ON ISOLATED SINGULARITIES AND GENERIC REGULARITY OF MIN-MAX CMC HYPERSURFACES

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Abstract

In compact Riemannian manifolds of dimension 3 or higher with positive Ricci curvature, we prove that every constant mean curvature hypersurface produced by the Allen–Cahn min-max procedure in [BW20a] (with constant prescribing function) is a local minimiser of the natural area-type functional around each isolated singularity. In particular, every tangent cone at each isolated singularity of the resulting hypersurface is area-minimising. As a consequence, for any real λ we show, through a surgery procedure, that for a generic 8-dimensional compact Riemannian manifold with positive Ricci curvature there exists a closed embedded smooth hypersurface of constant mean curvature λ; the minimal case ($\lambda = 0$) of this result was obtained in [CLS22].

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1 Introduction

The Allen–Cahn min-max procedure in [BW20a], with constant prescribing function, shows that in a compact 8-dimensional Riemannian manifold there exists a quasi-embedded hypersurface of constant mean curvature, with a singular set consisting of finitely many points (see Subsection 1.2 below for a precise description). One may thus conjecture the existence of a smoothly embedded constant mean curvature hypersurface in all 8-dimensional Riemannian manifolds under some assumption on the metric; for example a genericity assumption. As a first step, we resolve this for manifolds with positive Ricci curvature:

Theorem 1. *Let N be a smooth compact 8-dimensional manifold and $\lambda \in \mathbb{R}$. There is an open and dense subset, $\mathcal{G}$, of the smooth metrics with positive Ricci curvature such that for each $g \in \mathcal{G}$, there exists a closed embedded smooth hypersurface of constant mean curvature λ in (N, g).*

We actually prove more general results valid in higher dimensions, showing the generic existence of a closed embedded hypersurface of constant mean curvature, with singular set of codimension 7, containing no isolated singularities with regular tangent cones. Indeed, Theorem 1 is a consequence of the following:

Theorem 2. *Let N be a smooth compact manifold of dimension $n+1 \geq 3$ and $\lambda \in \mathbb{R}$. There is a dense subset, $\mathcal{G}$, of the smooth metrics with positive Ricci curvature such that for each $g \in \mathcal{G}$, there exists a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set of Hausdorff dimension at most $n-7$, containing no isolated singularities with regular tangent cones in (N, g).*

Remark 1. *Let $\mathrm{Met}^{k,\alpha}_{\mathrm{Ric}_g > 0}(N)$ for each $k \geq 1$ and $\alpha \in (0,1)$ denote the open subset of Riemannian metrics of regularity $C^{k,\alpha}$ on N with positive Ricci curvature. In the proof of Theorem 2 we in fact establish that there exists a dense set, $\mathcal{G}_k \subset \mathrm{Met}^{k,\alpha}_{\mathrm{Ric}_g > 0}(N)$, such that for each $g \in \mathcal{G}_k$, the same existence conclusion of Theorem 2 holds.*

The focus of the present work concerns the generic regularity of constant mean curvature hypersurfaces in ambient dimensions 8 or higher and is related to previous work concerning minimal hypersurfaces. We now briefly summarise the works on generic regularity for minimal hypersurfaces in ambient dimension 8:

- The existence of the Simons cone, introduced in [Sim68], showed that stable minimal hypersurfaces may admit isolated singularities in dimension 8.
- The generic regularity of area-minimisers in each non-zero homology class was established in [Sma93], using the fundamental result of [HS85].
- In [CLS22], using the Almgren–Pitts min-max procedure and [HS85] (in particular the extension in [Liu19]), the existence of smooth minimal hypersurfaces was established in manifolds equipped with a generic metric of positive Ricci curvature.
- In [LW21] it was shown that every 8-dimensional closed manifold equipped with a generic metric (with no curvature assumption) admits a smooth minimal hypersurface. See also [LW22], where it is shown that for a generic metric, every embedded locally stable minimal hypersurface is smooth in dimension 8.

Both [Sma93] and [CLS22] exploit local foliations by area-minimising hypersurfaces, provided by [HS85], allowing for a surgery procedure to be established in order to perturb the metric locally. Recently an analogous foliation was established in [Les23] for constant mean curvature hypersurfaces that locally minimise a prescribed mean curvature functional to at least one side. Such a foliation, to one side of the hypersurface, provides a natural way to perturb away an isolated singularity via a surgery procedure.

In order to produce such a foliation, one needs to establish that the tangent cones to isolated singularities of a candidate hypersurface are area-minimising; we establish this for all hypersurfaces of constant mean curvature arising from the Allen–Cahn min-max procedure in [BW20a] (with constant prescribing function) in manifolds with positive Ricci curvature.

The Allen–Cahn min-max procedure in [BW20a] produces in the first instance a quasi-embedded hypersurface of constant mean curvature with a possibly non-empty singular set of codimension at least 7. The results of [BW22] then establish that in manifolds with positive Ricci curvature the above hypersurface of constant mean curvature in fact attains the min-max value (in a manner made precise below) and is embedded, with the above dimension bound on the singular set. We thus work with this hypersurface as a candidate to perturb away isolated singularities via a surgery procedure.

For a compact Riemannian manifold (N, g) and $\lambda \in \mathbb{R}$ we define the $\mathcal{F}_\lambda$ functional on a Caccioppoli set, $F \subset N$, by

$$\mathcal{F}_\lambda(F) = \mathrm{Per}_g(F) - \lambda \mathrm{Vol}_g(F).$$

Recall that, as shown in [BW20b, Proposition B.1], smooth constant mean curvature hypersurfaces are locally $\mathcal{F}_\lambda$-minimising. Our main technical result shows that this $\mathcal{F}_\lambda$-minimisation also holds in sufficiently small balls around isolated singularities for constant mean curvature hypersurfaces produced by the Allen–Cahn min-max in manifolds with positive Ricci curvature:

Theorem 3. *Let (N, g) be a smooth compact Riemannian manifold of dimension $n+1 \geq 3$, with positive Ricci curvature, and $\lambda \in \mathbb{R}$. The one-parameter Allen–Cahn min-max in [BW20a], with constant prescribing function λ, produces a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set of Hausdorff dimension at most $n-7$, locally $\mathcal{F}_\lambda$-minimising in balls around each isolated singularity. Precisely, this hypersurface arises as the boundary of a Caccioppoli set, $E \subset N$, and for each isolated singularity, $p \in \overline{\partial^* E} \setminus \partial^* E$, there exists an $r > 0$ such that*

$$\mathcal{F}_\lambda(E) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(G) \mid G \setminus B_r(p) = E \setminus B_r(p)\},$$

where $\mathcal{C}(N)$ is the set of Caccioppoli sets in N. Consequently, the hypersurface has area-minimising tangent cones at each isolated singularity.

Remark 2. *In the case that $\lambda = 0$, Theorem 3 shows that minimal hypersurfaces produced by the Allen–Cahn min-max procedure in [Gua18] in manifolds with positive Ricci curvature are in fact locally area-minimising (to both sides), as opposed to just one-sided homotopy minimising as obtained via the Almgren–Pitts min-max procedure in the results of [CLS22]. These minimising properties immediately pass to tangent cones. Stable regular minimal cones that do not minimise area are known to exist, for example the Simons cone*

$$C^{1,5} = \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^6 \mid 5|x|^2 = |y|^2\right\},$$

is stable and one-sided area-minimising, but is not area-minimising to the other side (see [Law91]). Such a cone is not explicitly ruled out in [CLS22] from arising as a tangent cone to a min-max minimal hypersurface at an isolated singularity. Theorem 3 precludes such tangent cones. We also note that in the Allen–Cahn framework, obtaining an absolute area-minimisation property, as opposed to a homotopic minimisation property, appears to be natural; indeed the relevant space where the min-max is carried out is $W^{1,2}(N)$, which is contractible.

The present work fits into the broader context of generic regularity for solutions to geometric variational problems. In addition to the works on generic regularity for minimal hypersurfaces in dimension 8 as discussed above, we provide a non-exhaustive summary of work on generic regularity for geometric variational problems:

- In [CMS23a] and [CMS23b] the generic regularity of area-minimising minimal hypersurfaces is established, in various settings, up to ambient dimension 10.
- In [Whi85] and [Whi19] it is shown that for a generic ambient metric, every 2-dimensional surface (integral current or flat chain mod 2) without boundary that minimises area in its homology class has support equal to a smoothly embedded minimal surface.
- In [Moo06] and [Moo17] it is shown that for a generic ambient metric, parameterised 2-dimensional minimal surfaces are free of branch points.
- In [CCMS20] and [CCMS22] an analogy was established between mean curvature flow with generic initial data and the generic regularity of area-minimising hypersurfaces.
- In [FROS20] the generic regularity of free boundaries for the obstacle problem is established up to ambient dimension 4.

1.1 Notation

We now collect some notation and definitions that will be used throughout the paper:

- Unless otherwise stated, throughout the paper we let (N^{n+1}, g) be a compact (with empty boundary), Riemannian manifold of dimension $n+1 \geq 3$, with positive Ricci curvature, $\mathrm{Ric}_g > 0$. We will implicitly assume N is connected.
- We let $M \subset N$ be a non-empty, smooth, two-sided, separating, embedded hypersurface of constant mean curvature $\lambda \in \mathbb{R}$, with closed singular set, $\mathrm{Sing}(M) = \overline{M} \setminus M$, of Hausdorff dimension at most $n-7$. As M is a separating hypersurface, we may write $N \setminus \overline{M}$ as two disjoint open sets, E and $N \setminus E$, with common boundary $\overline{M}$.
- We say that $p \in \mathrm{Sing}(M)$ is an *isolated singularity* of M if there exists some $R_p > 0$ such that $\mathrm{Sing}(M) \cap \overline{B_{R_p}}(p) = \{p\}$, i.e. such that $M \cap \overline{B_{R_p}}(p)$ is smooth. Moreover, we say that a multiplicity one tangent cone, C_p, to M at an isolated singularity, $p \in \mathrm{Sing}(M)$, is a *regular tangent cone* if $\mathrm{Sing}(C_p) = \{0\}$, where 0 here denotes the origin in $\mathbb{R}^{n+1}$. We note that the tangent cone to M at an isolated singularity with regular tangent cone is necessarily unique by the work of [Sim83].
- A measurable set $F \subset N$ is a *Caccioppoli set* if

$$\mathrm{Per}_g(F) = \sup\left\{ \int_E div_g \varphi \,\middle|\, \varphi \in \Gamma(TN), ||\varphi||_\infty \leq 1 \right\} < \infty,$$

  where div_g is the divergence with respect to the metric g, $\Gamma(TN)$ is the set of vector fields on N and $||\cdot||_\infty$ denotes the supremum norm. We denote by $\mathcal{C}(N)$ the set of Caccioppoli sets in N.
- By De Giorgi's Structure Theorem we have that the distributional derivative $D_g\chi_F$ (a Radon measure) of a Caccioppoli set F is given by $D_g\chi_F = -\nu_F \mathcal{H}^n \llcorner \partial^* F$, where $\partial^* F$ is the reduced boundary of F (an n-rectifiable set), $\mathcal{H}^n$ is the n-dimensional Hausdorff measure, ν_F is the normal direction to $\partial^* F$ pointing inside F defined $\mathcal{H}^n$-a.e. and χ_F is the indicator function of the set F. Our main reference for Caccioppoli sets will be [Mag12].
- We define the following prescribed mean curvature functional on measurable subsets of N: for a measurable set $F \subset N$ we let

$$\mathcal{F}_\lambda(F) = \mathrm{Per}_g(F) - \lambda \mathrm{Vol}_g(F) + \frac{\lambda}{2}\mathrm{Vol}_g(N),$$

where Vol_g denotes the $\mathcal{H}^{n+1}$ measure with respect to the metric g. This definition differs from that of Section 1 by the addition of the constant $\frac{\lambda}{2}\mathrm{Vol}_g(N)$. Note however this does not affect the set of critical points of the functional $\mathcal{F}_\lambda$ and is made purely for convenience of notation in forthcoming computations.

- With the above two definitions in mind, we take throughout ν to be the global unit normal to M pointing into E and write $M = \partial^* E$; by viewing $M = \partial^* E$ as the reduced boundary of the Caccioppoli set E we have that E is a critical point for $\mathcal{F}_\lambda$.

- We will use the notions of integral currents and varifolds throughout the paper; a reference for the notation and definitions used here may be found in [Sim84].

- Let dist_N denote the Riemannian distance on N and define the function $d_{\overline{M}}$ on N by setting $d_{\overline{M}}(\cdot) = \mathrm{dist}_N(\cdot, \overline{M})$; we then have that $d_{\overline{M}}$ is Lipschitz on N (with Lipschitz constant equal to 1) and, as N is complete, for each $x \in N$ there exists a geodesic realising the value $d_{\overline{M}}(x)$. Let $d(N) = \sup_{x,y\in N} d_N(x,y)$ be the diameter of N, which is finite as N is closed.

- We fix $R_l > 0$ such that for every $R \in (0, R_l)$ and each point $p \in N$ we have that the ball $B_R(p) \subset N$ of radius R centred at a point $p \in N$ is 2-bi-Lipschitz diffeomorphic, via a geodeisic normal coordinate chart, to the Euclidean ball, $B_R^{\mathbb{R}^{n+1}}(0) \subset \mathbb{R}^{n+1}$ of radius R centred at the origin in $\mathbb{R}^{n+1}$.

- For $\varepsilon \in (0,1)$ we denote the *Allen–Cahn energy* of a function $u \in W^{1,2}(N)$ by

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2\sigma}\int_N e_\varepsilon(u) = \frac{1}{2\sigma}\int_N \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon},$$

where W is a C^2 double well potential with non-degenerate minima at ± 1, $c_W \le W''(t) \le C_W$ for constants $c_W, C_W > 0$ for all $t \in \mathbb{R} \setminus [-2,2]$ and $\sigma = \int_{-1}^{1}\sqrt{W(t)/2}\,dt$. We then consider the following functional, which we shall refer to simply as the *energy*, defined on functions $u \in W^{1,2}(N)$ by

$$\mathcal{F}_{\varepsilon,\lambda}(u) = \mathcal{E}_\varepsilon(u) - \frac{\lambda}{2}\int_N u.$$

1.2 Allen–Cahn min-max

Let (N^{n+1}, g) a closed, connected Riemannian manifold of dimension $n+1 \ge 3$. The Allen–Cahn min-max procedure in [BW20a] produces a hypersurface with mean curvature prescribed by an arbitrary non-negative Lipschitz function, and provides sharp dimension bounds on the singular set.

We recall this procedure in the case relevant to this work, in which the metric is assumed to have positive Ricci curvature, and the prescribing function is a non-negative constant λ; for producing a candidate hypersurface of constant mean curvature $\lambda < 0$ one can consider $-\lambda$ in the results below. The constant mean curvature hypersurfaces produced will, after proving Theorem 3, be candidates for our surgery procedure established in Section 2.

For $\varepsilon \in (0,1)$ there exist two constant functions, a_ε and b_ε, on N, stable critical points of $\mathcal{F}_{\varepsilon,\lambda}$ with $-1 < a_\varepsilon < -1 + c\varepsilon$, $1 < b_\varepsilon < 1 + c\varepsilon$ and $a_\varepsilon \to -1$, $b_\varepsilon \to 1$ on N as $\varepsilon \to 0$, where $c > 0$ depends on W and λ. These functions are constructed in [BW20a, Section 5] by means of the negative gradient flow, through constant functions, of $\mathcal{F}_{\varepsilon,\lambda}$ starting at ± 1. In particular, there are continuous paths of functions in $W^{1,2}(N)$ connecting a_ε to -1 and b_ε to 1, provided by the negative gradient flow of $\mathcal{F}_{\varepsilon,\lambda}$, with energy along these paths bounded from above by $\mathcal{F}_{\varepsilon,\lambda}(-1)$ and $\mathcal{F}_{\varepsilon,\lambda}(1)$ respectively.

For every $\varepsilon \in (0,1)$ a min-max critical point, $u_\varepsilon \in W^{1,2}(N)$, of $\mathcal{F}_{\varepsilon,\lambda}$ may be constructed, with $\sup_N |u_\varepsilon|$ uniformly bounded and $\mathcal{E}_\varepsilon(u_\varepsilon)$ uniformly bounded from above and below by positive constants. This is done by applying a mountain pass lemma, for paths between the two stable critical points a_ε and b_ε, based on the fact that $\mathcal{F}_{\varepsilon,\lambda}$ satisfies a Palais–Smale condition. The Morse index of the u_ε will then automatically be equal 1 by virtue of the fact that if $\mathrm{Ric}_g > 0$ then, as noted in [BW20a, Remark 6.7], a_ε and b_ε are the only stable critical points of $\mathcal{F}_{\varepsilon,\lambda}$.

By general principles, the bounds above imply that there exist a sequence $\varepsilon_j \to 0$, a non-zero Radon measure, μ, on N and a function, $u_\infty \in BV(N)$ with $u_\infty = \pm 1$ for a.e. $x \in N$, such that for the min-max critical points, $\{u_{\varepsilon_j}\}_{j=1}^\infty$, we have as $\varepsilon_j \to 0$ that $e_{\varepsilon_j}(u_{\varepsilon_j}) \to \mu$ weakly as measures and $u_{\varepsilon_j} \to u_\infty$ strongly in $L^1(N)$. Defining $E = \{u_\infty = 1\}$, we note that E is a Caccioppoli set with its reduced boundary $\partial^* E \subset \mathrm{Spt}\mu$; moreover, as $\mathrm{Ric}_g > 0$ we have that $E \neq \emptyset$ by [BW20a, Remark 6.7].

In [BW20a], relying on the combined works of [HT00] and [RT07], it is then established that μ is the weight measure of an integral n-varifold V with the following properties:

- $V = V_0 + V_\lambda$.
- V_0 is a (possibly zero) stationary integral n-varifold on N with $\mathrm{Sing}(V_0)$ empty if $2 \leq n \leq 6$, $\mathrm{Sing}(V_0)$ discrete if $n = 7$ and $\mathrm{Sing}(V_0)$ of Hausdorff dimension $\leq n-7$ when $n \geq 8$.
- $V_\lambda = |\partial^* E| \neq 0$ (by [BW20a, Remark 6.7] as $\mathrm{Ric}_g > 0$), the multiplicity one n-varifold associated with the reduced boundary $\partial^* E$. Then $\mathrm{Spt}(V_\lambda) \setminus \mathrm{Sing}(V_\lambda) = \overline{\partial^* E} \setminus \mathrm{Sing}(V_\lambda)$ is a quasi-embedded hypersurface of constant mean curvature λ with respect to the unit normal pointing into E; moreover, $\mathrm{Sing}(V_\lambda)$ is empty if $2 \leq n \leq 6$, $\mathrm{Sing}(V_\lambda)$ is discrete if $n = 7$ and $\mathrm{Sing}(V_\lambda)$ is of Hausdorff dimension $\leq n-7$ when $n \geq 8$.

In the above, quasi-embedded means that near every non-embedded point of $\overline{\partial^* E} \setminus \mathrm{Sing}(V_\lambda)$, $\overline{\partial^* E}$ is the union of two embedded $C^{2,\alpha}$ disks intersecting tangentially, with each disk lying on one side of the other. Furthermore, for a varifold W (for example V_0 and V_λ above) by letting gen-reg(W) be the set of quasi-embedded points of $\mathrm{Spt}||W||$ we have defined $\mathrm{Sing}(W) = \mathrm{Spt}||W|| \setminus \text{gen-reg}(W)$ (thus for an embedded hypersurface this agrees with the definition in Subsection 1.1 above). For a more detailed description of the definitions and results above we refer to [BW20a, Sections 3 and 4].

As in [BW22, Theorem 2], the path we exhibit, for all $\varepsilon > 0$ sufficiently small, in Subsection 5.2 with the upper energy bounds provided by Lemma 5 (depicted by the dashed lines in Figure 2) between 1 and -1, along with short paths of constant functions connecting 1 to b_ε and -1 to a_ε, proves that $V_0 = 0$ (i.e. that the min-max procedure produces no minimal piece in manifolds with positive Ricci curvature). Using this we then note that as we have $e_{\varepsilon_j} \to \mu$ as $\varepsilon_j \to 0$ and $E = \{u_\infty = 1\}$ we have

$$\mathcal{F}_{\varepsilon_j,\lambda}(u_{\varepsilon_j}) \to \mathcal{F}_\lambda(E) \text{ as } \varepsilon_j \to 0, \tag{1}$$

i.e. that the constant mean curvature hypersurface attains the min-max value. In the proof of Theorem 3, under the contradiction assumption that our candidate hypersurface produced by the above procedure does not satisfy a local minimisation property, we will exploit (1) by constructing continuous paths of functions in $W^{1,2}(N)$ for all $\varepsilon > 0$ sufficiently small, admissible in the min-max construction above, with energy along the paths bounded above by a value strictly below $\mathcal{F}_\lambda(E)$ (independently of ε); thus violating the min-max characterisation of E.

