean and checked by the kernel. We note four further features of the development. First, the Poincaré inequality is derived from the one-dimensional estimate alone. Second, the compactness of the embedding is proved by means of the Fréchet–Kolmogorov criterion, so that the Fredholm alternative, the description of the spectrum, and the resolvent bound

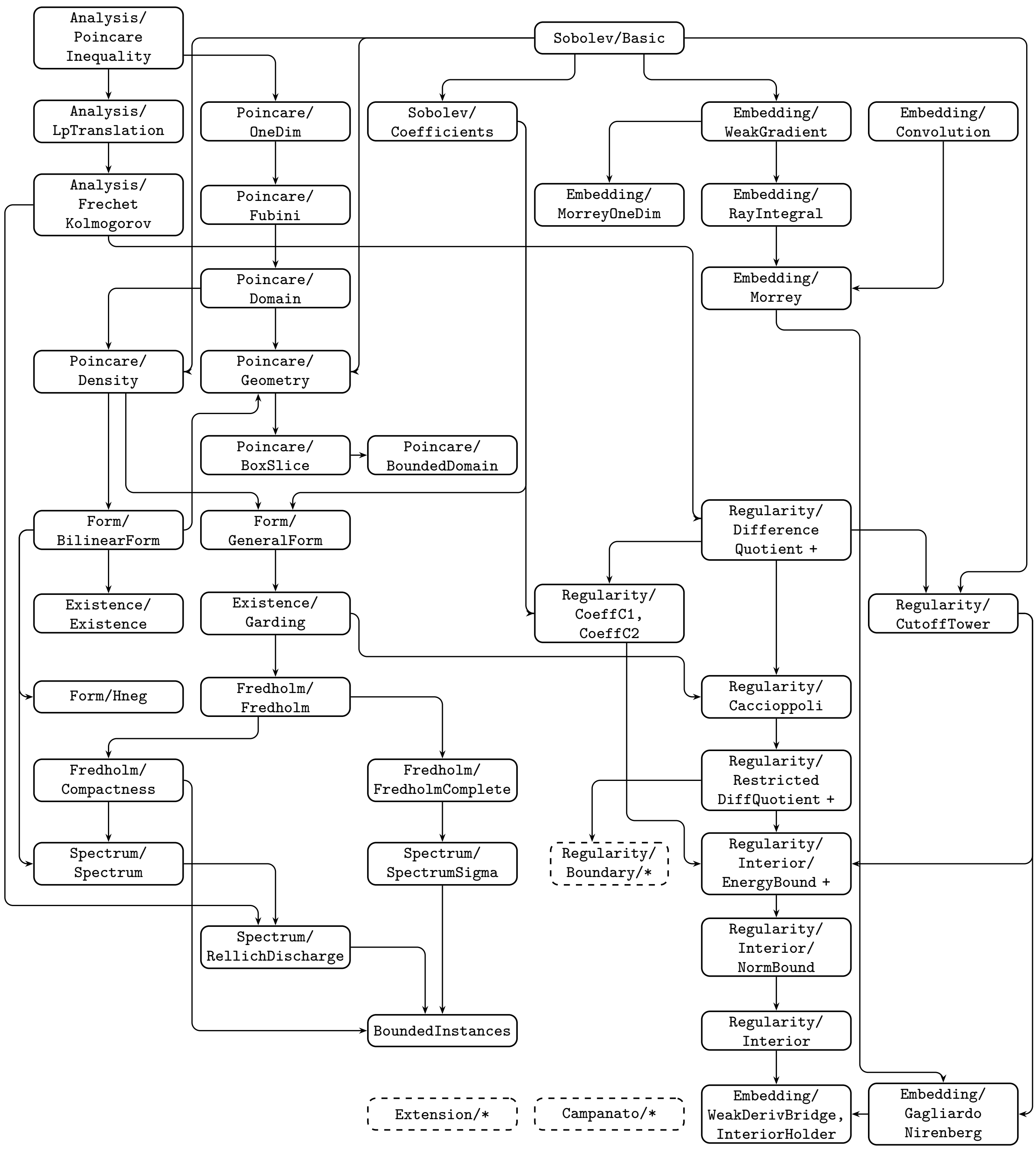

Figure 1: Modules of the library and the imports between them. Each arrow represents an `import` and module paths are given relative to `EllipticPdes/` with long module names broken across lines. Its upper part is the weak-derivative Sobolev layer and the Poincaré reduction, which are independent of the elliptic operator, and its lower part the modules of the solvability and regularity theory built on them. A starred path denotes a whole directory and a box marked `+` includes the continuation modules of the module that it names. Dashed boxes denote directories that no module in the tree imports. Two `Analysis` modules of auxiliary L^2 lemmas, `LpExtendByZero` and `EuclideanFunctionalNorm`, are omitted, as is `Analysis/LpTranslationContinuity`, which belongs to the preparatory results for the extension modules.