In fact, by [BW22, Theorem 4], for $\lambda \neq 0$ we have that $\partial^* E$ is embedded (rather than quasi-embedded). Moreover, $\partial^* E$ is connected, has index 1 and is separating in the sense that $N \setminus \partial E$ may be written as the union of two disjoint open sets whose common boundary is ∂E. The same results hold in the case that $\lambda = 0$ by combining [Gua18, Theorem A] with [Bel20, Theorem 1.8].

To summarise, we know that for (N^{n+1}, g) a closed, connected Riemannian manifold of dimension $n + 1 \geq 3$, with positive Ricci curvature, the properties of M as stated in Subsection 1.1 hold for any constant mean curvature hypersurface produced from the Allen–Cahn min-max construction of [BW20a] with prescribing function taken to be a constant.

1.3 Strategy

The main strategy of the present work can be split into three distinct steps:

1. **Surgery procedure:** We show how to perturb constant mean curvature hypersurfaces that are locally $\mathcal{F}_\lambda$-minimising around isolated singularities with regular tangent cones, resulting in a smooth hypersurface with constant mean curvature.
2. **Functions to geometry:** We relate the local geometric behaviour of hypersurfaces produced by the Allen–Cahn min-max procedure to the $\varepsilon \to 0$ energy properties of specific $W^{1,2}(N)$ functions.
3. **Paths of functions:** By exhibiting an admissible min-max path, we establish the energy properties for the functions from Step 2 hold, which enables us to conclude that hypersurfaces generated through the Allen-Cahn min-max procedure in positive Ricci curvature are locally $\mathcal{F}_\lambda$-minimising around isolated singularities.

We now sketch these steps in more detail:

1. **Surgery procedure:** As mentioned in Section 1, a local foliation around a hypersurface provides a natural way to perturb away an isolated singularity via a surgery procedure. Using the work of [HS85], in both [Sma93] and [CLS22], isolated singularities with regular tangent cones to locally area-minimising hypersurfaces are perturbed away by a "cut-and-paste" gluing along with a conformal change of the initial metric.

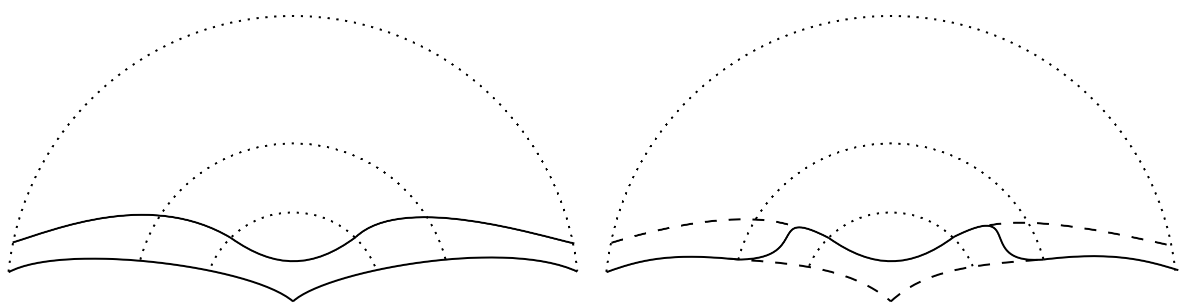

Figure 1: In both graphics the innermost two thinly dotted curves depict an annulus around an isolated singularity of a constant mean curvature hypersurface and the outermost thinly dotted curves depict the boundary of the ball in which the foliations will be defined. In the left-hand graphic the lower solid curve depicts an isolated singularity (assumed to have regular tangent cone) of a constant mean curvature hypersurface and the upper solid curve depicts, under the assumption that the lower singular hypersurface is locally $\mathcal{F}_\lambda$-minimising, a smooth constant mean curvature hypersurface in the one-sided foliation provided by [Les23]. The solid curve in the right-hand graphic depicts the smooth hypersurface constructed by gluing both of the hypersurfaces in the left-hand graphic. This gluing is done in such a way that the hypersurface outside of the larger ball in the annulus agrees with the singular one and inside of the smaller ball in the annulus agrees with the hypersurface provided by the foliation; with the thick dashed lines depicting the pieces of the hypersurfaces in the left-hand graphic not included in the construction in the right. The resulting construction in the right-hand graphic is then, after a suitable metric perturbation, the desired smooth constant mean curvature hypersurface.

We take a similar approach here. Near an isolated singularity with regular tangent cone of a locally $\mathcal{F}_\lambda$-minimising hypersurface of constant mean curvature λ, in [Les23] it is shown that there is a foliation around this hypersurface (to either side) by smooth hypersurfaces of constant mean curvature λ. In Section 2, using this foliation, we establish a surgery procedure to perturb away isolated singularities with regular tangent cones of locally $\mathcal{F}_\lambda$-minimising constant mean curvature hypersurfaces.

This is achieved by first constructing, via a "cut-and-paste" gluing, a smooth hypersurface close in Hausdorff distance to the original one inside a chosen ball (which may be taken arbitrarily small), with constant mean curvature λ outside of an annulus and inside which the original hypersurface is assumed to be smooth; this construction is depicted in Figure 1.

It then remains to perturb the metric inside of this annulus so that the newly constructed hypersurface has constant mean curvature λ everywhere. This is achieved by an appropriate choice of function for conformal change of the original metric; with the resulting metric arbitrarily close, in the $C^{k,\alpha}$-norm, to the original. The result is then a smooth hypersurface of constant mean curvature λ with respect to the new metric, agreeing with the original hypersurface outside of the chosen ball.

2. **Functions to geometry:** We aim to show that, when the ambient metric is assumed to have positive Ricci curvature, the hypersurfaces of constant mean curvature λ produced by the Allen–Cahn min-max procedure in [BW20a] are in fact locally $\mathcal{F}_\lambda$-minimising. In order to do this we first relate the local $\mathcal{F}_\lambda$-minimisation we desire to a property about the energy of specific $W^{1,2}(N)$ functions defined from such a hypersurface as the parameter $\varepsilon \to 0$.

Rather than work directly with the min-max critical points of $\mathcal{F}_{\varepsilon,\lambda}$ produced in [BW20a], we instead introduce, in Subsection 3.3, a function $v_\varepsilon \in W^{1,2}(N)$, which we call the *one-dimensional profile*. This function is constructed by placing a truncated version of the one-dimensional solution to the Allen–Cahn equation, $\overline{\mathbb{H}}^\varepsilon$, in the normal direction to an underlying hypersurface, M, of constant mean curvature λ as in Subsection 1.1:

$$v_\varepsilon = \overline{\mathbb{H}}^\varepsilon \circ d^{\pm}_{\overline{M}},$$

where here $d^{\pm}_{\overline{M}}$ is the Lipschitz *signed distance function* to $\overline{M}$, taking positive values in E and negative values in $N \setminus E$. The function v_ε is then shown to act as an approximation of the hypersurface M in the sense that, analogously to (1), we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \to \mathcal{F}_\lambda(E) \text{ as } \varepsilon \to 0. \tag{2}$$

In Subsection 4.2, for an isolated singularity, $p \in \mathrm{Sing}(M)$, $\varepsilon > 0$ sufficiently small and radius $\rho > 0$, we minimise $\mathcal{F}_{\varepsilon,\lambda}$ over a class of functions $\mathcal{A}_{\varepsilon,\rho}(p)$. The set $\mathcal{A}_{\varepsilon,\rho}(p)$ is, roughly speaking, all $W^{1,2}(N)$ functions agreeing with v_ε outside of the ball of radius ρ centred at p. The minimiser of this problem is thus a function, $g_\varepsilon \in W^{1,2}(N)$, that agrees with v_ε outside of the ball $B_\rho(p)$ and is such that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_{\varepsilon,\rho}(p)} \mathcal{F}_{\varepsilon,\lambda}(u). \tag{3}$$

Note that the notation used for g_ε suppresses the dependence on $p \in \mathrm{Sing}(M)$ and $\rho > 0$ used in the construction; in each instance that the functions g_ε are utilised the choice of isolated singularity and radius in question will be made explicit. We then produce, in Subsection 4.3, a sequence of "recovery functions", admissible in the minimisation problem that produced g_ε above, for any local $\mathcal{F}_\lambda$-minimiser. Precisely, in the vein of [KS89], for each local $\mathcal{F}_\lambda$-minimiser, $F \in \mathcal{C}(N)$, agreeing with E outside of $B_{\frac{\rho}{2}}(p)$, we show that there exists a sequence of functions, $f_\varepsilon \in \mathcal{A}_{\varepsilon,\rho}(p)$, for all $\varepsilon > 0$ sufficiently small such that

$$\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F) \text{ as } \varepsilon \to 0. \tag{4}$$

As $f_\varepsilon \in \mathcal{A}_{\varepsilon,\rho}(p)$ we conclude that by (3) we have $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon)$. In particular, by (2), (3) and (4), if it holds that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \text{ for some sequence } \tau_\varepsilon \to 0 \text{ as } \varepsilon \to 0, \tag{5}$$

then

$$\mathcal{F}_\lambda(E) \leq \mathcal{F}_\lambda(F),$$

so that E is $\mathcal{F}_\lambda$-minimising in $B_{\frac{\rho}{2}}(p)$. In this manner we have related, via (5), the $\varepsilon \to 0$ energy behaviour of specific $W^{1,2}(N)$ functions, namely v_ε and g_ε, defined from a hypersurface as produced by the Allen–Cahn min-max procedure, to the geometric behaviour of the underlying hypersurface; precisely, if (5) holds then E is locally $\mathcal{F}_\lambda$-minimising. In order to prove Theorem 3 we will turn our attention, in Step 3 below, to establishing that (5) holds for all constant mean curvature hypersurfaces produced by the Allen–Cahn min-max procedure in manifolds with positive Ricci curvature.

In order to produce the "recovery functions" above we first establish a local smoothing procedure for Caccioppoli sets that are smooth in an annular region. This is done by another "cut-and-paste" argument using Sard's Theorem on the level sets of mollified indicator functions for the Caccioppoli set, full details of which can be found in Subsection 4.1.

We emphasise that this local smoothing procedure we exhibit for Caccioppoli sets is not, in and of itself, sufficient to establish Theorem 1. The reason for this is that the local smoothing we produce records no information about the mean curvature near the isolated singularity, and this lack of information on the mean curvature makes it difficult to perturb the metric. The foliation of [Les23] however ensures there exists a sequence of smooth hypersurfaces of constant mean curvature λ converging to the singular hypersurface, allowing for the perturbation to be constructed in the manner as described in Step 1.

3. **Paths of functions:** Similarly to the strategy employed in previous works on hypersurfaces produced by the Allen–Cahn min-max procedure, for example in [Bel20], [BW20a], and [BW22], the proof of Theorem 3 is achieved by exhibiting a suitable continuous path in $W^{1,2}(N)$.

Under the assumption that a hypersurface produced by the min-max procedure violates a desired property, one basic idea is to exploit its min-max characterisation as follows. If a path admissible in the min-max procedure may be produced, with energy along this path bounded above by a constant strictly less than the min-max value, then one contradicts the assumption that such a hypersurface arose from the min-max. Thus, the desired property must hold for all hypersurfaces produced by the min-max procedure.

We first emphasise that the paths we construct in $W^{1,2}(N)$ reflect the underlying geometry imposed by the assumption of positive Ricci curvature. We denote the super-level sets and level sets of the signed distance function, for each $s \in \mathbb{R}$, by

$$E(s) = \{x \in N \mid d^{\pm}_{\overline{M}}(x) > s\} \text{ and } \Gamma(s) = \{x \in N \mid d^{\pm}_{\overline{M}}(x) = s\},$$

respectively; so that $E(0) = E$ and $\Gamma(0) = \overline{M}$. Formally computing we have, for almost every $s \in \mathbb{R}$, that

$$\frac{d}{ds}\mathcal{H}^n(\Gamma(s)) = -\int_{\Gamma(s)} H(x,s)\, d\mathcal{H}^n(x) \text{ and } \frac{d}{ds}\mathrm{Vol}_g(E(s)) = \mathcal{H}^n(\Gamma(s)), \tag{6}$$

where $H(x,s)$ denotes the mean curvature of the level set $\Gamma(s)$ at a point x. By denoting $m = \min_N \mathrm{Ric}_g > 0$, the assumption of positive Ricci curvature implies the following relation between the mean curvature of M and the level sets $\Gamma(s)$:

$$\begin{cases} H(x,s) \geq \lambda + ms \text{ for } s > 0 \\ H(x,0) = \lambda \\ H(x,s) \leq \lambda + ms \text{ for } s < 0 \end{cases}, \tag{7}$$

see Subsection 3.2. Therefore, by (6) and (7), for each $t \in \mathbb{R} \setminus \{0\}$ we compute that

$$\begin{aligned} \mathcal{F}_\lambda(E(t)) - \mathcal{F}_\lambda(E) &= \int_0^t \frac{d}{ds}\mathcal{H}^n(\Gamma(s)) - \lambda \frac{d}{ds}\mathrm{Vol}_g(E(s))\, ds \\ &= \int_0^t \int_{\Gamma(s)} \lambda - H(x,s)\, d\mathcal{H}^n(x)\, ds < 0, \end{aligned}$$

and so from the assumption of positive Ricci curvature we conclude that for each $t \in \mathbb{R} \setminus \{0\}$ we have

$$\mathcal{F}_\lambda(E(t)) < \mathcal{F}_\lambda(E). \tag{8}$$

We then consider the continuous path of *sliding functions*, $v_\varepsilon^t \in W^{1,2}(N)$, produced by sliding the zero level set of v_ε from $\overline{M}$ to $\Gamma(t)$ for each $t \in \mathbb{R}$; that is

$$v_\varepsilon^t = \overline{\mathbb{H}}^\varepsilon \circ (d^\pm_{\overline{M}} - t),$$

so that for each $t \in \mathbb{R}$ we have $\{v_\varepsilon^t = 0\} = \Gamma(t)$, $v_\varepsilon^0 = v_\varepsilon$, $v_\varepsilon^{2d(N)} = -1$ and $v^{-2d(N)} = +1$ (whenever $\varepsilon > 0$ is sufficiently small). The geometric relation (8), induced by the positive Ricci curvature assumption, then translates to the level of functions and allows us to compute, in Lemma 5, that the v_ε^t have the following property:

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) \leq \mathcal{F}_\lambda(v_\varepsilon) + E(\varepsilon) \text{ where } E(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

As the nodal sets of the v_ε^t are the level sets of the signed distance function to $\overline{M}$, the functions v_ε^t may be seen as a path of functions analogous to a sweep-out of N by the level sets of the signed distance function to $\overline{M}$.

Thus, the path provided by the sliding functions, along with the energy reducing paths from -1 and $+1$ provided by negative gradient flow of the energy to a_ε and b_ε respectively, provides a "recovery path" for the value $\mathcal{F}_\lambda(E)$; this path connects a_ε to b_ε, passing through v_ε, with the maximum value of the energy along this path approximately $\mathcal{F}_\lambda(E)$ (by virtue of (2)). Approximate upper energy bounds along this path are depicted by the thick dashed lines in Figure 2 below. In this manner, as mentioned in Subsection 1.2, such a path establishes that the Allen–Cahn min-max procedure in positive Ricci curvature produces no minimal piece.

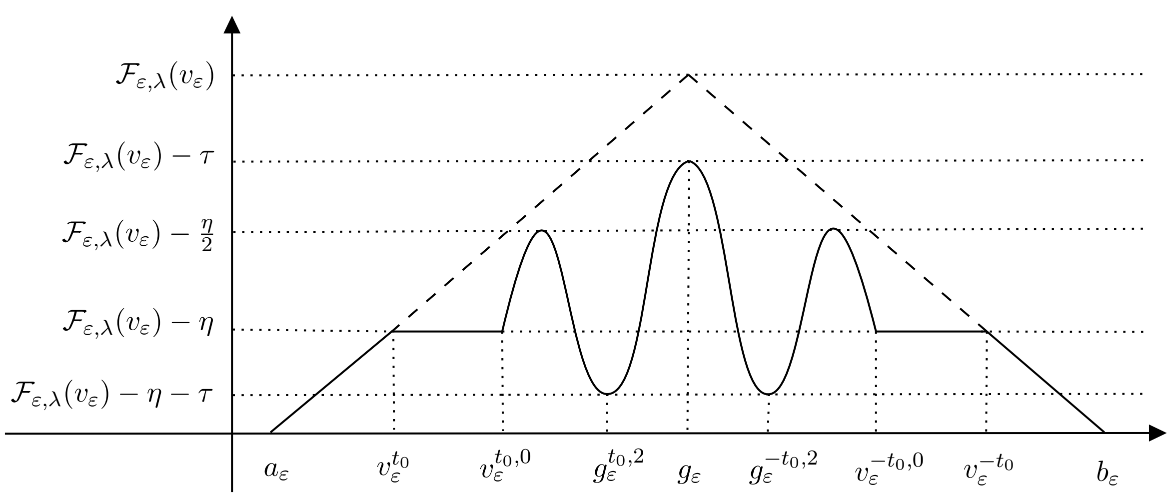

Figure 2: The solid curve depicts approximate (i.e. up to the addition of a term that converges to zero as $\varepsilon \to 0$) upper energy bounds along the path taken from a_ε to b_ε constructed for the proof of Theorem 3 in Subsection 6.1. The horizontal axis identifies some specific functions in the path and the vertical axis depicts the approximate upper bound on the energy of the path between each identified function. The thick dashed lines depict an approximate upper energy bound along the path of functions in Lemma 5.

We now exhibit (for all $\varepsilon > 0$ sufficiently small) a continuous path in $W^{1,2}(N)$ between a_ε and b_ε which, under the assumption that E is not $\mathcal{F}_\lambda$-minimising a small ball around an isolated singularity, contradicts the min-max characterisation of E and proves Theorem 3. We emphasise that this path is constructed with energy bounded above by a value strictly below $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ (independently of ε); approximate upper energy bounds are depicted by the solid curve in Figure 2 above. We thus conclude, by the arguments in Step 2, that the Allen–Cahn min-max procedure in positive Ricci curvature produces a hypersurface which is locally $\mathcal{F}_\lambda$-minimising.

The approximate upper energy bounds along the portions of the path we construct are computed explicitly in Section 5. However, in this section, we now sketch the various path constructions and motivate the upper energy bounds as a diffuse reflection of the underlying geometry.

First, for a given isolated singularity, $p \in \mathrm{Sing}(M)$, we are able to continuously deform the sets $E(t_0)$ (for a fixed $t_0 > 0$ sufficiently small) locally around p so that the resulting deformation agrees with E inside of a fixed ball $B_{r_0}(p)$ (for some $r_0 > 0$ determined only by the area of M). This is done by exploiting (8) in such a way that the $\mathcal{F}_\lambda$-energy of the deformations remain a fixed amount below $\mathcal{F}_\lambda(E)$. At the diffuse level this is replicated by placing $\overline{\mathbb{H}}^\varepsilon$ in the normal direction to the deformations as in the construction of v_ε, producing a $W^{1,2}(N)$ continuous path of functions with controlled energy; depicted in Figure 3 below.

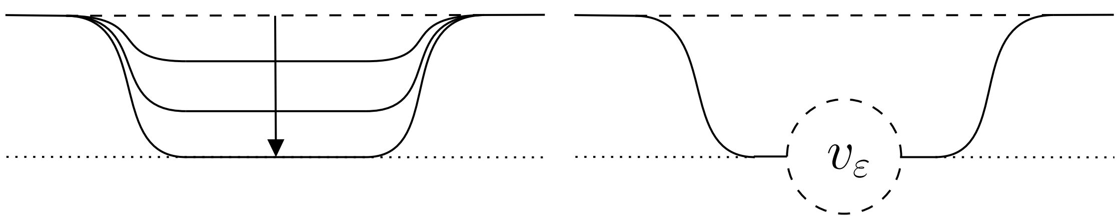

Figure 3: In both graphics above the lower thin dashed horizontal line depicts $\overline{M}$, the zero level set of the function v_ε, and the upper thick dashed horizontal line depicts the zero level set, $\Gamma(t_0)$, of $v_\varepsilon^{t_0} = v_\varepsilon^{t_0,1}$ which is deformed in the construction of the shifted functions. In the left-hand graphic the solid lines depict various zero level sets of the $v_\varepsilon^{t_0,s}$ as we vary s from 0 to 1. In the right-hand graphic the solid line depicts the zero level set of $v_\varepsilon^{t_0,0}$ and the thick dashed circle depicts the boundary of the ball, $B_{r_0}(p)$, in which $v_\varepsilon^{t_0,0} = v_\varepsilon$.

Specifically, we produce a continuous path of *shifted functions*, $s \in [0,1] \to v_\varepsilon^{t_0,s} \in W^{1,2}(N)$, which satisfy $v_\varepsilon^{t_0,1} = v_\varepsilon^{t_0}$ on N and $v_\varepsilon^{t_0,0} = v_\varepsilon$ inside $B_{r_0}(p)$. Furthermore, there exists an $\eta > 0$ such that for each $s \in [0,1]$ we have the following upper energy bound

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t_0,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta,$$

as depicted in Figure 2. In this manner we have exhibited a continuous path in $W^{1,2}(N)$ from $v_\varepsilon^{t_0}$ to a function $v_\varepsilon^{t_0,0}$, equal to v_ε in a fixed ball $B_{r_0}(p)$, with the energy along this path a fixed amount below the min-max value.