require no assumption on Ω beyond boundedness. Third, the declaration on which Theorem 12 rests takes $a^{ij} \in W^{m+1,\infty}(\Omega)$ and $b^i, c \in W^{m,\infty}(\Omega)$, which is weaker than the hypothesis $C^{m+1}(\Omega)$ of the classical statement. Fourth, the variational construction of the eigenvalues is independent of the spectral theorem for the solution operator and also applies to a semilinear equation.

Every declaration is proved with no `sorry` and depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`. The one-dimensional Poincaré inequality and the Cauchy–Schwarz inequality on an interval from which it is derived have been added to a fork of Mathlib in a form suitable for submission upstream.

We measure the extent of a formalisation by its discharge ratio, defined as the fraction of claimed results that are established by a named Lean declaration in place of a warrant or an appeal to routine. By a claimed result we mean a theorem, lemma, corollary or proposition of Section 2, whether it is stated there as a numbered result or in the prose with its declaration given in a footnote. Sixteen of these results are numbered there, and the two lemmas on which Theorem 1 rests are stated in Section 3. Since every claimed result is discharged by a named Lean declaration and no leaf step among them is warranted, routine or open, the discharge ratio of the present development is 1.

4.1 Conduct of the formalisation

Within the workflow and the acceptance checks described in Subsection 1.3, the authors made three decisions that the agents did not reach independently, namely the proof of the Poincaré inequality on a box (Theorem 1) in place of the representation formula on a ball, the bootstrap in half-steps of $1/(2n)$ in place of the full step (Theorem 13 and the iterations on which it depends), and the use of a family closed under weak differentiation in place of an induction on the order (Theorem 12 and the bootstrap that uses it).

The four statuses of Subsection 1.3 are defined with reference to the one question that the kernel cannot decide, namely whether a formal statement is the intended one. A discharged obligation corresponds to a Lean declaration whose type a human reader has compared with the prose statement. Whereas a warranted obligation is justified by a located citation and a routine obligation by shared competence, an open obligation has no justification at all. The kernel verifies only the first component of a discharge, since a statement that is vacuous, that quantifies in the wrong order or that permits a constant to depend on the data still elaborates with only the three standard axioms.

A defect of this kind arose in the present development and was detected in review before the manuscript was written. Every estimate in the regularity theory had quantified the solution and the datum before the constant, so that the type permitted the constant to depend on both and the inequality imposed no constraint. We corrected the statements so that the constant is quantified first, in the order now stated in Section 2, and specified in each statement the quantities on which the constant depends. Since a successful build is compatible with either order, the discharge ratio stated above depends on a comparison, made by a human reader, of each type with its prose statement.

4.2 Relation to prior work

Every general ingredient used in the assembly is available in Mathlib, namely the Lax–Milgram theorem in the form of the coercive-form equivalence `IsCoercive.continuousLinearEquivOfBilin` together with its uniqueness counterpart, the Fréchet–Riesz representation theorem, the theory of product measures and Fubini's theorem, and the spectral theory of compact operators including the Fredholm alternative. The bootstrap uses the Gagliardo–Nirenberg–Sobolev inequality for compactly supported maps in the form `MeasureTheory.eLpNorm_le_eLpNorm_fderiv_of_eq` formalised in Mathlib in [DM24]. The extension of this inequality from compactly supported C^1 maps to the Sobolev classes on a domain defined through weak derivatives is carried out in the present work.

The Sobolev spaces themselves are also constructed in the present work. The Sobolev spaces available in Mathlib are the Bessel potential spaces in [Dol25], defined on the whole space through the Fourier transform. They therefore provide no predicate of weak differentiability on a domain of the kind required by the weak formulation of an equation in divergence form. The De Giorgi–Nash–Moser development in [AK26] contains such a theory, together with a structure for uniformly elliptic coefficients and a Poincaré inequality on balls proved by integrating the fundamental theorem of calculus along rays. Our library was built independently of that development. The boundary between the two was determined by a query over the combined declaration graph of both Lean environments that locates each premise together with the module in which it is declared.

The disclosure in Subsection 1.3 agrees with that given by [CLQ26] for a development of comparable size. Their library is also the closest point of comparison in a second respect, since it is a differential-geometric development on Mathlib that contains the divergence theorem on a compact manifold with boundary, Green's identities, a surface measure and an outward normal. It therefore provides the closest formal counterpart to the integration by parts with a boundary term that the boundary theory of the present work would require. Subsection 4.3 returns to this comparison.

4.3 Further directions

The most immediate direction concerns the interior theory itself. Theorem 11 and Corollary 3 take $a^{ij} \in C^1(\Omega)$, although the declarations on which they rest require only a Lipschitz bound on the principal coefficients. Removing this hypothesis requires the identification of $W^{1,\infty}$ functions with their Lipschitz representative, which Mathlib does not yet contain.

Beyond the interior theory, the parts of [Eva10, Chapter 6] and [GT01, Chapters 6 and 8] that remain to be formalised are the boundary theory, Harnack's inequality and the Schauder theory, which require differing amounts of further work. Of these, boundary H^2 regularity in the sense of [Eva10, §6.3.2, Theorem 4] is the closest to completion, since the geometry of the half-ball, the tangential difference quotients together with the cutoff that ensures their admissibility (`cutoffMulOn_tangDiffQuotG_mem_H01`) and the rule for dividing a C^1 weight out of a weak derivative (`HasWeakDerivOn.of_mul_contDiff_left`) are already formalised. It remains to prove the energy and norm bounds on the half-ball and to rewrite the equation in nondivergence form so that the tangential estimate yields a bound on the normal second derivative. This rewriting has no analogue in the interior argument and constitutes the principal difficulty of the boundary case.

The global theory requires an integration by parts with a boundary term against surface measure on $\partial\Omega$, a tool that the interior theory never uses. No such result is available in Mathlib. The closest formal counterpart is the differential-geometric library in [CLQ26], in which the divergence theorem on a compact manifold with boundary is proved (`stokes_compact_via_pou`). That library also proves Green's first and second identities in the same form, providing a surface measure and outward normal on the boundary. The identification of the face sum with $\int_{\partial M} \langle \nu, X \rangle \, d\sigma$ against the induced surface measure enters as a hypothesis of the theorems that state it (its `boundaryFaceSum_eq_surface_integral_of_chartIdentification` and `green_first_eq_boundary_surface_integral`), so that this identification remains to be proved there. Since that library is directed at closed manifolds, its main theorem is stated for a model without boundary and the results on the boundary are developed separately from it. The boundary H^2 estimate on the flat half-ball requires no such theorem, because the test functions vanish where the boundary term would appear. Its difficulty therefore lies in the rewriting in nondivergence form described above. The trace theory needed for a global estimate, by contrast, requires this identification as the first result to be established.

For the Schauder theory the library contains only Campanato's characterisation of Hölder continuity and its converse, both of which are proved but neither of which is yet used. The library contains the maximum principles of [Eva10, §6.4], namely Theorems 1 and 2 of §6.4.1 for a C^2 subsolution and both clauses of Theorem 3 of §6.4.2, together with the Hopf boundary point lemma and the weak maximum principle for a weak subsolution.[54] Harnack's inequality is the furthest from the present library, since our library contains no form of Moser iteration.

A separate direction concerns the regularity of the coefficients. For equations in divergence form with bounded measurable coefficients, the interior De Giorgi–Nash–Moser theory has already been formalised in [AK26]. At the other end of the scale, the Schauder and Calderón–Zygmund theories for smoother coefficients have as yet no formal counterpart.

A further direction is the completion of the eigenvalue theory. In the present library the eigenvalues are defined through Rayleigh quotients (Theorem 7 and the sequence constructed from it in Subsection 2.1). Some results that remain to be proved include the completeness of the variational family in $L^2(\Omega)$, the divergence of the eigenvalue sequence, and two clauses concerning the principal eigenvalue that depend on a maximum principle, that is, the positivity of the first eigenfunction in Ω and the simplicity of λ_1. Neither clause is proved, because the library states the strong maximum principle only for a C^2 subsolution and establishes only the weak maximum principle for a weak subsolution.