Next we construct a continuous path of functions, from $v_\varepsilon^{t_0,0}$ to the local energy minimiser $g_\varepsilon \in \mathcal{A}_{\varepsilon,\frac{R}{2}}(p)$ (recall Step 2) for a fixed $R \in (0, r_0)$. This is done in such a way that we only alter the functions inside $B_R(p)$; thus, in the following description we will only consider functions in the ball $B_R(p)$. The radius $R > 0$ here is chosen sufficiently small based on the energy drop, $\eta > 0$, above achieved outside of $B_{r_0}(p)$, ensuring that the energy along the constructed path will remain a fixed amount below the min-max value.

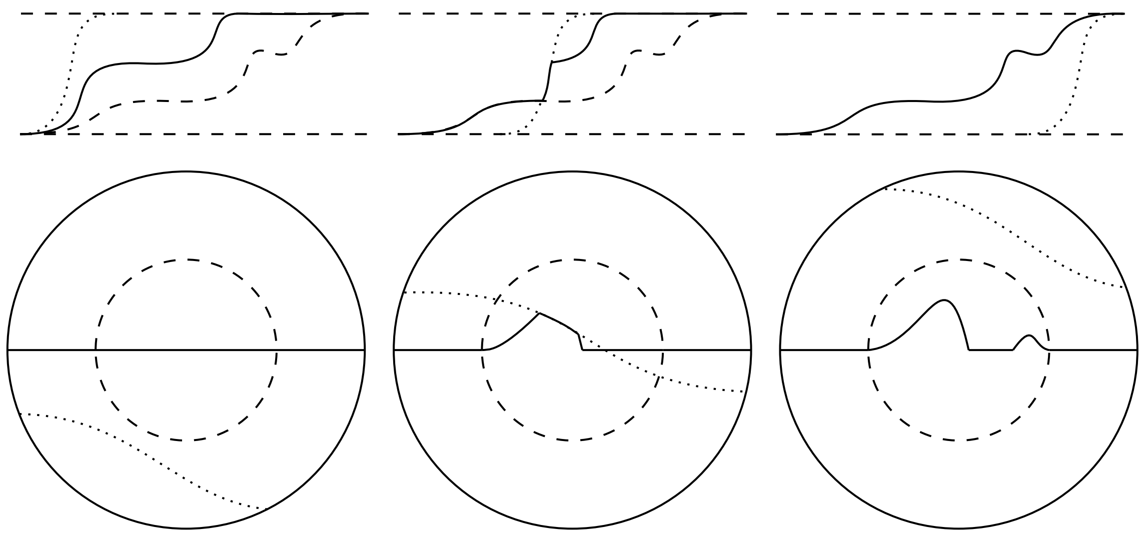

Figure 4: The graphics above depict various stages of the transition, in $W^{1,2}(N)$, between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$ in $B_R(p)$. The first row depicts schematics of this transition and the second row depicts the geometry of the zero level sets of the local path functions; in this way, each column of the figure contains a schematic for the local path function with a corresponding depiction of the geometry of its zero level set below it. For the first row, in all three of the images the solid black line depicts the local path function in question (equal to v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$ in the left-hand and right-hand graphic respectively), the thick dashed curve depicts portions of $\min\{v_\varepsilon, g_\varepsilon\}$ that are not yet included in the path, the thin dashed curve depicts the diffuse sweep-out function in question, and the thick dashed upper and lower horizontal lines depict the functions ± 1 respectively. Notice that the local path function depicted in the middle graphic includes portions of v_ε that lie to the right of the diffuse sweep-out function, portions of $\min\{v_\varepsilon, g_\varepsilon\}$ that lie to the left of the diffuse sweep-out function and uses the diffuse sweep-out function itself to interpolate between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$. The explicit construction of the local path functions replicating the behaviour of these schematics involves taking various maxima and minima of the functions in the diffuse sweep-out, v_ε and g_ε; the diffuse sweep-out allows for this choice of maxima and minima (corresponding geometrically to a choice of zero level set) to be made continuously. For the second row, in all three graphics the outer solid circle depicts the boundary of $B_R(p)$ and the thick dashed inner circle depicts the boundary of $B_{\frac{R}{2}}(p)$. In the left-hand graphic the solid line depicts the zero level set of v_ε, which is $\overline{M}$, in the right-hand graphic the solid curve depicts the zero level set of $\min\{g_\varepsilon, v_\varepsilon\}$, and in the middle graphic the solid curve depicts the zero level set of a local path function in the transition between v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$. Each the thin dashed curves in the three graphics depict a given hypersurface in the "planar" sweep-out of $B_R(p)$, each separating $B_R(p)$ into two open sets (one above and one below it). The zero level set of the function in the local path associated to this hypersurface is chosen to be the portions of the zero level set of $\min\{v_\varepsilon, g_\varepsilon\}$ that are beneath the hypersurface, the portions of $\overline{M}$ that are above the hypersurface, and when the the hypersurface lies between the nodal sets of v_ε and $\min\{v_\varepsilon, g_\varepsilon\}$, the hypersurface itself is chosen as the zero level set. The same ideas described above are used for the construction of the portion of the path from $\min\{v_\varepsilon, g_\varepsilon\}$ to g_ε in $B_R(p)$.

In order to construct this local path from v_ε to g_ε inside of $B_R(p)$ we utilise a sweep-out of the ball by images of Euclidean planes via a geodesic normal coordinate chart; depicted by the thin dashed curves in the second row of Figure 4 above. The hypersurfaces in this sweep-out are used to continuously transition from our hypersurface and the local $\mathcal{F}_\lambda$-minimiser at the diffuse level; precisely, the planes facilitate the construction of a path between the diffuse representatives v_ε and g_ε.

Remark 3. *In the setting of the Almgren–Pitts min-max homotopy sweep-outs of N by cycles are considered, as opposed to continuous paths in $W^{1,2}(N)$ in the Allen–Cahn min-max. However, there does not necessarily exist a homotopy of cycles between our hypersurface and any local $\mathcal{F}_\lambda$-minimiser. We overcome this for the Allen–Cahn min-max by directly exploiting the topology of $W^{1,2}(N)$, showing that the local path may be seen as a diffuse analogue of [CLS22, Lemma 1.12], and illustrating why we guarantee local $\mathcal{F}_\lambda$-minimisation (see Remark 2).*

In the same manner as in the construction of v_ε, by placing $\overline{\mathbb{H}}^\varepsilon$ in the normal direction to the hypersurfaces in this "planar" sweep-out we construct a sweep-out at the diffuse level that is continuous in $W^{1,2}(N)$; the diffuse sweep-out thus acts as an approximation for the underlying "planar" sweep-out. The diffuse sweep-out of $B_R(p)$ is utilised twice, first for the construction of a path from v_ε to $\min\{g_\varepsilon, v_\varepsilon\}$, and second for the construction of a path from $\min\{g_\varepsilon, v_\varepsilon\}$ to g_ε. By taking a combination of maxima and minima of functions in the diffuse sweep-out, v_ε and g_ε (which ensure the resulting functions are in $W^{1,2}(N)$), we are able to produce a local path from v_ε to g_ε; see Figure 4 above for a description of this construction.

We note that $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(\min\{v_\varepsilon, g_\varepsilon\}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ (by local energy minimisation of g_ε) and that $R \in (0, r_0)$ is chosen based on $\eta > 0$ to ensure that the total energy contribution of the diffuse sweep-out functions in the ball is at most $\frac{\eta}{2}$. As a consequence of these two facts, the energy in $B_R(p)$ of any local path function can be estimated to be at most the energy of v_ε in $B_R(p)$ plus $\frac{\eta}{2}$; from there one obtains the energy estimate in the whole of N. Specifically, we produce a continuous path of *local functions*, $s \in [-2, 2] \to g_\varepsilon^{t_0,s} \in W^{1,2}(N)$, which satisfy $g_\varepsilon^{t_0,-2} = v_\varepsilon^{t_0,0}$ on N, $g^{t_0,2} = g_\varepsilon$ in $B_R(p)$ and are all equal to $v_\varepsilon^{t_0,0}$ outside of $B_R(p)$. Furthermore, for each $s \in [-2, 2]$ we have the following upper energy bound

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t_0,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\eta}{2},$$

as depicted in Figure 2. In this manner we have exhibited a continuous path in $W^{1,2}(N)$ from $v_\varepsilon^{t_0,0}$ to a function $g_\varepsilon^{t_0,2}$, changing $v_\varepsilon^{t_0,0}$ only inside of $B_R(p)$ (from v_ε to the local energy minimiser g_ε).

In order to establish that E is locally $\mathcal{F}_\lambda$-minimising, we now argue by contradiction and assume that there exists a $\tau > 0$ such that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau \text{ for all } \varepsilon > 0 \text{ sufficiently small.} \tag{9}$$

Note that by the results discussed in Step 2, contradicting (9) will establish that M is $\mathcal{F}_\lambda$-minimising in $B_{\frac{R}{4}}(p)$ for $R > 0$ as chosen above (as (5) must hold). Using the contradiction assumption, in addition to keeping the energy of the local path functions a fixed amount, $\frac{\eta}{2}$, below the min-max value, as $g_\varepsilon^{t_0,2} = g_\varepsilon$ in $B_R(p)$, we thus also have that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t_0,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta - \tau,$$

as depicted in Figure 2. We now directly exploit this extra energy drop afforded by the contradiction assumption to construct the next portion of the path.

To this end, we deform the rest of the set $E(t_0)$ entirely onto E outside of $B_{r_0}(p)$. This is done in such a way that the deformations fix the inside of $B_{r_0}(p)$, thus preserving a drop in $\mathcal{F}_\lambda$-energy under the assumption that E is not locally $\mathcal{F}_\lambda$-minimising. At the diffuse level this is replicated by placing $\overline{\mathbb{H}}^\varepsilon$ in the normal direction to these deformations, keeping the functions equal to the local energy minimiser g_ε (which yields an energy drop by (9)) inside of $B_{r_0}(p)$ and producing a $W^{1,2}(N)$ continuous path of functions with controlled energy; this is depicted in Figure 5.

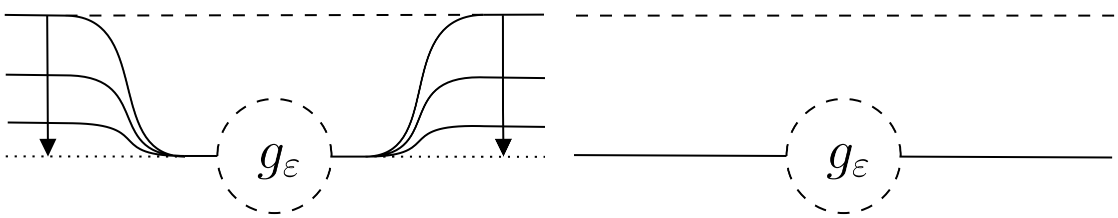

Figure 5: In both graphics above the lower horizontal lines depict $\overline{M}$, the upper horizontal line depicts the zero level set, $\Gamma(t_0)$, of $v_\varepsilon^{t_0} = v_\varepsilon^{t_0,1}$ which is deformed in the construction of the shifted functions and the thick dashed circle depicts the boundary of the ball, $B_{r_0}(p)$, in which the shifted functions $g_\varepsilon^{t,2} = g_\varepsilon$. In the left-hand graphic the solid lines depict various zero level sets of the $g_\varepsilon^{t,2}$ as we vary t from t_0 to 0. In the right-hand graphic the solid line depicts the zero level set of $g_\varepsilon^{0,2} = g_\varepsilon$.

Specifically, we produce (under the assumption that (9) holds) a continuous path of *shifted functions*, $t \in [0, t_0] \to g_\varepsilon^{t,2} \in W^{1,2}(N)$, which are equal to the local function $g_\varepsilon^{t_0,2}$ when $t = t_0$ (justifying notation), equal to g_ε when $t = 0$ and are all equal to g_ε in $B_{r_0}(p)$. Furthermore, for each $t \in [0, t_0]$ have the following upper energy bound

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) - \tau,$$

as depicted Figure 2. In this manner we have exhibited a continuous path in $W^{1,2}(N)$ from $g_\varepsilon^{t_0,2}$ to the local energy minimiser g_ε, only changing $g_\varepsilon^{t_0,2}$ outside of $B_{r_0}(p)$.

To summarise all of the above, as the endpoints of each of the paths described agree with the start of the next, for all $\varepsilon > 0$ sufficiently small we have exhibited a continuous paths in $W^{1,2}(N)$ connecting a_ε to the local energy minimiser g_ε. We also demonstrated that the energy along this path is bounded above by a value strictly below $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ (independently of ε), depicted in Figure 2.

To complete the desired path from a_ε to b_ε it remains to construct the portion from g_ε to b_ε. By considering $-t_0$ instead of t_0 in each of the paths sketched above, we ensure that through symmetric (with respect to the underlying hypersurfaces) deformations of the sets $E(-t_0)$, the relevant symmetric portions of the path may constructed with identical upper energy bounds; this portion of the path is depicted in Figure 2, where the symmetry of the path with respect to g_ε is made apparent.

To conclude the proof of Theorem 3 we concatenate the path from a_ε to g_ε and the path from g_ε to b_ε, completing the desired continuous path in $W^{1,2}(N)$ between the two stable critical points a_ε and b_ε. Under the assumption that (9) holds, the upper energy bounds along this path (depicted by the solid curve in Figure 2) and (2) ensure that for $\varepsilon > 0$ sufficiently small we have

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_\lambda(E) - \min\left\{\frac{\eta}{4}, \frac{\tau}{2}\right\}.$$

As (1) holds, as mentioned in Subsection 1.2, and as the above path is admissible in the Allen–Cahn min-max construction of E, we must contradict the assumption that (9) holds. We thus conclude that any such E as produced by the Allen–Cahn min-max procedure in Ricci positive curvature must be such that E is locally $\mathcal{F}_\lambda$-minimising (as (5) holds), proving Theorem 3. We then establish Theorems 1 and 2 by applying the results described in Step 1.

1.4 Structure and remarks

We now proceed as follows:

- Section 2 recalls the result of [Les23] and uses it to perturb away isolated singularities of constant mean curvature hypersurfaces with regular tangent cones.

- Section 3 analyses the signed distance function and introduces the one-dimensional profile, before showing that it approximates the underlying hypersurface in a suitable sense.
- Section 4 establishes a procedure to locally smooth Caccioppoli sets which are smooth in annular regions. This procedure is then used to relate local energy minimisation to the local geometric behaviour of the hypersurface.
- Section 5 provides constructions of the various continuous paths in $W^{1,2}(N)$ along with calculations of upper bounds of the energy along these paths.
- Section 6 ties together the results of the previous sections in order to prove the main results.

The following remarks should be kept in mind throughout:

Remark 4. *As mentioned above, in [BW20b, Proposition B.1] it is shown that smooth constant mean curvature hypersurfaces are locally $\mathcal{F}_\lambda$-minimising. In proving Theorem 3 we aim to show that this also holds around isolated singularities for constant mean curvature hypersurfaces produced by the Allen–Cahn min-max in manifolds with positive Ricci curvature. We may thus restrict to the dimensions in which these objects may be singular and work under the assumptions that N is of dimension $n+1 \geq 8$.*

Remark 5. *The assumption of positive Ricci curvature is used only to ensure that the upper energy bounds on the paths constructed in Section 5 remain a fixed amount below the min-max value, $\mathcal{F}_\lambda(E)$. In particular, we note that the results of Sections 2, 3 and 4 as well as the paths constructed (but not their upper energy bounds) in Section 5 make no use of the assumption of positive Ricci curvature.*

Remark 6. *We make the choice of a positive upper bound on $\varepsilon > 0$ finitely many times throughout the construction of the paths in the proof of Theorem 3, ultimately constructing the paths for all $\varepsilon > 0$ smaller than a fixed positive constant. The specific choice of upper bound utilised in each instance may change, but we implicitly assume that a correct upper bound for which the desired property holds is used in each case. This remark will apply each time we choose $\varepsilon > 0$ sufficiently small.*

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2 Surgery procedure for isolated singularities with regular tangent cone

We show how the recent result of [Les23] can be combined with a local perturbation of the metric to regularise a hypersurface of constant mean curvature around an isolated singularity with area-minimising regular tangent cone. We first collect some notation and definitions, phrased in notation in keeping with this paper, before stating the main theorem from [Les23] and using it to establish the surgery procedure.

We reset notation for this section, letting (N^{n+1}, g) be a Riemannian manifold with no curvature assumption. Throughout this section $T = (\partial[A]) \llcorner B_1(p)$ will denote the current associated to a Caccioppoli set, $A \in \mathcal{C}(N)$, restricted to a ball $B_1(p)$ about a point $p \in N$, for which the following properties hold:

- $\mathrm{Spt}(T)$ is connected.
- $\mathrm{Sing}(T) = \{p\}$ (so that p is an isolated singularity of T).
- T has a regular tangent cone at p.

For a given $\lambda \in \mathbb{R}$ we fix a choice of $0 < r_1 < r_2 < 1$ sufficiently small so that the following properties hold:

- $\Gamma_0 = \partial(T \llcorner B_{r_1}(p))$ is a closed embedded, connected $(n-1)$-dimensional sub-manifold of $\partial B_{r_1}(p)$.
- $\mathrm{Spt}(T) \cap \partial B_{r_1}(p)$ is a transverse intersection.
- $\overline{A \cap B_{r_2}(p)}$ has connected complement in $B_{r_2}(p)$.

Let $\phi_j : \Gamma_0 \to \partial B_{r_1}(p)$ denote C^2 maps with

$$|\phi_j - i_{\Gamma_0}|_{C^2} \leq \frac{1}{j},$$

and for $\Gamma_j = (\phi_j)_*\Gamma_0$ we assume that $\Gamma_j \cap A^o \neq \emptyset$. Here we denote by i_{Γ_0} the identity map on Γ_0, $(\phi_j)_*\Gamma_0$ the push-forward of Γ_0 by the map ϕ_j and A^o the interior of A. We now recall the definitions used in [Les23]:

Definition 1. *For $\lambda \in \mathbb{R}$ we say that $T = (\partial[A]) \llcorner B_1(p)$ is one-sided minimising for $\mathcal{F}_\lambda$ in $B_1(p)$ if both of the following properties hold:*

- *A is a critical point of $\mathcal{F}_\lambda$ in $B_1(p)$.*
- *We have that*

$$\mathcal{F}_\lambda(A) \leq \mathcal{F}_\lambda(X).$$

for any integral n-current X with $\mathrm{Spt}(X) \subset \overline{A \cap B_1(p)}$ and $\partial X = \partial T$.

Using the above notation and definition, [Les23] then proves the following foliation result, generalising the results of [HS85] to the case of constant mean curvature hypersurfaces.

Theorem 4. *[Les23] Let $\lambda \in \mathbb{R}$, with T as defined above such that in addition the following two properties hold:*

- *T is a critical point for $\mathcal{F}_\lambda$ in $B_1(p)$.*
- *T is one-sided minimising for $\mathcal{F}_\lambda$ in $B_1(p)$.*

Then, for every $j \geq 1$, there exist integral n-currents, S_j, that minimise $\mathcal{F}_\lambda$ in $\overline{A \cap B_{r_2}(p)}$, subject to the boundary condition $\partial S_j = \Gamma_j$, that satisfy the following properties:

- *$\mathrm{Spt}(S_j) \subset \overline{B_{r_1}(p)}$.*
- *There exist sets of finite perimeter B_j, with $\overline{B_j} \subset \overline{A \cap B_{r_1}(p)}$, such that $S_j = \partial[B_j] \llcorner B_{r_1}(p)$.*
- *$\Gamma_j = \mathrm{Spt}(S_j) \cap \partial B_{r_1}(p)$.*
- *Each S_j is a critical point of $\mathcal{F}_\lambda$ in $B_{r_1}(p)$.*
- *For the measures associated to the supports of the S_j and T we have $\mu_{S_j} \to \mu_{T \llcorner B_{r_1}(p)}$. Thus, in particular on compact sets we have that $\mathrm{Spt}(S_j) \to \mathrm{Spt}(T)$ in the Hausdorff distance.*
- *The $\mathrm{Spt}(S_j)$ are smooth hypersurfaces.* □

Using Theorem 4 we now establish the desired surgery procedure, the proof of which is similar to [CLS22, Proposition 4.1].