A Supporting definitions

We collect here the supporting definitions used in the statements of Section 2, each given in mathematical prose beside its Lean definition.

`lpExtendByZero` **Extension by zero** as a linear isometry `Lp ℝ p (μ.restrict s) →ₗᵢ[ℝ] Lp ℝ p μ`. A class on the restricted measure is sent to the `Lp` class of its extension by zero, with the `Lp` norm preserved.

`gradCLM` The continuous linear functional whose coordinate values are the entries of `g` at `y`. This is the form that `HasFDerivAt` requires, assembled from the form in which a weak gradient is given.

```
def gradCLM (g : Fin d → EuclideanSpace ℝ (Fin d) → ℝ) (y : EuclideanSpace ℝ (Fin d)) : EuclideanSpace
ℝ (Fin d) →L[ℝ] ℝ := ∑ k, g k y • (EuclideanSpace.proj k : EuclideanSpace ℝ (Fin d) →L[ℝ] ℝ)
```

[54] `EllipticPdes.Classical.weak_maximum_principle`, `EllipticPdes.Classical.weak_maximum_principle_of_nonneg`, `EllipticPdes.Classical.strong_maximum_principle`, `EllipticPdes.Classical.hopf_lemma` and `EllipticPdes.Sobolev.weak_maximum_principle`. The last of these is stated for a weak subsolution in $H^1(\Omega)$ and follows [GT01, Theorem 8.1].

`HasWeakGradOn` `g` is the pointwise weak gradient of `u` on `B`: integration by parts holds against every smooth compactly supported test function whose support lies in `B`. This mirrors `EllipticPdes.Regularity.HasWeakDerivOn` component-wise but for pointwise functions $u, g_k : \mathbb{R}^d \to \mathbb{R}$ (`u, gₖ : EuclideanSpace ℝ (Fin d) → ℝ`).

`HasC1Boundary` **Ω (`Ω`) has C^1 (`C¹`) boundary.** Every boundary point admits a chart, which is the hypothesis of Guo's Theorem III.2.2 and of Evans's §5.4 Theorem 1.

```
def HasC1Boundary (Ω : Set (EuclideanSpace ℝ (Fin d))) : Prop := ∀ x ∈ frontier Ω, ∃ c : C1Chart d,
c.Fits Ω x
```

`laplaceBilin` The bilinear form of the Laplacian on $H_0^1(\Omega)$ (`H₀¹(Ω)`) as a bounded (continuous) bilinear form, with operator-norm bound `d`.

`embL2` The coordinate-`0` embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ (`H₀¹(Ω) ↪ L²(Ω)`), $U \mapsto U0$ (`U ↦ U 0`), as a continuous linear map: the `PiLp` projection onto coordinate `0` precomposed with the submodule inclusion.

```
def embL2 (Ω : Set (EuclideanSpace ℝ (Fin d))) : H01 Ω →L[ℝ] L2D Ω := (PiLp.proj (𝕜 := ℝ) 2 (fun _ :
Fin (d + 1) => L2D Ω) (0 : Fin (d + 1))).comp (H01 Ω).subtypeL
```

`IteratedL2Bound` **Uniform L^2 (`L²`) bound on an iterated family.** Every derivative up to order `k` is bounded by `C` in $L^2(V)$ (`L²(V)`). Kept apart from `HasIteratedWeakDerivOn` so that existence and estimate can be proved and used separately, matching the shape of `interior_H2_estimate`, which returns the derivative and its bound as separate conjuncts.

```
def IteratedL2Bound {u : L2D V} (hu : HasIteratedWeakDerivOn V k u) (C : ℝ) : Prop := ∀ α : List (Fin
d), α.length ≤ k → ‖hu.D α‖ ≤ C
```

`extendL2` **Extension by zero**, $L^2(\Omega) \hookrightarrow L^2(\mathbb{R}^d)$ (`L2D Ω →ₗᵢ[ℝ] EucL2 d`). A class on the restricted measure `volume.restrict Ω` is sent to the whole-space $L^2(\mathbb{R}^d)$ (`L²(ℝ^d)`) class of its extension by zero, with the L^2 (`L²`) norm preserved. This extension lets the whole-space difference quotients act on gradient data given on a restricted domain [Eva10, §6.3.1].

```
def extendL2 {Ω : Set (EuclideanSpace ℝ (Fin d))} (hΩm : MeasurableSet Ω) : L2D Ω →ₗᵢ[ℝ] EucL2 d :=
lpExtendByZero volume 2 Ω hΩm
```