Proposition 1. *Let (N^{n+1}, g) be a Riemannian manifold and $T = (\partial[A]) \llcorner B_1(p)$ be a current associated to $A \in \mathcal{C}(N)$ with the properties as stated above and satisfying the hypotheses of Theorem 4. Given $r \in (0, r_1)$ and any $\varepsilon > 0$ there exists a current $\tilde{T}$ and metric $\tilde{g}$ with the following properties:*

- $\mathrm{Sing}(\tilde{T}) = \emptyset$.
- *$\tilde{T}$ is a critical point of $\mathcal{F}_\lambda$ in $B_1(p)$ with respect to the metric $\tilde{g}$.*
- $\mathrm{Spt}(\tilde{T}) \setminus B_r(p) = \mathrm{Spt}(T) \setminus B_r(p)$ *and* $\partial(\tilde{T} \llcorner B_r(p)) = \partial(T \llcorner B_r(p))$.
- $d_{\mathcal{H}}(\mathrm{Spt}(T), \mathrm{Spt}(\tilde{T})) < \varepsilon$*, where here $d_{\mathcal{H}}$ denotes the Hausdorff distance.*
- $||\tilde{g} - g||_{C^{k,\alpha}} < \varepsilon$ *for any $k \geq 1$ and $\alpha \in (0,1)$, with $g = \tilde{g}$ on $N \setminus B_r(p)$.*

Proof. The case $\lambda = 0$ is precisely the content of [CLS22, Proposition 4.1]. We thus consider the case $\lambda \in \mathbb{R} \setminus \{0\}$.

As $\mathrm{Sing}(T) = \{p\}$, for each $r \in (0, r_1)$ we have that $(\mathrm{Spt}(T) \cap B_r(p)) \setminus \{p\}$ is smooth. We apply Theorem 4 to see that there exists some sequence, S_j, of smooth constant mean curvature hypersurfaces such that the S_j converge as currents to $T \llcorner B_r(p)$. Allard's Theorem, (see [Sim84, Chapter 5] and [HS85, Lemma 1.14]) implies that we may write (for j sufficiently large) the intersection of $\mathrm{Spt}(S_j)$ with the annulus $A(p, \frac{r}{4}, \frac{3r}{4}) = B_{\frac{3r}{4}}(p) \setminus \overline{B_{\frac{r}{4}}(p)}$ as a smooth graph over $\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4})$.

Explicitly, let $u_j \in C^2(\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4}))$ denote the graphing function of $\mathrm{Spt}(S_j)$ over $\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4})$ and let φ be a smooth cutoff function taking values in $[0, 1]$ such that $\varphi = 1$ on $B_{\frac{3r}{8}}(p)$ and $\varphi = 0$ outside $B_{\frac{5r}{8}}(p)$. We then denote by $T + \varphi u_j$ the image of the normal graph of the function φu_j over $\mathrm{Spt}(T) \cap A(p, \frac{r}{4}, \frac{3r}{4})$.

We now define $\widetilde{T} = (T \setminus B_r(p)) \cup (S_j \cap B_{\frac{r}{4}}) \cup ((T + \varphi u_j) \cap A(p, \frac{r}{4}, \frac{3r}{4}))$ for j large enough to ensure that $\mathrm{Spt}(S_j)$ is smooth and graphical as above. Note that the $\mathrm{Spt}(\widetilde{T})$ is smooth as $\mathrm{Spt}(S_j)$, $\mathrm{Spt}(T) \setminus B_r(p)$ and $(\mathrm{Spt}(T) + \varphi u_j) \cap A(p, \frac{r}{4}, \frac{3r}{4})$ are smooth; hence $\mathrm{Sing}(\widetilde{T}) = \emptyset$. By construction we ensure that $\mathrm{Spt}(\tilde{T}) \setminus B_r(p) = \mathrm{Spt}(T) \setminus B_r(p)$ and $\partial(\tilde{T} \llcorner B_r(p)) = \partial(T \llcorner B_r(p))$. Note that for j sufficiently large we have that $d_{\mathcal{H}}(\mathrm{Spt}(T), \mathrm{Spt}(\tilde{T})) < \varepsilon$ by the properties in the conclusion of Theorem 4.

Let $H_g(x)$ denote the mean curvature of the hypersurface $\widetilde{T}$ with respect to the metric g at a point $x \in \widetilde{T}$. By construction, H_g may not be equal to λ only on $A(p, \frac{r}{4}, \frac{3r}{4})$ (as both T and S_j are critical points of $\mathcal{F}_\lambda$). It remains to construct a new metric $\tilde{g}$, close to g, and show that $\widetilde{T}$ has constant mean curvature λ with respect to this new metric. In the construction of $\widetilde{T}$ above, as $\lambda \in \mathbb{R} \setminus \{0\}$ and the graphing functions, u_j, converge to 0, we may choose j large enough to ensure that H_g and λ have the same sign, so that $\frac{H_g(x)}{\lambda} > 0$ for each $x \in \mathrm{Spt}(\widetilde{T})$.

For some smooth function f on N to be determined we set $\tilde{g} = e^{2f} g$; by standard results for conformal change of metric (e.g. see [Sak96, Chapter 2]) we then have that the mean curvature, $H_{\tilde{g}}$, of $\widetilde{T}$ with respect to the metric $\tilde{g}$ satisfies $H_{\tilde{g}}(x) = e^{-f}(H_g(x) + \frac{\partial f}{\partial \nu})$ at each point $x \in \widetilde{T}$, where ν is the unit normal on $\widetilde{T}$ (agreeing with T outside of $A(p, \frac{r}{4}, \frac{3r}{4})$).

We define a smooth cutoff function, z, such that $z(x) \equiv 1$ if $\mathrm{dist}_N(y, \mathrm{Spt}(\widetilde{T}) \cap A(p, \frac{r}{4}, \frac{3r}{4})) < \frac{r}{20}$ and $z \equiv 0$ whenever $\mathrm{dist}_N(y, \widetilde{T} \cap A(p, \frac{r}{4}, \frac{3r}{4})) > \frac{r}{10}$. We denote by $\Pi(y)$ the closest point projection to $\widetilde{T}$ so that $H_g(\Pi(y))$ is a well defined function, supported in a tubular neighbourhood of $\mathrm{Spt}(\widetilde{T}) \cap A(p, \frac{r}{4}, \frac{3r}{4})$. We now solve for $H_{\tilde{g}} = \lambda$ for each $x \in \mathrm{Spt}(\tilde{T})$ by setting $f(y) = \log\left(\frac{H_g(\Pi(y))}{\lambda}\right) z(y)$; note then by construction $\mathrm{Spt}(f) \subset A(p, \frac{r}{4}, \frac{3r}{4})$ and $\frac{\partial f}{\partial \nu} = 0$ on $\mathrm{Spt}(\widetilde{T}) \cap A(p, \frac{r}{4}, \frac{3r}{4})$ (as f is constant along normal geodesics to this region).

Thus we have that $H_{\tilde{g}}(\widetilde{T}) = \lambda$ for the choice of f as above, so in particular $\widetilde{T}$ is a critical point of $\mathcal{F}_\lambda$, with the change in metric occurring only in $A(p, \frac{r}{4}, \frac{3r}{4})$. Furthermore, by the smoothness of z and the fact that the graphing functions, u_j, are converging to 0 ensures $||H_g(\Pi(y)) - \lambda||_{C^{2,\alpha}} \to 0$ as $j \to \infty$, we see that for each $k \geq 1$ and $\alpha \in (0,1)$ we have

$$||e^{2f} - 1||_{C^{k,\alpha}} = \left|\left| \left(\frac{H_g(\Pi(y))}{\lambda}\right)^{2z(y)} - 1 \right|\right|_{C^{k,\alpha}} \to 0 \text{ as } j \to \infty,$$

and so

$$||\tilde{g} - g||_{C^{k,\alpha}} \leq ||e^{2f} - 1||_{C^{k,\alpha}} ||g||_{C^{k,\alpha}} \to 0 \text{ as } j \to \infty.$$

Hence by choosing j sufficiently large we ensure the metric change is arbitrarily small. □

Remark 7. *In the proof of Theorem 3 we will establish that the one-parameter Allen–Cahn min-max procedure of [BW20a] with constant prescribing function λ, in a compact Riemannian manifold of dimension 3 or higher with Ricci positive curvature, produces a closed embedded hypersurface, M, of constant mean curvature λ which is locally $\mathcal{F}_\lambda$-minimising around isolated singular points. For M as above, around isolated singularities with regular tangent cones the results of [Sim83] imply that M is locally a graph over its unique tangent cone. Thus, M will satisfy all the required properties of T specified throughout this section in a sufficiently small ball around an isolated singular point with regular tangent cone. With this in hand, as any one-sided minimiser as in Definition 1 is required to be a Caccioppoli set (see [Liu19], [Les23] for more details), we may thus apply the results of this section to M.*

3 Signed distance and the one-dimensional profile

3.1 Singular behaviour of the distance function

Recall that $M = \partial^* E \subset N$ is assumed to be a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set, denoted $\mathrm{Sing}(M) = \overline{M} \setminus M$, of Hausdorff dimension at most $n - 7$. We now adapt some of the analysis in [Bel20, Section 3] to the setting of hypersurfaces of constant mean curvature.

Denoting by $S_{d_{\overline{M}}}$ the set of points in $N \setminus \overline{M}$ where the distance function $d_{\overline{M}}$ fails to be differentiable (precisely the set of points $x \in N \setminus \overline{M}$ such that there exist two or more geodesics realising $d_{\overline{M}}(x)$), we then have that $d_{\overline{M}}$ is C^1 on $N \setminus (\overline{M} \cup \overline{S_{d_{\overline{M}}}})$ and that $S_{d_{\overline{M}}}$ is countably n-rectifiable (by [Alb94]). We now show that $\overline{S_{d_{\overline{M}}}}$ is a countably n-rectifiable set; this fact will allow us to work with the smooth portions of the signed level sets of $d_{\overline{M}}$. To establish this we will need the following lemma; the proof of which is identical to that of [Bel20, Lemma 3.1] replacing the use of the sheeting theorem in [Wic14] by the one in [BW20c].

Lemma 1. *(Geodesic Touching) Let $x \in N \setminus \overline{M}$, then any minimising geodesic connecting x to $\overline{M}$ (i.e. a geodesic whose length realises $d_{\overline{M}}(x)$) with endpoint $y \in \overline{M}$ is such that $y \in M$ (i.e. is a regular point of M).* □

Let $F(y,t) = \exp_y(t\nu(y))$ where $y \in M$, $t \in \mathbb{R}$ and ν the choice of unit normal to M pointing into E. We then have that $F(y,t)$ is a geodesic emanating from M orthogonally; here we interpret $F(y,t)$ for $t < 0$

by $\exp_y(t(\nu^-(y)))$ where ν^- is the unit normal to M pointing into $N \setminus E$. We define $\sigma^+(y), \sigma^-(y) \in \mathbb{R}$ with $\sigma^+(y) > 0$ and $\sigma^-(y) < 0$ chosen so that $F(y,t)$ is the minimising geodesic between its endpoint y, on M, and $F(y,t)$ for all $\sigma^-(y) \le t \le \sigma^+(y)$ but fails to be the minimising geodesic, between y and $F(y,t)$, for $t > \sigma^+(y)$ or $t < \sigma^-(y)$. With this definition we define the *cut locus* of M to be

$$\mathrm{Cut}(M) = \{F(y, \sigma^\pm(y)) \ : \ y \in M, \sigma^\pm(y) < \infty\}. \tag{10}$$

Standard theory (e.g. [Sak96]) for geodesics characterises the cut locus in the following manner: if $x = F(y, \sigma^\pm(y)) \in \mathrm{Cut}(M)$ then either their exist (at least) two distinct geodesics realising $d_{\overline{M}}(x)$ or $F : M \times (0, \infty) \to N$ is such that $dF(y, \sigma^\pm(y))$ is not invertible. We then see that $\overline{S_{d_{\overline{M}}}} \cap (N \setminus \overline{M}) = \mathrm{Cut}(M)$ (c.f. proposition 4.6. in [MM02]) and so in order to establish the countable n-rectifiability of $\overline{S_{d_{\overline{M}}}}$ it is sufficient to show that $\mathrm{Cut}(M) \setminus S_{d_{\overline{M}}}$ is countably n-rectifiable in $N \setminus \overline{M}$; this fact is the analogue of proposition 4.9. in [MM02] in our setting (where the surface M may be singular). Observe that $\overline{S_{d_{\overline{M}}}} \cap \overline{M} \subset \mathrm{Sing}(M)$ and as noted above $\mathrm{Sing}(M)$ has zero $\mathcal{H}^n$ measure, so this set does not affect the rectifiability.

The proof of [MM02, Proposition 4.9.] may be adapted (exactly as in [Bel20, Section 3]) to our setting by virtue of the fact that their arguments are local to points away from $\overline{M}$, and hence we may apply the arguments used in their proof in our situation without change. Thus we conclude that $\overline{S_{d_{\overline{M}}}}$ is countably n-rectifiable, consequently its $\mathcal{H}^{n+1}$ measure is zero.

The level sets of $d_{\overline{M}}$ are smooth on $N \setminus (\overline{M} \cup \overline{S_{d_{\overline{M}}}})$ by virtue of the Implicit Function Theorem, invertibility of F and differentiability of $d_{\overline{M}}$ on this set. The arguments in the proof of [MM02, Proposition 4.6.] show further that F restricts to a diffeomorphism,

$$F : \{(y,t) \ : \ y \in M, t \in (\sigma^-(y), \sigma^+(y))\} \to N \setminus (\overline{M} \cup \overline{S_{d_{\overline{M}}}}).$$

We then extend F to $M \times \{0\}$ by setting $F(y, 0) = y$ so that the image of the extension of F is now $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$. Finally, by defining

$$V_M = \{(y,t) \,|\, y \in M, t \in \big(\sigma^-(y), \sigma^+(y)\big)\}. \tag{11}$$

we have that V_M is diffeomorphic to $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$.

Remark 8. *For each compact $K \subset M$ the continuity of the functions $\sigma^\pm(x)$ (defined in the paragraph before (10)) on M implies that there exists some $c_K > 0$ such that $0 < c_K < \min_{x \in K}\{\sigma^+(x), |\sigma^-(x)|\}$. For such a set K, as M itself is a two-sided hypersurface, there is a two-sided tubular neighbourhood of K when viewed as a subset of V_M (the coordinates defined in (11)), given by $K \times (-c_K, c_K)$ with its closure a subset of V_M. Furthermore, the image under the map F of this two-sided tubular neighbourhood, $F(K \times (-c_K, c_K)) \subset N$, is such that $F(K \times (-c_K, c_K)) \cap (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}) = \emptyset$, by the definition of c_K.*

Remark 9. *We define a projection to M on $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$. For each $y \in N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$ there exists a unique geodesic in N with endpoint $x \in M$ realising $d_{\overline{M}}(y)$ (i.e. such that $d_{\overline{M}}(y) = d_N(x,y)$). We denote by Π the smooth projection from a point in $N \setminus (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}})$ to its unique endpoint in M. Note that we may express this as $\Pi(x) = F \circ \Pi_{V_M} \circ F^{-1}(x)$, where Π_{V_M} is the smooth projection map on V_M sending a point $(x,s) \in V_M$ to $(x,0) \in V_M$.*

3.2 Level sets of the signed distance function

We now define the *signed distance*, corresponding to our choice of unit normal, ν, to the hypersurface M pointing into E as

$$d^\pm_{\overline{M}} = \begin{cases} +d_{\overline{M}}(x), \ \text{if } x \in E \\ 0, \ \text{if } x \in \overline{M} \\ -d_{\overline{M}}(x), \ \text{if } x \in N \setminus E \end{cases},$$

so that the positive sign corresponds to our point lying in E. Denoting by $S_{d^{\pm}_{\overline{M}}}$ the set of points in $N \setminus \overline{M}$ where $d^{\pm}_{\overline{M}}$ fails to be differentiable, we then have that $\overline{S_{d^{\pm}_{\overline{M}}}} = \overline{S_{d_{\overline{M}}}}$ and so $\overline{S_{d^{\pm}_{\overline{M}}}}$ is countably n-rectifiable with zero $\mathcal{H}^{n+1}$ measure.

We now consider the level sets

$$\Gamma(s) = \{x \in N \mid d^{\pm}_{\overline{M}} = s\}.$$

These level sets, $\Gamma(s)$, of $d^{\pm}_{\overline{M}}$ are smooth in the open set $N \setminus (\overline{S_{d^{\pm}_{\overline{M}}}} \cup \overline{M})$ by the Implicit Function Theorem, as $d^{\pm}_{\overline{M}}$ is smooth and the exponential map is invertible on this open set. Note that we have $\Gamma(s) = \emptyset$ for $|s| > d(N)$, where $d(N)$ is the diameter of N.

We use the following notation to refer to the smooth parts of the level sets of $d^{\pm}_{\overline{M}}$, setting $\widetilde{\Gamma}(0) = M$, and for $s \neq 0$,

$$\widetilde{\Gamma}(s) = \Gamma(s) \setminus \overline{S_{d^{\pm}_{\overline{M}}}}.$$

As $\overline{S_{d^{\pm}_{\overline{M}}}}$ is countably n-rectifiable, and hence has vanishing $\mathcal{H}^{n+1}$ measure, we may apply the co-area formula (slicing with $d^{\pm}_{\overline{M}}$) to conclude that

$$\text{for almost every } s \in \mathbb{R} \text{ we have } \mathcal{H}^n\left(\Gamma(s) \cap \overline{S_{d^{\pm}_{\overline{M}}}}\right) = 0. \tag{12}$$

Recall the diffeomorphism $F : V_M \to N \setminus (\mathrm{Sing}(M) \cup Cut(M))$ defined by $F(y,t) = \exp_y(t\nu(y))$ and the coordinates V_M on $N \setminus (\mathrm{Sing}(M) \cup Cut(M))$ as defined in (11). We equip V_M with the pull-back metric via the map F, giving the usual induced metric for M on $M \times \{0\} \subset V_M$. Note that we have $F^{-1}(\widetilde{\Gamma}(s)) = M \times \{s\} \subset V_M$. We now work on V_M to establish facts about the level sets of $d^{\pm}_{\overline{M}}$.

We choose local coordinates, $(x_1, \ldots, x_n, s)$, on V_M such that $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are a local orthonormal frame around a point $x_0 \in M$ such that $\frac{\partial}{\partial s}$ is the unit speed of geodesics with constant base-point in M. The pullback metric (via F) then induces a volume form on V_M, and hence, at a point $(x_0, s_0) \in V_M$, an area element, $\theta(x,s)$, on the set $M \times \{s_0\} \subset V_M$. We set $\theta(x,s) = 0$ for $(x,s) \in (M \times \mathbb{R}) \setminus V_M$. We then have that by the structure of V_M and F that

$$\int_M \theta(x,s) dx_1 \cdots dx_n = \mathcal{H}^n(\widetilde{\Gamma}(s)). \tag{13}$$

Note that as the pull-back metric gives the induced metric for M on $M \times \{0\}$ we have that

$$\int_M \theta(x,0) dx_1 \cdots dx_n = \mathcal{H}^n(M).$$

As the volume form is smooth on V_M we have that the induced area element, $\theta(x,s)$, is continuous in both variables, in particular we have for any $x \in M$ that

$$\theta(x,s) \to \theta(x,0) \text{ as } s \to 0. \tag{14}$$

We have from [Gra04, Theorem 3.11.] that

$$\frac{\partial}{\partial s} \log(\theta(x,s)) = -H(x,s), \tag{15}$$

so that

$$\partial_s \theta(x,s) = -H(x,s)\theta(x,s). \tag{16}$$

Here $H(x,s)$ denotes the mean curvature of the the pullback of $\Gamma(s)$ to V_M at the point $(x,s) \in V_M$. Note that $H(x,0) = \lambda$ as M is a hypersurface of constant mean curvature λ.