`iteratedNorm` **`H^k(V)` norm of an iterated family**, summed over lists of directions of length at most `k`.

```
def iteratedNorm (H : HasIteratedWeakDerivOn V k u) : ℝ := Real.sqrt (∑ m ∈ Finset.range (k + 1), ∑ α
: Fin m → Fin d, ‖H.D (List.ofFn α)‖ ^ 2)
```

`LocalWeakSol` **Local weak solution on `W`, on plain function representatives.** `u` with gradient `G`, tested against smooth functions compactly supported in `W`. This is the plain-integral form in which Evans §6.3.1 states the equation, as opposed to the `H01`-and-`fullBilin` formulation that `interior_smooth` takes; `isLocalWeakSolution_iff_localWeakSol` connects it to the predicate `IsLocalWeakSolution` on the ambient space.

`partialD` The `i`-th classical partial derivative of φ (`φ`) (a directional `fderiv`).

```
def partialD (i : Fin d) (φ : EuclideanSpace ℝ (Fin d) → ℝ) : EuclideanSpace ℝ (Fin d) → ℝ := fun x =>
(fderiv ℝ φ x) (EuclideanSpace.single i 1)
```

`IsTestFn` A smooth, compactly supported test function whose support lies inside Ω (`Ω`).

```
def IsTestFn (Ω : Set (EuclideanSpace ℝ (Fin d))) (φ : EuclideanSpace ℝ (Fin d) → ℝ) : Prop :=
ContDiff ℝ (⊤ : ℕ∞) φ ∧ HasCompactSupport φ ∧ tsupport φ ⊆ Ω
```

`testGraphSet` The set of test-function graphs over Ω (`Ω`).

```
def testGraphSet (Ω : Set (EuclideanSpace ℝ (Fin d))) : Set (H1amb Ω) := { U | ∃ (φ : EuclideanSpace ℝ
(Fin d) → ℝ) (h : IsTestFn Ω φ), U = h.testGraph }
```

`H01` $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ (`H₀¹(Ω) = W₀^{1,2}(Ω)`): the closure of the smooth compactly supported functions inside the ambient H^1 (`H¹`) space. As a topological closure it is automatically a closed, complete, real Hilbert space.

```
def H01 (Ω : Set (EuclideanSpace ℝ (Fin d))) : Submodule ℝ (H1amb Ω) := (Submodule.span ℝ
(testGraphSet Ω)).topologicalClosure
```

`solOp` The **solution operator** on $L^2(\Omega)$ (`L²(Ω)`) of a coercive form `B`: $G = \iota \circ (B^\sharp)^{-1} \circ \iota^\dagger$ (`G = ι ∘ (B♯)⁻¹ ∘ ι†`), with `ι = embL2 Ω` the Rellich embedding and $(B^\sharp)^{-1}$ (`(B♯)⁻¹`) the Lax–Milgram inverse of `B`.

```
def solOp (B : H01 Ω →L[ℝ] H01 Ω →L[ℝ] ℝ) (hco : IsCoercive B) : L2D Ω →L[ℝ] L2D Ω := (embL2 Ω).comp
((hco.continuousLinearEquivOfBilin.symm : H01 Ω →L[ℝ] H01 Ω).comp (embL2 Ω).adjoint)
```

`rayleighValues` The values a bilinear form takes on the unit L^2 (`L²`) sphere.

```
def rayleighValues (B : H01 Ω →L[ℝ] H01 Ω →L[ℝ] ℝ) : Set ℝ := (fun U => B U U) '' rayleighSphere Ω
```

`principalEigenvalue` **Principal eigenvalue** of a symmetric coercive form on $H_0^1(\Omega)$ (`H₀¹(Ω)`): the infimum of the Rayleigh quotient `B[U, U]` over the functions of unit L^2 (`L²`) norm.

```
def principalEigenvalue (B : H01 Ω →L[ℝ] H01 Ω →L[ℝ] ℝ) : ℝ := sInf (rayleighValues B)
```

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