We also recall the Ricatti equation, [Gra04, Corollary 3.6.], in the following form

$$\frac{\partial}{\partial s} H(x,s) \geq \min_N \operatorname{Ric}_g. \tag{17}$$

Let us hereafter denote $m = \min_N \operatorname{Ric}_g > 0$. We then have that

$$\begin{cases} H(x,s) \geq \lambda + ms \text{ for } s > 0 \\ H(x,0) = \lambda \\ H(x,s) \leq \lambda + ms \text{ for } s < 0 \end{cases}. \tag{18}$$

Combining (15), (18) and applying the Fundamental Theorem of Calculus we see that for each $t \in \mathbb{R}$ we have

$$\log(\theta(x,t)) \leq -\int_0^t (ms + \lambda)\, ds,$$

from which we conclude that

$$\theta(x,t) \leq e^{-t\left(\frac{mt}{2}+\lambda\right)}. \tag{19}$$

Noting that the quadratic $-t\left(\frac{mt}{2}+\lambda\right)$ is maximised for $t = -\frac{\lambda}{m}$, from (19) we see that

$$\theta(x,s) \leq e^{\frac{\lambda^2}{2m}}. \tag{20}$$

We then apply the Dominated Convergence Theorem to see that by (14) and (20) we have

$$\int_M \theta(x,s) dx_1 \cdots dx_n \to \int_M \theta(x,0) dx_1 \cdots dx_n,$$

so in particular by (13)

$$\mathcal{H}^n(\widetilde{\Gamma}(s)) \to \mathcal{H}^n(M) \text{ as } s \to 0. \tag{21}$$

Let us record for later use that (21) above implies that

$$\operatorname{ess\,inf}_{s \in [-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} \mathcal{H}^n(\widetilde{\Gamma}(s)) \to \mathcal{H}^n(M) \text{ as } \varepsilon \to 0, \tag{22}$$

and

$$\operatorname{ess\,sup}_{s \in [-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} \mathcal{H}^n(\widetilde{\Gamma}(s)) \to \mathcal{H}^n(M) \text{ as } \varepsilon \to 0. \tag{23}$$

3.3 Approximation with the one-dimensional profile

Let $\overline{\mathbb{H}}^\varepsilon$ denote the smooth increasing truncation of the one-dimensional solution to the Allen–Cahn equation as in [Bel20]. Note that in particular if $s > 2\varepsilon\Lambda_\varepsilon$ then $\overline{\mathbb{H}}^\varepsilon(s) = 1$ and if $s < -2\varepsilon\Lambda_\varepsilon$ then $\overline{\mathbb{H}}^\varepsilon(s) \equiv -1$; thus as $|\overline{\mathbb{H}}^\varepsilon(s)| = 1$ for all $|s| \geq 2\varepsilon\Lambda_\varepsilon$ where $\Lambda_\varepsilon = 3|\log(\varepsilon)|$, we have that $e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(s)) = 0$ for $|s| \geq 2\varepsilon\Lambda_\varepsilon$. As shown in [Bel20, Section 2.2], we have

$$\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \to 1 \text{ as } \varepsilon \to 0, \tag{24}$$

where the specific convergence is such that for fixed $\beta > 0$ we have

$$1 - \beta\varepsilon^2 \leq \mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \leq 1 + \beta\varepsilon^2. \tag{25}$$

We define the one-dimensional profile to be

$$v_\varepsilon(x) = \overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)).$$

As $\overline{\mathbb{H}}^{\varepsilon}$ is smooth and the distance function is Lipschitz, it follows that $v_\varepsilon \in W^{1,2}(N)$. We now prove that (2) holds, showing that v_ε acts as an Allen–Cahn "approximation" of M in the sense that it recovers the $\varepsilon \to 0$ limit of the energies of the critical points obtained by the min-max in [BW20a]. We will exploit this fact directly in the construction of the paths in Section 5, eventually allowing us to establish Theorem 3.

By the co-area formula to $d^{\pm}_{\overline{M}}$, noting that $|\nabla d^{\pm}_{\overline{M}}| = 1$, we compute, similarly to [BW22, Section 3.6], that

$$\begin{aligned}\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) &= \mathcal{E}_\varepsilon(v_\varepsilon) - \frac{\lambda}{2}\int_N v_\varepsilon \\ &= \frac{1}{2\sigma}\int_N \frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{W(v_\varepsilon)}{\varepsilon} - \frac{\lambda}{2}\int_N v_\varepsilon \\ &= \frac{1}{2\sigma}\int_{\mathbb{R}}\int_{\Gamma(t)} \frac{\varepsilon}{2}\left(\left(\overline{\mathbb{H}}^{\varepsilon}\right)'(t)\right)^2 + \frac{W(\overline{\mathbb{H}}^{\varepsilon}(t))}{\varepsilon} d\mathcal{H}^n dt - \frac{\lambda}{2}\int_{\mathbb{R}}\int_{\Gamma(t)} \overline{\mathbb{H}}^{\varepsilon}(t) d\mathcal{H}^n dt.\end{aligned}$$

Using (12) and the definition of e_ε as in Subsection 1.1, we then have that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) = \frac{1}{2\sigma}\int_{\mathbb{R}} e_\varepsilon(\overline{\mathbb{H}}^{\varepsilon}(t))\mathcal{H}^n(\widetilde{\Gamma}(t))dt - \frac{\lambda}{2}\int_{\mathbb{R}} \overline{\mathbb{H}}^{\varepsilon}(t)\mathcal{H}^n(\widetilde{\Gamma}(t))dt.$$

From the properties of $\overline{\mathbb{H}}^{\varepsilon}$ as stated above we may obtain the following bounds

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \operatorname{ess\,inf}_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\widetilde{\Gamma}(s))\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^{\varepsilon}) - \frac{\lambda}{2}\int_{-2\varepsilon\Lambda_\varepsilon}^{\infty} \mathcal{H}^n(\widetilde{\Gamma}(t))dt + \frac{\lambda}{2}\int_{-\infty}^{-2\varepsilon\Lambda_\varepsilon} \mathcal{H}^n(\widetilde{\Gamma}(t))dt,$$

and also

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \operatorname{ess\,sup}_{s\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\widetilde{\Gamma}(s))\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^{\varepsilon}) - \frac{\lambda}{2}\int_{2\varepsilon\Lambda_\varepsilon}^{\infty} \mathcal{H}^n(\widetilde{\Gamma}(t))dt + \frac{\lambda}{2}\int_{-\infty}^{2\varepsilon\Lambda_\varepsilon} \mathcal{H}^n(\widetilde{\Gamma}(t))dt.$$

Observe that

$$\mathrm{Vol}_g(E) = \mathcal{H}^{n+1}(\{x \in N \mid d^{\pm}_{\overline{M}} > 0\}) = \int_0^{\infty} \mathcal{H}^n(\widetilde{\Gamma}(t))dt = \lim_{\varepsilon\to 0}\int_{\pm 2\varepsilon\Lambda_\varepsilon}^{\infty} \mathcal{H}^n(\widetilde{\Gamma}(t))dt,$$

and

$$\mathrm{Vol}_g(N \setminus E) = \mathcal{H}^{n+1}(\{x \in N \mid d^{\pm}_{\overline{M}} < 0\}) = \int_{-\infty}^{0} \mathcal{H}^n(\widetilde{\Gamma}(t))dt = \lim_{\varepsilon\to 0}\int_{-\infty}^{\pm 2\varepsilon\Lambda_\varepsilon} \mathcal{H}^n(\widetilde{\Gamma}(t))dt.$$

Combining these two identities with (22), (23) and (24) in the above two bounds on $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ we conclude that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \to \mathcal{H}^n(M) - \lambda\mathrm{Vol}_g(E) + \frac{\lambda}{2}\mathrm{Vol}_g(N) = \mathcal{F}_\lambda(E) \text{ as } \varepsilon \to 0,$$

as desired.

4 Relating local properties of the energy to the geometry

This section relates local behaviour of the energy of the one-dimensional profile, $v_\varepsilon = \overline{\mathbb{H}}^{\varepsilon} \circ d^{\pm}_{\overline{M}}$, to the local geometric properties of E. Recall that $M = \partial^* E \subset N$ is assumed to be a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set, denoted $\mathrm{Sing}(M) = \overline{M} \setminus M$, of Hausdorff dimension at most $n-7$. Furthermore, M separates $N \setminus \overline{M}$ into open sets, E and $N \setminus E$, with common boundary $\overline{M}$ and $E \in \mathcal{C}(N)$ with $M = \partial^* E$. For an isolated singularity $p \in \overline{M}$ we fix throughout this section some $0 < r_1 < r_2 < \min\{R_p, R_l\}$ such that $M \cap \overline{B_{r_2}(p)} \setminus B_{r_1}(p)$ is smooth.

4.1 Local smoothing of Caccioppoli sets

We first establish a procedure to locally perturb Caccioppoli sets that are assumed to be smooth in an annular region to ensure that they are smooth in the entire ball.

Proposition 2. *Suppose that $F \in \mathcal{C}(N)$ is such that $F \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)$. Fix some $\tilde{r}_2 \in (r_1, r_2)$, then, for each $\delta > 0$, there exists $F_\delta \in \mathcal{C}(N)$ with the following properties:*

- *∂F_δ is smooth in $B_{r_2}(p)$.*
- *$F_\delta \setminus B_{\tilde{r}_2}(p) = E \setminus B_{\tilde{r}_2}(p)$.*
- *$|\text{Per}_g(F_\delta) - \text{Per}_g(F)| \leq \delta$.*
- *$Vol_g(F_\delta \Delta F) \leq \delta$.*

Thus, in particular we have that F_δ agrees with F outside of $B_{\tilde{r}_2}(p)$ and is such that

$$|\mathcal{F}_\lambda(F_\delta) - \mathcal{F}_\lambda(F)| \leq (1+\lambda)\delta.$$

Remark 10. *Though Proposition 2 is phrased in our setting for a constant mean curvature hypersurface, the proof makes no use of the variational assumption on M. Thus, the same result holds for any Caccioppoli set which satisfies the same properties as M, without any condition on the mean curvature.*

Proof. Recall that, as $r_2 < R_l$, $B_{r_2}(p)$ is 2-bi-Lipschitz to the Euclidean ball $B^{\mathbb{R}^{n+1}}_{r_2}(0)$ via some a geodeisic normal coordinate chart, ϕ, with $\phi(p) = 0$ and so that $\frac{1}{2} \leq \sqrt{|g|} \leq 2$ on $B_{r_2}(p)$. We consider a radially symmetric mollifier $\rho \in C^\infty_c(B^{\mathbb{R}^{n+1}}_1(0))$ such that $\int_{\mathbb{R}^{n+1}} \rho = 1$ and for each $\theta > 0$ define $\rho_\theta(x) = \frac{1}{\theta^{n+1}}\rho(\frac{x}{\theta})$. We fix $\tilde{r}_2 \in (r_1, r_2)$ and consider, for $\theta < r_2 - \tilde{r}_2$, the function $s_\theta = (\chi_{\phi(F)} * \rho_\theta) \circ \phi \in C^\infty(B_{\tilde{r}_2}(p))$. We now show that s_θ approximates χ_F in the BV norm. First, by standard properties of mollifiers we have

$$\begin{aligned} ||\chi_F - s_\theta||_{L^1(B_{\tilde{r}_2}(p)))} &= \int_{B_{\tilde{r}_2}(p)} |\chi_F - (\chi_{\phi(F)} * \rho_\theta) \circ \phi| d\mathcal{H}^n_g \\ &= \int_{\phi(B_{\tilde{r}_2}(p))} |\chi_{\phi(F)} - (\chi_{\phi(F)} * \rho_\theta)| \sqrt{|g|} d\mathcal{L}^n \\ &\leq 2||\chi_{\phi(F)} - (\chi_{\phi(F)} * \rho_\theta)||_{L^1(\phi(B_{\tilde{r}_2}(p)))} \to 0 \text{ as } \theta \to 0. \end{aligned}$$

Hence by the lower semi-continuity of the perimeter we have that

$$\text{Per}_g(F) = |D_g \chi_F|(B_{\tilde{r}_2}(p)) \leq \liminf_{\theta \to 0} |D_g s_\theta|(B_{\tilde{r}_2}(p)).$$

We let $X \in \Gamma^1_c(TB_{\tilde{r}_2}(p))$ with $|X|_g \leq 1$ and compute

$$\begin{aligned} \int_{B_{\tilde{r}_2}(p)} s_\theta div_g X d\mathcal{H}^n_g &= \int_{\phi(B_{\tilde{r}_2}(p))} (\chi_{\phi(F)} * \rho_\theta) \partial_i (\sqrt{|g|} \hat{X}^i) d\mathcal{L}^n \\ &= \int_{\phi(B_{\tilde{r}_2}(p))} \chi_{\phi(F)} \partial_i (\rho_\theta * \sqrt{|g|} \hat{X}^i) d\mathcal{L}^n \\ &= \int_{B_{\tilde{r}_2}(p)} \chi_F div_g Y \leq |D_g \chi_F|(B_{\tilde{r}_2}(p)), \end{aligned}$$

where here $\hat{X} = \hat{X}^i \partial^\phi_i$ with $\hat{X}^i = X^i \circ \phi^{-1}$, and $Y = Y^i \partial^\phi_i$ with $Y^i = \frac{1}{\sqrt{|g|}}(\rho_\theta * \sqrt{|g|}\hat{X}^i) \circ \phi$ (so $|Y|_g \leq 1$). As the choice of X above was arbitrary here we thus conclude that

$$\lim_{\theta \to 0} |D_g s_\theta|(B_{\tilde{r}_2}(p)) = |D_g \chi_F|(B_{\tilde{r}_2}(p)).$$

Arguing identically to the proof of [Mag12, Theorem 13.8] we conclude that the level sets of s_θ (which have smooth boundary for almost every $t \in (0,1)$ by Sard's Theorem), $L^t_\theta = \{s_\theta > t\}$ for a.e. $t \in (0,1)$ provide a sequence of open sets with smooth boundary such that $Vol_g(L^t_\theta \Delta F) \to 0$ and $Per(L^t_\theta; A) \to Per(F; A)$ for each open set $A \subset B_{\tilde{r}_2}(p)$, whenever $Per(F; \partial A) = 0$, as $\theta \to 0$.

As the exponential map is a radial isometry we have that the geodeisic normal coordinate chart ϕ is such that $B^{\mathbb{R}^{n+1}}_{r_2}(0) = \phi(B_{r_2}(p))$ and $S = \phi\left(M \cap (\overline{B}_{r_2}(p) \setminus B_{r_1}(p))\right)$ is smooth in $B^{\mathbb{R}^{n+1}}_{r_2}(0) \setminus B^{\mathbb{R}^{n+1}}_{r_1}(0)$. In particular we may choose $r_1 < r_a < r_b < \tilde{r}_2$ so that

$$T = \{x + b\nu_S(x) \mid x \in S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0)), b \in [-\delta, \delta]\} \subset\subset B^{\mathbb{R}^{n+1}}_{\tilde{r}_2}(0) \setminus B^{\mathbb{R}^{n+1}}_{r_1}(0),$$

for some $\delta > 0$. Here above we are denoting the unit normal induced on S from M by ν_S; note then that

$$F \cap \phi^{-1}(T) = \phi^{-1}(\{x + b\nu_S(x) \in T \mid b > 0\}). \tag{26}$$

We consider normal derivatives of the function $\chi_{\widehat{F}} * \rho_\theta$ for $\theta < \delta$ in the tubular neighbourhood T defined above. For $x + b\nu_S(x) \in T$ and $\theta > 0$ small enough so that $\nu_S(y) \cdot \nu_S(x) \geq \frac{1}{2}$ for any $y \in B_{2\theta}(x)$ (S is smooth so ν_S varies continuously) we compute that when $b \in (-\theta, \theta)$ we have

$$\begin{aligned}
\nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x) &= (\nabla\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_s(x)) \cdot \nu_S(x) \\
&= -((\mathcal{H}^n \llcorner S)\nu_S * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x) \\
&= -\left(\int_{S \cap B_\theta(x + b\nu_S(x))} \rho_\theta(x + b\nu_S(x) - y)\nu_S(y) \cdot \nu_S(x) d\mathcal{H}^n(y)\right) \\
&\leq -\frac{1}{2}\left(\int_{S \cap B_\theta(x + b\nu_S(x))} \rho_\theta(x + b\nu_S(x) - y) d\mathcal{H}^n(y)\right) < 0.
\end{aligned}$$

Here we've used that $D_g\chi_{\widehat{F}} = -(\mathcal{H}^n \llcorner S)\nu_S$ in T as S is smooth and by the triangle inequality we have both $B_\theta(x + b\nu_S(x)) \subset B_{2\theta}(x)$ for $b < \theta$ and that the strict inequality holds as $S \cap B_{\theta - b}(x) \subset S \cap B_\theta(x + b\nu_S(x))$.

Thus for $b \in (-\theta, \theta)$ we have that $\nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_s(x)) \cdot \nu_S(x) < 0$ and $\nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) = 0$ for $b \notin (-\theta, \theta)$ (as then $\chi_{\widehat{F}} * \rho_\theta \in \{0, 1\}$). We conclude that for each $t \in (0,1)$ and $x \in S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ there exists a unique $b^t_\theta(x) \in (-\theta, \theta)$ such that $(\chi_{\widehat{F}} * \rho_\theta)(x + b^t_\theta(x)\nu_S(x)) = t$; with $b^t_\theta \to 0$ point-wise on $S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ as $\theta \to 0$. Note also that by the structure of T we have

$$L^t_\theta \cap \phi^{-1}(T) = \phi^{-1}(\{x + b\nu_S(x) \in T \mid b < b^t_\theta(x)\}). \tag{27}$$

Consider the function $h_\theta : S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0)) \times (-\theta, \theta) \to T$ defined by

$$h^t_\theta(x, b) = (\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) - t.$$

By the above calculation we have that $\frac{\partial h_\theta}{\partial b}(x + b\nu_S(x)) = \nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x) < 0$ and so by the Implicit Function Theorem (working in charts in which $S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ is locally a graph) we have that the function b^t_θ is smooth on $S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ (noting that $h^t_\theta(x, b^t_\theta(x)) = 0$ for x in this set).

We now show that the directional derivatives of b^t_θ converge to zero as $\theta \to 0$. At a point $x \in S \cap (\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)} \setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$, denoting directional derivatives by $\frac{\partial}{\partial x_i}$, we compute

$$\left|\frac{\left(\frac{\partial h^t_\theta}{\partial x_i}\right)}{\left(\frac{\partial h^t_\theta}{\partial b}\right)}\right| = \left|\frac{\nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \frac{\partial}{\partial x_i}(x + b\nu_S(x))}{\nabla(\chi_{\widehat{F}} * \rho_\theta)(x + b\nu_S(x)) \cdot \nu_S(x))}\right|$$

$$= \left| \frac{\int_{S\cap B_\theta(x+b\nu_S(x))} \rho_\theta(x+b\nu_S(x)-y)\nu_S(y)\cdot \frac{\partial}{\partial x_i}(x+b\nu_S(x))d\mathcal{H}^n(y)}{\int_{S\cap B_\theta(x+b\nu_S(x))} \rho_\theta(x+b\nu_S(x)-y)\nu_S(y)\cdot \nu_S(x)d\mathcal{H}^n(y)} \right|$$
$$\leq 2 \sup_{y\in S\cap B_\theta(x+b\nu_S(x))} \left| \nu_S(y)\cdot \frac{\partial}{\partial x_i}(x+b\nu_S(x)) \right| \to 0 \text{ as } \theta\to 0.$$

Here we see that the last term converges to zero as $\theta \to 0$ by using the triangle inequality, noting that $\nu_S(y)\cdot \frac{\partial}{\partial x_i}x \to 0$ by the continuity of ν_S and $\nu_S(y)\cdot b\frac{\partial \nu_S(x)}{\partial x_i} \leq C\theta$ for a constant C depending only on the normal of S. We then have from the Implicit Function Theorem that

$$\left|\frac{\partial b^t_\theta}{\partial x_i}\right|(x) = \left|\frac{\left(\frac{\partial h^t_\theta}{\partial x_i}\right)}{\left(\frac{\partial h^t_\theta}{\partial b}\right)}\right|(x, b^t_\theta(x)) \to 0 \text{ as } \theta \to 0.$$

We conclude that for $t\in(0,1)$ the function b^t_θ defined on $S\cap(\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)}\setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))$ as above is smooth and in fact converges to zero uniformly in the C^1 norm as $\theta\to 0$.

We fix $\tilde{r}_a\in(r_a,r_b)$ so that for $\theta<\tilde{r}_a-r_a$ we ensure (by the triangle inequality) that $x+b^t_\theta(x)\nu_S(x)\in B_{\tilde{r}_a}(p)$ whenever $x\in\partial B_{r_a}(p)$ (i.e. when $|x|=r_a$). Fix a further $\tilde{r}_b\in(\tilde{r}_a,r_b)$ so that for $\theta<r_b-\tilde{r}_b$ we ensure (by the triangle inequality) that $x+b^t_\theta(x)\nu_S(x)\in N\setminus B_{\tilde{r}_b}(p)$ whenever $x\in\partial B_{r_b}(p)$ (i.e. when $|x|=r_b$).

We then define $\eta\in C^\infty_c(\mathbb{R}^{n+1})$ to be a radially symmetric cut-off function with the following properties

$$\begin{cases} 0\leq\eta\leq 1 \text{ on } \mathbb{R}^{n+1}\\ |\nabla\eta|\leq C_\eta \text{ on } \mathbb{R}^{n+1}\\ \eta(x)=1 \text{ if } |x|\leq\tilde{r}_a\\ \eta(x)=0 \text{ if } |x|\geq\tilde{r}_b,\end{cases}$$

for some $C_\eta=C_\eta(\tilde{r}_a,\tilde{r}_b)>0$.

Before proceeding to define the local smoothing we let

$$H^t_\theta = \phi^{-1}\left(\{x+b\nu_S(x)\in T \mid b<\eta(x)b^t_\theta(x)\}\right),$$

and, slightly abusing notation, we denote

$$\partial H^t_\theta = \phi^{-1}\left(\left\{x+\eta(x)b^t_\theta(x)\nu_S(x) \mid x\in S\cap(\overline{B^{\mathbb{R}^{n+1}}_{r_b}(0)}\setminus B^{\mathbb{R}^{n+1}}_{r_a}(0))\right\}\right),$$

where the notation for boundary is justified by virtue of the fact that, as M separates T, the "graph" given by $\phi^{-1}(x+\eta(x)b^t_\theta(x)\nu_S(x))$ separates $\phi^{-1}(T)$.

We then consider the following hypersurface

$$\partial G^t_\theta = (M\setminus B_{\tilde{r}_b}(p))\cup(\partial L^t_\theta\cap B_{\tilde{r}_a}(p))\cup\left(\partial H^t_\theta\cap(B_{\tilde{r}_b}(p)\setminus B_{\tilde{r}_a}(p))\right)$$

which is smooth inside $B_{r_b}(p)$ as $H^t_\theta\cap\partial B_{\tilde{r}_a}(p)\in\partial L^t_\theta$ and $H^t_\theta\cap\partial B_{\tilde{r}_b}(p)\in M$. We note also that as $\overline{M}$ satisfies the geodesic touching property from Lemma 1, ∂G^t_θ also satisfies the conclusions of Lemma 1.

Furthermore, by (26) and (27), ∂G^t_θ arises as the boundary of the open set

$$G^t_\theta = (E\setminus((B_{\tilde{r}_b}(p)\cup\phi^{-1}(T)))\cup(L^t_\theta\cap B_{\tilde{r}_b}(p)\setminus\phi^{-1}(T))\cup(H^t_\theta\cap\phi^{-1}(T)),$$

and hence $G^t_\theta \setminus B_{\tilde r_2}(p) = E \setminus B_{\tilde r_2}(p)$ as $\phi^{-1}(T) \subset\subset B_{\tilde r_2}(p) \setminus B_{r_1}(p)$.

We may write

$$(L^t_\theta \Delta E) \cap \phi^{-1}(T) = \phi^{-1}\left(\left\{ x + b\nu_S(x) \in T \;\middle|\; \begin{cases} 0 < b < b^t_\theta(x), \text{ if } b^t_\theta(x) > 0 \\ b^t_\theta < b < 0, \text{ if } b^t_\theta(x) < 0 \end{cases} \right\}\right).$$

As $0 \le \eta \le 1$, $\eta(x) b^t_\theta(x)$ has the same sign as b^t_θ and $|\eta(x)b^t_\theta(x)| \le |b^t_\theta(x)|$. Therefore, by construction of H^t_θ, we have that $(G^t_\theta \Delta E) \cap \phi^{-1}(T) \subset (L^t_\theta \Delta E) \cap \phi^{-1}(T)$. As G^t_θ agrees with either E or L^t_θ outside of $\phi^{-1}(T)$ we conclude that $G^t_\theta \Delta F \subset L^t_\theta \Delta F$ and hence $Vol_g(G^t_\theta \Delta F) \le Vol_g(L^t_\theta \Delta F) \to 0$ as $\theta \to 0$.

Thus we may choose $\theta > 0$ to ensure that

$$Vol_g(G^t_\theta \Delta F) \le \delta.$$

We note that we may always ensure that the various radii $r_a, \tilde r_a, \tilde r_b$ and r_b as chosen above are done so to ensure that

$$\mathrm{Per}_g(F; \partial(B_{\tilde r_b}(p) \setminus \phi^{-1}(T))) = 0,$$

so that $\mathrm{Per}_g(L^t_\theta; B_{\tilde r_b}(p) \setminus \phi^{-1}(T)) \to \mathrm{Per}_g(F; B_{\tilde r_b}(p) \setminus \phi^{-1}(T))$ as $\theta \to 0$; in particular for $\theta > 0$ sufficiently small we ensure that

$$|\mathrm{Per}_g(L^t_\theta; B_{\tilde r_b}(p) \setminus \phi^{-1}(T)) - \mathrm{Per}_g(F; B_{\tilde r_b}(p) \setminus \phi^{-1}(T))| \le \frac{\delta}{2}.$$

As we have $G^t_\theta = E$ outside of $B_{\tilde r_b} \cup \phi^{-1}(T)$ we have that

$$\mathrm{Per}_g(G^t_\theta; N \setminus (B_{\tilde r_b} \cup \phi^{-1}(T))) = \mathrm{Per}_g(E; N \setminus (B_{\tilde r_b} \cup \phi^{-1}(T))).$$

Note that as $|\nabla \eta| < C_\eta$ and $b^t_\theta \to 0$ uniformly in the C^1 norm we may choose $\theta > 0$ potentially smaller to ensure that

$$\sup_{y \in M \cap (B_{r_b}(p) \setminus B_{r_a}(p))} |J_{\phi^{-1} \circ (Id + (\eta b^t_\theta)) \circ \phi}(y) - 1| \le \frac{\delta}{2\mathcal{H}^n(M \cap (B_{r_b}(p) \setminus B_{r_a}(p)))},$$

where here J_f denotes the Jacobian of a function f (which depends only on the C^1 norm of the function f), and so by the area formula we have

$$\begin{aligned} |\mathcal{H}^n(\partial H^t_\theta \cap \phi^{-1}(T)) - \mathcal{H}^n(M \cap \phi^{-1}(T))| &= \left| \int_{M \cap (B_{r_b}(p) \setminus B_{r_a}(p))} (J_{\phi^{-1} \circ (Id + (\eta b^t_\theta)) \circ \phi}(y) - 1) d\mathcal{H}^n(y) \right| \\ &\le \frac{\delta}{2}. \end{aligned}$$

We combine the above facts to compute that for $\theta > 0$ sufficiently small we have

$$\begin{aligned} |\mathrm{Per}_g(G^t_\theta) - \mathrm{Per}_g(F)| &= |\mathrm{Per}_g(G^t_\theta; \phi^{-1}(T)) - \mathrm{Per}_g(F; \phi^{-1}(T)) \\ &\quad + \mathrm{Per}_g(G^t_\theta; N \setminus \phi^{-1}(T)) - \mathrm{Per}_g(F; N \setminus \phi^{-1}(T))| \\ &\le |\mathcal{H}^n(\partial H^t_\theta \cap \phi^{-1}(T)) - \mathcal{H}^n(M \cap \phi^{-1}(T))| \\ &\quad + |\mathrm{Per}_g(L^t_\theta; B_{\tilde r_b}(p) \setminus \phi^{-1}(T)) - \mathrm{Per}_g(F; B_{\tilde r_b}(p) \setminus \phi^{-1}(T))| \\ &\le \delta \end{aligned}$$

and so

$$|\mathcal{F}_\lambda(G^t_\theta) - \mathcal{F}_\lambda(F)| \le (1 + \lambda)\delta.$$

Setting $F_\delta = G^t_\theta$ for one choice of almost any $t \in (0, 1)$ and a choice of $\theta > 0$ sufficiently small then provides a Caccioppoli satisfying the desired conclusions. □

Remark 11. *As ∂F_δ is smooth in $B_{r_2}(p)$ and agrees with M outside of $B_{\tilde r_2}(p)$, we have that ∂F_δ satisfies the conclusions of Lemma 1. More specifically, for any $x \in N \setminus \partial F_\delta$, any minimising geodesic connecting x to ∂F_δ has endpoint in a regular point of ∂F_δ.*

4.2 Local energy minimisation

We now prove the existence of local energy minimisers that agree with our one-dimensional solution, $v_\varepsilon = d^{\pm}_{\overline{M}} \circ \overline{\mathbb{H}}^{\varepsilon}$, outside of a fixed ball in N. Minimisers of such problems will be used in the next subsection to conclude local statements about constant mean curvature hypersurfaces under appropriate assumptions on the behaviour of the energy of v_ε as $\varepsilon \to 0$.

Lemma 2. *Let $B_\rho(q) \subset N$ be a ball, of radius $\rho > 0$, centred at a point $q \in \overline{M}$ and define for $\varepsilon \in (0,1)$*

$$\mathcal{A}_{\varepsilon,\rho}(q) = \{u \in W^{1,2}(N) \mid |u| \leq 1, u = v_\varepsilon \text{ on } N \setminus B_\rho(q)\},$$

then there exists $g_\varepsilon \in \mathcal{A}_{\varepsilon,\rho}(q)$ such that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_{\varepsilon,\rho}(q)} \mathcal{F}_{\varepsilon,\lambda}(u).$$

Proof. Let $I_\varepsilon = \inf_{u \in \mathcal{A}_{\varepsilon,\rho}(q)} \mathcal{F}_{\varepsilon,\lambda}(u)$ and consider $\{u_k\}_{k=1}^{\infty} \in \mathcal{A}_{\varepsilon,\rho}(q)$ such that $\mathcal{F}_{\varepsilon,\lambda}(u_k) \to I_\varepsilon$. Note that for any $u \in \mathcal{A}_{\varepsilon,\rho}(q)$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(u) = \frac{1}{2\sigma} \int_N \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} - \frac{\lambda}{2} \int_N u \geq \frac{1}{2\sigma} \int_N \varepsilon \frac{|\nabla u|^2}{2} - \frac{\lambda}{2} \mathrm{Vol}(N),$$

using the fact that $|u| \leq 1$ $\mathcal{H}^{n+1}$-a.e. and $W \geq 0$; hence for k sufficiently large we have that

$$\int_N |\nabla u_k|^2 \leq \frac{4\sigma}{\varepsilon} \left(I_\varepsilon + 1 + \frac{\lambda}{2} \mathrm{Vol}(N) \right),$$

so $\sup_k ||\nabla u_k||_{L^2(N)} < \infty$. Using the triangle and Poincaré inequality (as $u_k - v_\varepsilon \in W^{1,2}_0(B_\rho(q))$ for each $k \in \mathbb{N}$) we have

$$\begin{aligned} ||u_k||_{L^2(N)} &= ||u_k - v_\varepsilon + v_\varepsilon||_{L^2(N)} \leq ||u_k - v_\varepsilon||_{L^2(N)} + ||v_\varepsilon||_{L^2(N)} \leq C_P ||\nabla u_k - \nabla v_\varepsilon||_{L^2(N)} + ||v_\varepsilon||_{L^2(N)} \\ &\leq C_P (\sup_k ||\nabla u_k||_{L^2(N)} + ||\nabla v_\varepsilon||_{L^2(N)}) + ||v_\varepsilon||_{L^2(N)}, \end{aligned}$$

(where C_P is the Poincaré constant for N) so $\sup_k ||u_k||_{L^2(N)} < \infty$. Thus $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in $W^{1,2}(N)$ and hence, by Rellich-Kondrachov compactness, there exists $u \in W^{1,2}(N)$ and a sub-sequence of the $\{u_k\}_{k=1}^{\infty}$ (not relabelled) such that u_k converges to u, weakly in $W^{1,2}(N)$, strongly in $L^2(N)$ (hence strongly in $L^1(N)$) and $\mathcal{H}^{n+1}$-*a.e.* in N. We then have that $|u| \leq 1$ and $u = v_\varepsilon$ on $N \setminus B_\rho(q)$ at $\mathcal{H}^{n+1}$-*a.e.* point, thus $u \in \mathcal{A}_{\varepsilon,\rho}(q)$. Combining the weak convergence of the u_k in $W^{1,2}(N)$, almost everywhere convergence of the u_k with Fatou's lemma (using the continuity of W and that $W \geq 0$) and the strong $L^1(N)$ convergence of the u_k, we have $\mathcal{F}_{\varepsilon,\lambda}(u) = I_\varepsilon$.

The above arguments show that, for sufficiently small $\varepsilon > 0$, the energy minimisation problem in the class $\mathcal{A}_{\varepsilon,\rho}(q)$ is well posed. Hence, for each $\varepsilon \in (0,1)$ we may produce a function $g_\varepsilon \in W^{1,2}(N)$ such that $|g_\varepsilon| \leq 1$ on N, $g_\varepsilon = v_\varepsilon$ on $N \backslash B_\rho(q)$ and $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_\varepsilon} \mathcal{F}_{\varepsilon,\lambda}(u)$; in particular $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$. □

4.3 Recovery functions for local geometric properties

Using Proposition 2 and our analysis from Subsection 3.3 we now prove the following recovery type Lemma relating the energy of v_ε to local geometric properties of constant mean curvature hypersurfaces. Recall that for an isolated singularity $p \in \overline{M}$ we fixed $0 < r_1 < r_2 < \min\{R_p, R_l\}$ such that $M \cap \overline{B_{r_2}(p)} \setminus B_{r_1}(p)$ is smooth.

Lemma 3. *Let $F \in \mathcal{C}(N)$ be such that $F \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)$ and*

$$\mathcal{F}_\lambda(F) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(F) \mid G \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)\}.$$

Then, for $\varepsilon > 0$ sufficiently small, there exist functions $f_\varepsilon \in \mathcal{A}_{\varepsilon,r_2}(p)$, so that $|f_\varepsilon| \leq 1$ and $f_\varepsilon = v_\varepsilon$ on $N \setminus B_{r_2}(p)$, such that

$$\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F).$$

Furthermore, if the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon$, for some sequence $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$, where the g_ε are defined as in Lemma 2 for $B_{r_2}(p)$, then E solves the above minimisation problem.

Remark 12. *There exist minimisers to the variational problem in Lemma 3 by [Mag12, Section 12.5].*

Proof. Let F solve the minimisation problem and consider, for each $\varepsilon, \delta > 0$, the functions $f_{\varepsilon,\delta} = \overline{\mathbb{H}}^\varepsilon \circ d^{\pm}_{\partial F_\delta}$ where F_δ is a local smoothing of F as in Proposition 2 and we define the signed distance function to ∂F_δ by

$$d^{\pm}_{\partial F_\delta} = \begin{cases} +d_{\partial F_\delta}(x), \text{ if } x \in F_\delta \\ 0, \text{ if } x \in \partial F_\delta \\ -d_{\partial F_\delta}(x), \text{ if } x \in N \setminus F_\delta \end{cases},$$

where here $d_{\partial F_\delta}$ is the usual distance function to the set ∂F_δ. Note that the functions $f_{\varepsilon,\delta} \in \mathcal{A}_{\varepsilon,r_2}(p)$ for $\varepsilon > 0$ small enough so that $2\varepsilon\Lambda_\varepsilon < r_2 - \tilde{r}_2$, where $\tilde{r}_2$ is chosen as in Proposition 2. To see this we note that as $F_\delta = E$ outside of $B_{\tilde{r}_2}(p)$ we have that $d_{\partial F_\delta} = d_{\overline{M}}$ on the set $(N \setminus B_{r_2}(p)) \cap \{d_{\partial F_\delta} \leq 2\varepsilon\Lambda_\varepsilon\}$ and hence $f_{\varepsilon,\delta} = v_\varepsilon$ on $N \setminus B_{r_2}(p)$.

By Remark 11 the hypersurfaces ∂F_δ satisfy the conclusions of Lemma 1. Hence we may repeat the analysis for ∂F_δ (of the cut locus, level sets etc.) as we did for M in Subsection 3.3 in order to conclude that $\mathcal{F}_{\varepsilon,\lambda}(f_{\varepsilon,\delta}) \to \mathcal{F}_\lambda(F_\delta)$ as $\varepsilon \to 0$. By Proposition 2 we have that $|\mathcal{F}_\lambda(F_\delta) - \mathcal{F}_\lambda(F)| \leq (1+\lambda)\delta$ and thus by setting $f_\varepsilon = f_{\varepsilon,\frac{\varepsilon}{1+\lambda}}$ we ensure that this sub-sequence is such that $\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F)$ as desired.

Under the assumption that the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon$, for some sequence $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$, and the fact that $f_\varepsilon \in \mathcal{A}_{\varepsilon,r_2}(p)$ we see that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \leq \mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) + \tau_\varepsilon,$$

where here we are using that $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) = \inf_{u \in \mathcal{A}_{\varepsilon,r_2}(p)} \mathcal{F}_{\varepsilon,\lambda}(u) \leq \mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon)$ by construction. Hence, by (2) and the fact that $\mathcal{F}_{\varepsilon,\lambda}(f_\varepsilon) \to \mathcal{F}_\lambda(F)$ from the above, letting $\varepsilon \to 0$ we conclude that

$$\mathcal{F}_\lambda(E) \leq \mathcal{F}_\lambda(F).$$

In particular we have that

$$\mathcal{F}_\lambda(E) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(F) \mid G \setminus B_{r_1}(p) = E \setminus B_{r_1}(p)\},$$

as desired. □

Remark 13. *Lemma 3 combined with the results of [Mag12, Section 28.2] imply that if the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon$, for some sequence $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$, then any tangent cone to the hypersurface M at a point of $B_{r_1}(p) \cap \partial F$ is area-minimising (in the sense that the cone is a perimeter minimiser in $\mathbb{R}^{n+1}$).*

5 Construction of paths

In order to prove Theorem 3 we will work under the assumption that the functions $v_\varepsilon = \overline{\mathbb{H}}^\varepsilon \circ d^\pm_{\overline{M}}$, defined in Subsection 3.3, are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$ for some $\tau > 0$, where the g_ε are defined as in Lemma 2 in a ball centred on an isolated singularity of M. We then exhibit a continuous path in $W^{1,2}(N)$, from a_ε to b_ε, with energy along the path bounded a fixed amount below $\mathcal{F}_\lambda(E)$, independent of ε. In this manner we will have exhibited, see Section 6, an admissible path in the min-max procedure of [BW20a] which contradicts the assumption that M arose from this construction and allows us to establish Theorem 3. In the following Subsections we construct the separate pieces of our desired path, with maximum energy along these pieces bounded a fixed amount below $\mathcal{F}_\lambda(E)$, independently of ε.

5.1 Paths with energy drop fixing a ball

We first construct the shifted functions described in Step 2 of Subsection 1.3 and compute upper energy bounds.

5.1.1 Choosing radii for local properties

We fix various radii, dependent only on the geometry of M around an isolated singularity, in order to later define functions for our path.

Let $p \in \text{Sing}(M)$ be an isolated singularity of M, there then exists an $R_p > 0$ such that $\overline{B_{R_p}(p)} \cap \text{Sing}(M) = \{p\}$, i.e. such that $M \cap \overline{B_{R_p}(p)} \setminus \{p\}$ is smooth.

The bounded mean curvature of M (specifically as the monotonicity formula holds at all points of $\overline{M}$) provides Euclidean volume growth, namely that there exists two constants $C_p, r_p > 0$ both depending on the point p such that for all $r < r_p$ we have

$$\mathcal{H}^n(M \cap B_{r_p}(p)) \leq C_p r^n. \tag{28}$$

We first fix $r_0 \in (0, \min\{r_p, R_p\})$ such that

$$\mathcal{H}^n(M \setminus B_{r_0}(p)) > \frac{3\mathcal{H}^n(M)}{4}. \tag{29}$$

Set $\widehat{K} = M \cap (\overline{B_{R_p}(p)} \setminus B_{r_0}(p))$, which is compact in M, so by Remark 8 we ensure that there exists a $c_{\widehat{K}} > 0$ such that $F(\widehat{K} \times (-c_{\widehat{K}}, c_{\widehat{K}})) \cap (\text{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}) = \emptyset$. Defining on $K \times (-c_{\widehat{K}}, c_{\widehat{K}}) \subset V_M$ the function

$$h(x, a) = \text{dist}_N(F(\Pi_{V_M}(x, a)), p) \tag{30}$$

we ensure that for $(x, a) \in K \times (-c_{\widehat{K}}, c_{\widehat{K}}) \subset V_M$ we have, by Remark 9, that

$$|\nabla h(x, a)|_{(x,a)} \leq C_h, \tag{31}$$

where here we are computing the gradient, ∇, on V_M with respect to the pullback metric F^*g, for some constant $C_h > 0$ dependent only on $\widehat{K}$, N and g.

We may now fix $r < \frac{1}{4}(\min\{r_p, R_p\} - r_0)$ sufficiently small to ensure that on the annulus $\mathcal{A} = M \cap (B_{r_0+3r}(p) \setminus B_{r_0+2r})$ we have

$$\frac{\mathcal{H}^n(\mathcal{A})}{r^2} \leq \left(\frac{me^{-\frac{\lambda^2}{2m}} \mathcal{H}^n(M)}{8C_h^2} \right); \tag{32}$$

where here we are using the Euclidean volume growth of the mass of M (which follows from the monotonicity formula) to ensure that $\mathcal{H}^n(\mathcal{A})$ is of order r^n, combined with the assumption that, from

Remark 4, $n \geq 7$. The above choice of r will become clear when calculating the energy of the functions $v_\varepsilon^{t,s}$ in Subsection 5.1.3.

We now define the balls $B_i = B_{r_0+ir}(p)$ for $i \in \{1,2,3,4\}$ which are such that $B_1 \subset\subset B_2 \subset\subset B_3 \subset\subset B_4$; with this notation we have that $\mathcal{A} = M \cap (\overline{B_3} \setminus B_1)$.

5.1.2 Defining the shifted functions

We define, for $t \in [-t_1, t_1]$ where $t_1 > 0$ is to be chosen and all $\varepsilon > 0$ sufficiently small, functions $v_\varepsilon^{t,s} \in W^{1,2}(N)$ for $s \in [0,1]$ with the paths $t \in [-t_1, t_1] \to v_\varepsilon^{t,s}$ and $s \in [0,1] \to v_\varepsilon^{t,s}$ continuous in $W^{1,2}(N)$. The functions $v_\varepsilon^{t,s}$ will be defined so that the following properties hold for all $t \in [-t_1, t_1]$ and $s \in [0,1]$

$$\begin{cases} v_\varepsilon^{t,1} = \overline{\mathbb{H}}^\varepsilon \circ (d^{\pm}_{\overline{M}} - t) \text{ on } N \\ v_\varepsilon^{0,s} = v_\varepsilon \text{ on } N \\ v_\varepsilon^{t,0} = v_\varepsilon \text{ in } B_{r_0}(p) \subset B_1 \end{cases}, \tag{33}$$

and in such a way that, for some $E(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon).$$

Furthermore, we will show in Lemma 4 that there exists some $0 < t_0 < t_1$ and $\eta > 0$ such that for all $s \in [0,1]$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon).$$

Finally, by Lemma 5, for $t > t_0$ we will have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,1}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon).$$

These upper bounds on the energy along the paths provided by the functions $v_\varepsilon^{t,s}$ will be calculated explicitly in Subsection 5.1.3.

Thus, assuming the above, and combined with the path provided in Subsection 5.3, which changes the functions $v_\varepsilon^{t,0}$ only in B_1 to decrease their energy, we construct a path from a_ε to b_ε in $W^{1,2}(N)$ whose energy along the path remains bounded strictly below $\mathcal{F}_\lambda(E)$. We now proceed to define the $v_\varepsilon^{t,s}$ explicitly.

Consider $B_1 \subset\subset B_2 \subset\subset B_3 \subset\subset B_4$ centred at $p \in M$ as specified in Subsection 5.1.1. Let $K = M \cap \overline{B_4} \setminus B_1$, which is compact in M, and fix $c_K = c_{M \cap \overline{B_4} \setminus B_1} > 0$ as in Remark 8; we then have that $K \times (-c_K, c_K) \subset V_M$ is a two-sided tubular neighbourhood of $M \cap \overline{B_4} \setminus B_1$ in N under the map F. Note that we have $F(K \times (-c_K, c_K)) \cap (\mathrm{Sing}(M) \cup \overline{S_{d_{\overline{M}}}}) = \emptyset$ by the choice of c_K and as $K \subset \widehat{K}$ we ensure that $c_K \leq c_{\widehat{K}}$ as in Subsection 5.1.1.

For $r > 0$ as chosen we define the following function on M,

$$f(y) = \begin{cases} 1 \text{ if } y \in M \setminus \overline{B_3} \\ 0 \text{ if } y \in M \cap \overline{B_2} \\ \frac{1}{r}(\mathrm{dist}_N(y,p) - r) \text{ if } y \in \mathcal{A}. \end{cases} \tag{34}$$

Then we have $0 \leq f \leq 1$ and set

$$f_s(x) = s + (1-s)f(x).$$

Recall, from Subsection 3.2, the definition of the smooth projection, Π to M defined on $N\setminus(\mathrm{Sing}(M)\cup \overline{S_{d_{\overline{M}}}})$. Fix $t_1>0$ and an $0<\varepsilon_2<\varepsilon_1$ to ensure that $2\varepsilon\Lambda_\varepsilon+t_1<\min\{c_K,\frac{r}{2}\}$, for all $0<\varepsilon<\varepsilon_2$. We now define, for $t\in[-t_1,t_1]$, $s\in[0,1]$ and $0<\varepsilon<\varepsilon_2$, the functions

$$v_\varepsilon^{t,s}(x)=\begin{cases}\overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-tf_s(\Pi(x)))) \text{ if } x\in F(K\times[-c_K,c_K])\\ \overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-ts) \text{ if } x\in B_{r_0+\frac{3r}{2}}(p)\\ \overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-t) \text{ if } x\in N\setminus B_{r_0+\frac{7r}{2}}(p)\\ 1 \text{ if } x\in E\cap\{d^{\pm}_{\overline{M}}>2\varepsilon\Lambda_\varepsilon+t_1\}\\ -1 \text{ if } x\in(N\setminus E)\cap\{d^{\pm}_{\overline{M}}<-2\varepsilon\Lambda_\varepsilon-t_1\}\end{cases}.$$

We now show that the $v_\varepsilon^{t,s}$ are well defined Lipschitz functions on N so that in particular, as $F(\mathcal{A}\times[-2\varepsilon_2\Lambda_{\varepsilon_2}-t_1,2\varepsilon_2\Lambda_{\varepsilon_2}+t_1])\subset\subset\overline{B_{r_0+\frac{7r}{2}}(p)}\setminus B_{r_0+\frac{3r}{2}}(p)$ for $2\varepsilon_2\Lambda_{\varepsilon_2}+t_1<\frac{r}{2}$, we have

$$v_\varepsilon^{t,s}(x)=\overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-tf_s(\Pi(x))) \text{ on } N\setminus(\mathrm{Sing}(M)\cup\overline{S_{d_{\overline{M}}}}).$$

Note that if $x\in\overline{B_{r_0+\frac{7r}{2}}(p)}\setminus B_{r_0+\frac{3r}{2}}(p)$ and $|d^{\pm}_{\overline{M}}|\le 2\varepsilon\Lambda_\varepsilon+t_1$ then by the triangle inequality we see that $x\in F(K\times[-c_K,c_K])$. In $F(K\times[2\varepsilon_2\Lambda_{\varepsilon_2}+t_1,c_K])$ we have that $v_\varepsilon^{t,s}\equiv 1$ and in $F(K\times[-c_K,-(2\varepsilon_2\Lambda_{\varepsilon_2}+t_1)])$ we have that $v_\varepsilon^{t,s}\equiv -1$. Combining these two facts we see that the $v_\varepsilon^{t,s}$ are in fact Lipschitz functions in $\overline{B_{r_0+\frac{7r}{2}}(p)}\setminus B_{r_0+\frac{3r}{2}}(p)$.

Noting that $F(\mathcal{A}\times[-2\varepsilon_2\Lambda_{\varepsilon_2}-t_1,2\varepsilon_2\Lambda_{\varepsilon_2}+t_1])\subset\subset\overline{B_{r_0+\frac{7r}{2}}(p)}\setminus B_{r_0+\frac{3r}{2}}(p)$ as $2\varepsilon_2\Lambda_{\varepsilon_2}+t_1<\frac{r}{2}$, we have by definition of f that if $x\in B_{r_0+\frac{3r}{2}}(p)\cap F(K\times[-c_K,c_K])$ then $v_\varepsilon^{t,s}=\overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-ts)$, and if $x\in(N\setminus B_{r_0+\frac{7r}{2}}(p))\cap F(K\times[-c_K,c_K])$ then $v_\varepsilon^{t,s}=\overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-t)$.

Combining the above two paragraphs, we conclude that the $v_\varepsilon^{t,s}$ are indeed well defined continuous functions on N. Furthermore, as noted above, the $v_\varepsilon^{t,s}$ are Lipschitz in $\overline{B_{r_0+\frac{7r}{2}}(p)}\setminus B_{r_0+\frac{3r}{2}}(p)$, and in the sets $N\setminus B_{r_0+\frac{7r}{2}}(p)$ and $B_{r_0+\frac{3r}{2}}(p)$ by the $v_\varepsilon^{t,s}$ are Lipschitz by definiton (as on these sets $v_\varepsilon^{t,s}=\overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-t)$ and $v_\varepsilon^{t,s}=\overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x)-ts)$ respectively).

In conclusion we have that the $v_\varepsilon^{t,s}$ are continuous on N and Lipschitz on $\overline{B_{r_0+\frac{7r}{2}}(p)}\setminus B_{r_0+\frac{3r}{2}}(p)$, $N\setminus B_{r_0+\frac{7r}{2}}(p)$ and $B_{r_0+\frac{3r}{2}}(p)$; hence we have that the $v_\varepsilon^{t,s}\in W^{1,\infty}(N)\subset W^{1,2}(N)$. Note also that by the definition of the $v_\varepsilon^{t,s}$ we have that (33) holds as desired.

5.1.3 Energy calculations and continuity of paths

We calculate an upper bound on the energy of the functions $v_\varepsilon^{t,s}$ for $t\in[t_0,t_0]$, for some $0<t_0\le t_1$, all $s\in[0,1]$ and $\varepsilon>0$ sufficiently small. The calculation method here is similar to those in [BW22, Sections 4, 6.1 and 7.1].

Lemma 4. *There exists $t_0\in(0,t_1)$ such that for all $s\in[0,1]$ and $t\in[-t_0,t_0]$ we have that*

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s})\le\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)+E(\varepsilon)$$

where $E(\varepsilon)\to 0$ as $\varepsilon\to 0$. Furthermore, we have that, for all $\varepsilon>0$ sufficiently small there exists

$$\eta=\frac{m\sigma}{8}\mathcal{H}^n(M)t_0^2>0$$

such that for all $s\in[0,1]$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,s})\le\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)-\eta.$$

Proof. We define, for $(x,a) \in V_M$, $\hat{v}_\varepsilon^{t,s}((x,a)) = v_\varepsilon^{t,s}(F(x,a))$ so that, as $\Pi \circ F = F \circ \Pi_{V_M}$, we have that

$$\hat{v}_\varepsilon^{t,s}(x,a) = \overline{\mathbb{H}}^\varepsilon(a - tf_s(F(\Pi_{V_M}(x,a)))) \text{ on } V_M. \tag{35}$$

Note that by (30) and (34) we have for points $(x,a) \in V_M$ with $x \in \mathcal{A}$ that

$$\nabla\big(f_s(F(\Pi_{V_M}(x,a)))\big) = \frac{1-s}{r}\nabla h(x,a),$$

and if $x \in M \setminus \mathcal{A}$ then f is constant and so

$$\nabla\big(f_s(F(\Pi_{V_M}(x,a)))\big) = 0.$$

We then compute by the generalised Gauss' Lemma, [Gra04, Chapter 2.4], that we have

$$|\nabla \hat{v}_\varepsilon^{t,s}(x,a)|^2_{(x,a)} = \big((\overline{\mathbb{H}}^\varepsilon)'(a - tf_s(x))\big)^2 \left(1 + \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\chi_{\mathcal{A}\times\mathbb{R}}(x,a)\right), \tag{36}$$

where here $\chi_{\mathcal{A}\times\mathbb{R}}$ is the indicator function for the set $\mathcal{A}\times\mathbb{R}$ on V_M.

Using the co-area formula (slicing with a) and Fubini's Theorem we have

$$\begin{aligned}
2\sigma\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) &= \int_{V_M} e_\varepsilon(\hat{v}_\varepsilon^{t,s}) - \sigma\lambda\hat{v}_\varepsilon^{t,s} d\mathcal{H}^{n+1}_{F^*g} \\
&= \int_{V_M} \frac{\varepsilon}{2}|\nabla \hat{v}_\varepsilon^{t,s}|^2 + \frac{W(\hat{v}_\varepsilon^{t,s})}{\varepsilon} - \sigma\lambda\hat{v}_\varepsilon^{t,s} d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&= \int_{V_M} \frac{\varepsilon}{2}\big((\overline{\mathbb{H}}^\varepsilon)'(a-tf_s(x))\big)^2 + \frac{W(\overline{\mathbb{H}}^\varepsilon(a-tf_s(x))}{\varepsilon} - \sigma\lambda\overline{\mathbb{H}}^\varepsilon(a-tf_s(x))d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&\quad + \int_{V_M} \frac{\varepsilon}{2}\big((\overline{\mathbb{H}}^\varepsilon)'(a-tf_s(x))\big)^2 \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\chi_{\mathcal{A}\times\mathbb{R}}(x,a)d\mathcal{H}^{n+1}_{F^*g}(x,a) \\
&= \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-tf_s(x))\right) - \sigma\lambda\overline{\mathbb{H}}^\varepsilon(a-tf_s(x))\right]\theta(x,a)da\,d\mathcal{H}^n(x) \\
&\quad + \int_{\mathcal{A}} \int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}\big((\overline{\mathbb{H}}^\varepsilon)'(a-tf_s(x))\big)^2 \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\theta(x,a)da\,d\mathcal{H}^n(x).
\end{aligned}$$

Then, as $v_\varepsilon = v_\varepsilon^{0,s}$ for any $s \in [0,1]$, we have

$$2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)) = \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-tf_s(x))\right) - e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a)\right)\right]\theta(x,a)da\,d\mathcal{H}^n(x) \tag{37}$$

$$- \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda\left[\overline{\mathbb{H}}^\varepsilon(a-tf_s(x)) - \overline{\mathbb{H}}^\varepsilon(a)\right]\theta(x,a)da\,d\mathcal{H}^n(x) \tag{38}$$

$$+ \int_{\mathcal{A}} \int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}\big((\overline{\mathbb{H}}^\varepsilon)'(a-tf_s(x))\big)^2 \frac{(1-s)^2t^2}{r^2}|\nabla h(x,a)|^2_{(x,a)}\theta(x,a)da\,d\mathcal{H}^n(x) \tag{39}$$

We now analyse the three terms above separately.

First, consider (38), by the Fundamental Theorem of Calculus and Fubini's Theorem we see that

$$(38) = -\int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda\left[\overline{\mathbb{H}}^\varepsilon(a-tf_s(x)) - \overline{\mathbb{H}}^\varepsilon(a)\right]\theta(x,a)da\,d\mathcal{H}^n(x)$$

$$= \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr$$

Second, consider (37), by the Fundamental Theorem of Calculus, Fubini's Theorem and integrating by parts then, by setting $\theta(x, \sigma^{\pm}(x)) = \lim_{a\to\sigma^{\pm}(x)} \theta(x,a)$ and recalling (16), we obtain

$$\begin{aligned}
(37) &= \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - tf_s(x))\right) - e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a)\right)\right]\theta(x,a)da\, d\mathcal{H}^n(x) \\
&= -\int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} e'_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x))\right)\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&= \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\partial_a\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&\quad - \int_0^t \int_M f_s(x) e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x) - rf_s(x))\right)\theta(x,\sigma^+(x))da\, d\mathcal{H}^n(x)\, dr \\
&\quad + \int_0^t \int_M f_s(x) e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x) - rf_s(x))\right)\theta(x,\sigma^-(x))da\, d\mathcal{H}^n(x)\, dr \\
&= \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))(\lambda - H(x,a))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&\quad - \int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \\
&\quad - \int_0^t \int_M f_s(x) e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x) - rf_s(x))\right)\theta(x,\sigma^+(x))da\, d\mathcal{H}^n(x)\, dr \\
&\quad + \int_0^t \int_M f_s(x) e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x) - rf_s(x))\right)\theta(x,\sigma^-(x))da\, d\mathcal{H}^n(x)\, dr.
\end{aligned}$$

Note that in the last equality above we added and subtracted the term

$$\int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr$$

in order to introduce the quantity $\lambda - H(x,a)$ to the calculation; thus we consider the term

$$\int_0^t \int_M f_s(x) \int_{\sigma^-(x)}^{\sigma^+(x)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))(\lambda - H(x,a))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr, \tag{40}$$

which we now control by the assumption of positive Ricci curvature.

By the continuity of the $\sigma^{\pm}$ on M, Subsection 3.2 and (29), we may fix a compact $L \subset (M \setminus B_4)$, $l > 0$ sufficiently small so that $|\sigma^{\pm}(x)| > l$ for $x \in L$,

$$\mathcal{H}^n(\{(x,l) \in V_M \mid x \in L\}) > \frac{\mathcal{H}^n(M)}{2}, \tag{41}$$

and so that for some $t_0 \leq \min\{t_1, 2d(N)\}$ we ensure $\theta(x,a) \geq \theta(x,l)$ for all $a \in [-t_0, t_0]$. Note that as $L \cap B_4 = \emptyset$ we have that $f(x) = 1$ for all $x \in L$.

By (18) (relying on the positive Ricci curvature assumption) and the above paragraph we thus have, for $a \in [-t_0, t_0]$ and $x \in L$ that

$$\begin{cases} (\lambda - H(x,a))\theta(x,a) \leq -m\, a\, \theta(x,l) \text{ if } a \in [0, t_0] \\ (\lambda - H(x,a))\theta(x,a) \geq -m\, a\, \theta(x,l) \text{ if } a \in [-t_0, 0]. \end{cases}$$

With the above inequalities and (41) we compute for $t \in [-t_0, t_0]$ it holds that, using (25) and the fact that $e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a-r)) \neq 0$ only for $a \in (r - 2\varepsilon\Lambda_\varepsilon, r + 2\varepsilon\Lambda_\varepsilon)$, we have

$$\begin{aligned}(40) &\leq -\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon)\left[\frac{m\sigma}{2}\mathcal{H}^n(M)t^2 - 2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)\right] \\ &\leq -(1-\beta\varepsilon^2)\left[\frac{m\sigma}{2}\mathcal{H}^n(M)t^2 - 2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)\right].\end{aligned}$$

Third, we focus on (39). Using (20), (25), (31) (32) and noting that $s \in [0,1]$, we see that

$$(39) \leq 2\sigma t^2\left(\frac{\mathcal{H}^n(\mathcal{A})C_h^2 e^{\frac{\lambda^2}{2m}}}{r^2}\right)\mathcal{E}_\varepsilon(\overline{\mathbb{H}}^\varepsilon) \leq \frac{m\sigma}{4}\mathcal{H}^n(M)t^2(1-\beta\varepsilon^2).$$

We now simplify the terms as computed above and see that for $t \in [-t_0, t_0]$ we have

$$2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)) \leq -(1-\beta\varepsilon^2)\left[\frac{m\sigma}{4}\mathcal{H}^n(M)t^2 - 2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)\right] \tag{42}$$

$$+\int_0^t\int_M f_s(x)\int_{\sigma^-(x)}^{\sigma^+(x)}\sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \tag{43}$$

$$-\int_0^t\int_M f_s(x)\int_{\sigma^-(x)}^{\sigma^+(x)}\lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr \tag{44}$$

$$-\int_0^t\int_M f_s(x)e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x) - rf_s(x))\right)\theta(x,\sigma^+(x))da\, d\mathcal{H}^n(x)\, dr \tag{45}$$

$$+\int_0^t\int_M f_s(x)e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x) - rf_s(x))\right)\theta(x,\sigma^-(x))da\, d\mathcal{H}^n(x)\, dr. \tag{46}$$

The objective now is to show that the sum of the terms (43) to (46) are errors uniformly small in ε.

First we treat (43) and (44) together. We define

$$m_\varepsilon(rf_s(x), x) = \max_{a\in[rf_s(x)-2\varepsilon\Lambda_\varepsilon, rf_s(x)+2\varepsilon\Lambda_\varepsilon]}\theta(x,a) - \min_{a\in[rf_s(x)-2\varepsilon\Lambda_\varepsilon, rf_s(x)+2\varepsilon\Lambda_\varepsilon]}\theta(x,a)$$

so that by (20), $0 \leq m_\varepsilon(rf_s(x), x) \leq \max_{a\in\mathbb{R}}\theta(x,a) \leq e^{\frac{\lambda^2}{2m}}$ and $m_\varepsilon(rf_s(x), x) \to 0$ as $\varepsilon \to 0$. Hence, by the Dominated Convergence Theorem we see that

$$M_\varepsilon(t) = 2\sigma\lambda\int_M\int_0^t m_\varepsilon(rf_s(x), x)d\mathcal{H}^n(x)\, dr \to 0 \text{ as } \varepsilon \to 0.$$

Noting that for $0 < \tilde{\varepsilon} < \varepsilon$ we have that $0 \leq m_{\tilde{\varepsilon}}(rf_s(x), x) \leq m_\varepsilon(rf_s(x), x)$ we may apply Dini's Theorem (as the $M_\varepsilon(t)$ are increasing/decreasing in ε for positive/negative values of $t \in \mathbb{R}$ respectively) to the functions $M_\varepsilon(t)$ to conclude that

$$M_\varepsilon \to 0 \text{ uniformly on compact subsets of } \mathbb{R} \tag{47}$$

Using the above we compute that for (43) and (44), by $|f_s|, |\overline{\mathbb{H}}^\varepsilon| \leq 1$, (25) and Fubini's Theorem, we have, for $t \in [-t_0, t_0]$, that

$$\begin{aligned}&\left|\int_0^t\int_M f_s(x)\int_{\sigma^-(x)}^{\sigma^+(x)}\sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a) - \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a)da\, d\mathcal{H}^n(x)\, dr\right| \\ &\leq \int_0^t\int_M\left|\int_{\sigma^-(x)}^{\sigma^+(x)}\sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a - rf_s(x))\theta(x,a) - \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a - rf_s(x)))\theta(x,a)da\right| d\mathcal{H}^n(x)\, dr\end{aligned}$$

$$\leq 2\sigma\lambda \int_0^t \int_M m_\varepsilon(rf_s(x),x)d\mathcal{H}^n(x)\,dr + 2\sigma\lambda\mathcal{H}^n(M)e^{\frac{\lambda^2}{2m}}t_0\beta\varepsilon^2$$
$$\leq \max_{t\in[-2d(N),2d(N)]} M_\varepsilon(t) + 2\sigma\lambda\mathcal{H}^n(M)e^{\frac{\lambda^2}{2m}}t_0\beta\varepsilon^2.$$

So

$$|(43)+(44)| \leq \max_{t\in[-2d(N),2d(N)]} M_\varepsilon(t) + 2\sigma\lambda\mathcal{H}^n(M)e^{\frac{\lambda^2}{2m}}t_0\beta\varepsilon^2. \tag{48}$$

Finally we work on (45) and (46). If $t \geq 0$ then as f_s, θ and e_ε are non-negative we have (45) ≤ 0; similarly, if $t \leq 0$ then we have (46) ≤ 0. As $|f_s| \leq 1$ by definition, $\theta \leq e^{\frac{\lambda^2}{2m}}$ by (20) and $e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(\sigma^\pm(x) - rf_s(x)) \neq 0$ only when $\sigma^\pm(x) \in (rf_s(x) - 2\varepsilon\Lambda_\varepsilon, rf_s(x) + 2\varepsilon\Lambda_\varepsilon)$ by definition of $\overline{\mathbb{H}}^\varepsilon$, we conclude that

$$(45) \leq (2\sigma + \beta\varepsilon^2)t_0 e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^+(x) \leq 2\varepsilon\Lambda_\varepsilon\} \tag{49}$$

and

$$(46) \leq (2\sigma + \beta\varepsilon^2)t_0 e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^-(x) \geq -2\varepsilon\Lambda_\varepsilon\}. \tag{50}$$

As $\mathcal{H}^n(\{x \in M \mid \sigma^\pm(x) = 0\}) = 0$ by the Dominated Convergence Theorem we conclude that

$$\mathcal{H}^n(\{x \in M \mid \sigma^+(x) \leq 2\varepsilon\Lambda_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0 \tag{51}$$

and

$$\mathcal{H}^n(\{x \in M \mid \sigma^-(x) \geq -2\varepsilon\Lambda_\varepsilon\} \text{ as } \varepsilon \to 0. \tag{52}$$

Hence we conclude that for $t \in [-t_0, t_0]$

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{m\sigma}{4}\mathcal{H}^n(M)t^2 + E(\varepsilon)$$

where by (42), (48), (49) and (50) we have noting that as $t_0 \leq 2d(N)$ we have

$$\begin{aligned}
2\sigma E(\varepsilon) = &\frac{m\sigma}{4}\mathcal{H}^n(M)(2d(N))^2\beta\varepsilon^2 + (1-\beta\varepsilon^2)\left[2\varepsilon\Lambda_\varepsilon m\sigma\mathcal{H}^n(M)\right] \\
&+ \max_{t\in[-2d(N),2d(N)]} M_\varepsilon(t) + 2\sigma\lambda\mathcal{H}^n(M)e^{\frac{\lambda^2}{2m}}(2d(N))\beta\varepsilon^2 \\
&+ (2\sigma+\beta\varepsilon^2)(2d(N))e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^+(x) \leq 2\varepsilon\Lambda_\varepsilon\} \\
&+ (2\sigma+\beta\varepsilon^2)(2d(N))e^{\frac{\lambda^2}{2m}}\mathcal{H}^n(\{x \in M \mid \sigma^-(x) \geq -2\varepsilon\Lambda_\varepsilon\}
\end{aligned}$$

and so $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ by virtue of (47), (51) and (52). Setting once and for all

$$\eta = \frac{m\sigma}{8}\mathcal{H}^n(M)t_0^2$$

we conclude, taking $\varepsilon > 0$ sufficiently small so that $E(\varepsilon) < \eta$, the energy bounds as desired. □

We now conclude this subsection by showing that the paths $t \in [-t_1, t_1] \to v_\varepsilon^{t,s}$ and $s \in [-1,1] \to v_\varepsilon^{t,s}$ are continuous, for fixed $s \in [0,1]$ and $t \in [-t_1, t_1]$ respectively, in $W^{1,2}(N)$. Recall that the functions F, Π_{V_M} and f_s are continuous. For each $t \in [-t_1, t_1]$ consider that for $s, \tilde{s} \in [0,1]$ we have, by the Dominated Convergence Theorem (using $|\overline{\mathbb{H}}^\varepsilon| \leq 1$) and (35), that

$$\begin{aligned}
||v_\varepsilon^{t,s} - v_\varepsilon^{t,\tilde{s}}||^2_{L^2(N)} &= \int_{V_M} |\hat{v}_\varepsilon^{t,s} - \hat{v}_\varepsilon^{t,\tilde{s}}|^2 d\mathcal{H}^{n+1}_{F^*g} \\
&= \int_{V_M} |\overline{\mathbb{H}}^\varepsilon(a - tf_s(F(\Pi_{V_M}(x,a)))) - \overline{\mathbb{H}}^\varepsilon(a - tf_{\tilde{s}}(F(\Pi_{V_M}(x,a))))|^2 d\mathcal{H}^{n+1}_{F^*g}(x,a)
\end{aligned}$$

$\to 0$ as $\tilde{s} \to s$.

Also, we see that by the Dominated Convergence Theorem, (35) and (36) we have

$$\begin{aligned}||\nabla v_\varepsilon^{t,s} - \nabla v_\varepsilon^{t,\tilde{s}}||^2_{L^2(N)} &= \int_{V_M} |\nabla \hat{v}_\varepsilon^{t,s} - \nabla \hat{v}_\varepsilon^{t,\tilde{s}}|^2 d\mathcal{H}^{n+1}_{F^*g} \\ &\leq \int_{V_M} \Big((\overline{\mathbb{H}}^\varepsilon)'\big(a - tf_s(F(\Pi_{V_M}(x,a)))\big) \\ &\qquad\qquad - (\overline{\mathbb{H}}^\varepsilon)'\big(a - tf_{\tilde{s}}(F(\Pi_{V_M}(x,a)))\big)\Big)^2 d\mathcal{H}^{n+1}_{F^*g}(x,a) \\ &\quad + \int_{V_M} t^2 \big|\nabla\big(f_{\tilde{s}}(F(\Pi_{V_M}(x,a)))\big)(\overline{\mathbb{H}}^\varepsilon)'\big(a - tf_{\tilde{s}}(F(\Pi_{V_M}(x,a)))\big) \\ &\qquad\qquad - \nabla\big(f_s(F(\Pi_{V_M}(x,a)))\big)(\overline{\mathbb{H}}^\varepsilon)'\big(a - tf_s(F(\Pi_{V_M}(x,a)))\big)\big|^2 d\mathcal{H}^{n+1}_{F^*g}(x,a) \\ &\to 0 \text{ as } \tilde{s} \to s,\end{aligned}$$

(where we also use (31) to ensure we can apply the Dominated Convergence Theorem). Hence, for fixed $t \in [-t_1, t_1]$, the path

$$s \in [0,1] \to v_\varepsilon^{t,s} \text{ is continuous in } W^{1,2}(N). \tag{53}$$

Analogous arguments to those above show that, for fixed $s \in [0,1]$, the path

$$t \in [-t_1, t_1] \to v_\varepsilon^{t,s} \text{ is continuous in } W^{1,2}(N). \tag{54}$$

5.2 Sliding the one-dimensional profile

We now define, for $t \in \mathbb{R}$, the functions $v_\varepsilon^t \in W^{1,2}(N)$ by setting

$$v_\varepsilon^t(x) = \overline{\mathbb{H}}^\varepsilon(d^{\pm}_{\overline{M}}(x) - t),$$

and note that, by (33), for $t \in [-t_1, t_1]$ we have $v_\varepsilon^t = v_\varepsilon^{t,1}$. By the continuity of translations on L^p for $1 \leq p < \infty$ we have that $v_\varepsilon^t \in W^{1,2}(N)$ for all $t \in \mathbb{R}$, the path $t \to v_\varepsilon^t$ is continuous in $W^{1,2}(N)$ and that $v_\varepsilon^0 = v_\varepsilon$. By choosing $\varepsilon > 0$ sufficiently small so that $2\varepsilon\Lambda_\varepsilon < d(N)$ we ensure that both $v_\varepsilon^t = -1$ everywhere on N for all $t \geq 2d(N)$ and $v_\varepsilon^t = +1$ everywhere on N for all $t \leq -2d(N)$.

We now show that, using the assumption of positive Ricci curvature, the path provided by the functions v_ε^t, along with the energy reducing paths from -1 and $+1$ provided by negative gradient flow of the energy to a_ε and b_ε respectively, provides a "recovery path" for the value $\mathcal{F}_\lambda(E)$; this path connects a_ε to b_ε, passing through v_ε, with the maximum value of the energy along this path approximately $\mathcal{F}_\lambda(E)$.

Lemma 5. *For all $t \in [-2d(N), 2d(N)]$ we have that*

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon)$$

where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ is as in Lemma 4. Furthermore, we have that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm\tilde{t}}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t}) + E(\varepsilon),$$

whenever $\tilde{t} \geq t \geq 0$; thus, in particular for $t > t_0$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon).$$

Proof. We compute in an identical manner to the proof of Lemma 4, writing

$$2\sigma\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) = \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-t)\right) - \sigma\lambda\overline{\mathbb{H}}^\varepsilon(a-t)\right]\theta(x,a)da\,d\mathcal{H}^n(x).$$

Assuming that either $\tilde{t} \geq t > 0$ or $\tilde{t} \leq t < 0$, as before we compute

$$\begin{aligned}2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\tilde{t}}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t)) = &\int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \left[e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-\tilde{t})\right) - e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(a-t)\right)\right]\theta(x,a)da\,d\mathcal{H}^n(x)\\ &- \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda\left[\overline{\mathbb{H}}^\varepsilon(a-\tilde{t}) - \overline{\mathbb{H}}^\varepsilon(a-t)\right]\theta(x,a)da\,d\mathcal{H}^n(x)\end{aligned}$$

which yields, noting that $\lambda - H(x,a) \leq 0$ for all $a \in \mathbb{R}$ by the assumption of positive Ricci curvature, the expression

$$\begin{aligned}2\sigma(\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\tilde{t}}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t)) \leq &\int_t^{\tilde{t}} \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}^\varepsilon)'(a-r)\theta(x,a)da\,d\mathcal{H}^n(x)\,dr\\ &- \int_t^{\tilde{t}} \int_M \int_{\sigma^-(x)}^{\sigma^+(x)} \lambda e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a-r))\theta(x,a)da\,d\mathcal{H}^n(x)\,dr\\ &- \int_t^{\tilde{t}} \int_M e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^+(x)-r)\right)\theta(x,\sigma^+(x))da\,d\mathcal{H}^n(x)\,dr\\ &+ \int_t^{\tilde{t}} \int_M e_\varepsilon\left(\overline{\mathbb{H}}^\varepsilon(\sigma^-(x)-r)\right)\theta(x,\sigma^-(x))da\,d\mathcal{H}^n(x)\,dr.\end{aligned}$$

Near identical computations to those in the proof of Lemma 4 (for the error terms (43), (44) giving the bound (48), as well the bounds on (49), (50) giving (51), (52) respectively) for the above four terms give that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\tilde{t}}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) + E(\varepsilon),$$

where the expression for $E(\varepsilon)$ is identical to that of Lemma 4 (where here are exploiting the fact that $t \in [-2d(N), 2d(N)]$).

Recalling $v_\varepsilon^0 = v_\varepsilon$, $v_\varepsilon^{\pm t_0} = v_\varepsilon^{\pm t_0,1}$ and the bounds from Lemma 4 then completes the proof. □

5.3 Paths to local energy minimisers

Recall, $v_\varepsilon = \overline{\mathbb{H}}^\varepsilon \circ d_{\overline{M}}^{\pm}$ and that, by Lemma 4 and (33), for a fixed $\eta > 0$ and some $r_0 > 0$, as chosen in Subsection 5.1.1, there exists a $t_0 > 0$ and functions $v_\varepsilon^{t,0} \in W^{1,2}(N)$ for $t \in [-t_0, t_0]$, with the following properties:

- $v_\varepsilon^{t,0} = v_\varepsilon$ in $B_{r_0}(p)$.
- $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + E(\varepsilon)$, where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$.
- $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\pm t_0,0}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta$.

We now produce the local functions described in Step 3 of Subsection 1.3. These functions are constant outside of $B_{r_0}(p)$, providing a continuous path in $W^{1,2}(N)$ from v_ε to g_ε such that the maximum energy along this path is bounded from above by $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{\eta}{2}$.

Proposition 3. *For $\eta > 0$ as above there exists, for some $R \in (0, r_0)$ and $\varepsilon > 0$ sufficiently small, functions $g_\varepsilon^{t,s} \in W^{1,2}(N)$, for each $t \in [-t_0, t_0]$ and $s \in [-2, 2]$, such that the following properties hold:*

- *For each $t \in [-t_0, t_0]$ we have $g_\varepsilon^{t,-2} = v_\varepsilon^{t,0}$ on N and $g_\varepsilon^{t,s} = v_\varepsilon^{t,0}$ on $N \setminus B_R(p)$ for all $s \in [-2, 2]$.*
- *For each $t \in [-t_0, t_0]$ we have $g_\varepsilon^{t,2} = g_\varepsilon$ in $B_R(p)$ where $g_\varepsilon \in \mathcal{A}_{\varepsilon, \frac{R}{2}}(p)$ arises from Lemma 2 for $\rho = \frac{R}{2}$ and $q = p$.*
- *For each $s \in [-2, 2]$, $t \in [-t_0, t_0] \to g_\varepsilon^{t,s}$ is a continuous path in $W^{1,2}(N)$.*
- *For each $t \in [-t_0, t_0]$, $s \in [-2, 2] \to g_\varepsilon^{t,s}$ is a continuous path in $W^{1,2}(N)$.*
- *For each $t \in [-t_0, t_0]$ and $s \in [-2, 2]$ we have*

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{\eta}{2}.$$

Furthermore, if the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$ for some $\tau > 0$, we have for each $t \in [-t_0, t_0]$ that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) - \tau.$$

Remark 14. *The paths exhibited in Proposition 3 are "local" in the sense that they remain constant outside of $B_R(p)$, and additionally do not require the assumption of positive Ricci curvature of N. Specifically, in any ambient manifold (without any curvature assumption), given an $\eta > 0$ (not necessarily as fixed by Lemma 4) and $f \in W^{1,2}(N)$ such that $|f| \leq 1$ and f equal to v_ε in a ball bi-Lipschitz diffeomorphic to the Euclidean ball of the same radius, a path as in Proposition 3 may be constructed (replacing $v_\varepsilon^{t,0}$ by f in the conclusions) for a sufficiently small $R > 0$.*

Proof. Given some $\eta > 0$ we choose $R > 0$ sufficiently small to ensure that

$$2^{n+2} R^n \omega_n + C_p R^n + 2\lambda \mathrm{Vol}(B_{\frac{R}{2}}(p)) < \frac{\eta}{4},$$

where ω_n is the volume of the n-dimensional unit ball. We then potentially re-choose a smaller $0 < R \leq \min\{r_0, r_p\}$, such that the closed ball $\overline{B}_R(p)$ is 2-bi-Lipschitz diffeomorphic, via some smooth chart ψ on N, to the Euclidean ball, $\overline{B}_R^{\mathbb{R}^{n+1}}(0)$ of radius R, with $\psi(p) = 0$. Consider a sweep-out of $\overline{B}_R^{\mathbb{R}^{n+1}}$ by planes Π_l defined for $l \in [-1, 1]$ by

$$\Pi_l = \left\{ y \in \overline{B}_R^{\mathbb{R}^{n+1}} \,\middle|\, y = (y_1, \dots, y_n, lR) \right\},$$

which for each $l \in [-1, 1]$ is the intersection of $\overline{B}_R^{\mathbb{R}^{n+1}}$ with the plane $\{y_{n+1} = lR\}$. We consider the images, $P_l = \psi(\Pi_l)$ in N. The P_l then sweep-out the closure of $B_R(p)$, $\overline{B}_R(p)$ in the sense that $\overline{B}_R(p) = \cup_{l \in [-1,1]} P_l$ and that for $l \neq s$ we have $P_l \cap P_s = \emptyset$. Each Π_l divides $\overline{B}_R^{\mathbb{R}^{n+1}}$ into two connected regions $\{y \in \overline{B}_R^{\mathbb{R}^{n+1}} \,|\, y_{n+1} > lR\}$ and $\{y \in \overline{B}_R^{\mathbb{R}^{n+1}} \,|\, y_{n+1} < lR\}$; we denote the images of these sets under the diffeomorphism ψ as $E_l = \psi(\{y \in \overline{B}_R^{\mathbb{R}^{n+1}} \,|\, y_{n+1} > lR\})$ and $F_l = \psi(\{y \in \overline{B}_R^{\mathbb{R}^{n+1}} \,|\, y_{n+1} < lR\})$ in $\overline{B}_R(p)$ respectively. Note then that $\overline{B}_R(p) = E_l \cup F_l \cup P_l$, where the unions are all mutually disjoint.

We now define for $l \in [-1, 1]$ functions $p_\varepsilon^l \in W^{1,2}(B_R(p))$ given by

$$p_\varepsilon^l = \overline{\mathbb{H}}^\varepsilon(d_{P_l}^\pm),$$

where here, for d_{P_l} the usual distance function on N from the set P_l, we define the Lipschitz signed distance to P_l by

$$d_{P_l}^\pm = \begin{cases} +d_{P_l}, & \text{if } x \in E_l \\ 0, & \text{if } x \in P_l \\ -d_{P_l}, & \text{if } x \in F_l. \end{cases}$$

As the sets P_l vary continuously in the Haussdorff distance as $l \in [-1,1]$ varies, the functions p^l_ε vary continuously in $W^{1,2}(B_R(p))$ with respect to $l \in [-1,1]$.

Apply Lemma 2 for the choice of $B_{\frac{R}{2}}(p)$ for each $0 < \varepsilon < \varepsilon_2$ as in the minimisation section. For $l \in [-1,1]$ we define on $B_R(p)$ the functions

$$\check{h}^l_\varepsilon(x) = \max\Big\{ \min\{v_\varepsilon(x), p^l_\varepsilon(x)\}, \min\{g_\varepsilon(x), v_\varepsilon(x)\}\Big\},$$

and

$$\hat{h}^l_\varepsilon(x) = \max\Big\{ \min\{g_\varepsilon(x), p^l_\varepsilon(x)\}, \min\{g_\varepsilon(x), v_\varepsilon(x)\}\Big\}.$$

Note then that for $2\varepsilon\Lambda_\varepsilon < \frac{R}{2}$ we have $\check{h}^{-1}_\varepsilon = v_\varepsilon$, $\check{h}^1_\varepsilon = \hat{h}^1_\varepsilon = \min\{g_\varepsilon, v_\varepsilon\}$, $\hat{h}^{-1}_\varepsilon = g_\varepsilon$, and both $|\check{h}^l_\varepsilon| \leq 1$, $|\hat{h}^l_\varepsilon| \leq 1$.

For functions $f, g \in W^{1,2}(B_R(p))$ we may write $\max\{f,g\} = \max\{f-g,0\}+g = \frac{1}{2}(|f-g|+(f-g))+g$ and $\min\{f,g\} = \min\{f-g,0\}+g = \frac{1}{2}(|f-g|-(f-g))+g$; thus $\max\{f,g\}, \min\{f,g\} \in W^{1,2}(N)$. We also note that if $t \to f_t$ is a continuous path in $W^{1,2}(B_R(p))$, then the paths $t \to \max\{f_t, 0\}$ and $t \to \min\{f_t, 0\}$ are continuous in $W^{1,2}(B_R(p))$. To see this we write $\max\{f_t,0\} - \max\{f_s,0\} = \frac{1}{2}((|f_t| - |f_s|) + (f_t - f_s))$ which by reverse triangle inequality may be controlled by the $W^{1,2}(B_R(p))$ norm of $f_t - f_s$, showing continuity of the path in $W^{1,2}(B_R(p))$. The proof for the continuity of the path $t \to \min\{f_t, 0\}$ is identical. With the above in mind, we have that $\check{h}^l_\varepsilon, \hat{h}^l_\varepsilon \in W^{1,2}(B_R(p))$ for each $l \in [-1,1]$ and the paths $l \in [-1,1] \to \check{h}^l_\varepsilon$ and $l \in [-1,1] \to \hat{h}^l_\varepsilon$ are continuous in $W^{1,2}(B_R(p))$.

We have that $v_\varepsilon = g_\varepsilon$ on $B_R(p) \setminus \overline{B_{\frac{R}{2}}(p)}$ (as $v_\varepsilon = g_\varepsilon$ on $N \setminus B_{\frac{R}{2}}(p)$) and hence $\hat{h}^l_\varepsilon = \check{h}^l_\varepsilon = v_\varepsilon$ on $B_R(p) \setminus \overline{B_{\frac{R}{2}}(p)}$ also. Thus we may extend to well defined functions $\check{g}^{t,l}_\varepsilon, \hat{g}^{t,l}_\varepsilon \in W^{1,2}(N)$ for $t \in [-t_0, t_0]$ and $l \in [-1,1]$ by defining

$$\check{g}^{t,l}_\varepsilon(x) = \begin{cases} v^{t,0}_\varepsilon(x), & \text{if } x \in N \setminus B_R(p) \\ \check{h}^l_\varepsilon(x), & \text{if } x \in B_R(p), \end{cases}$$

and

$$\hat{g}^{t,l}_\varepsilon(x) = \begin{cases} v^{t,0}_\varepsilon(x), & \text{if } x \in N \setminus B_R(p) \\ \hat{h}^l_\varepsilon(x), & \text{if } x \in B_R(p), \end{cases}$$

This is well defined as we have that $v^{t,0}_\varepsilon = v_\varepsilon$ on the set $B_R(p)$, hence in defining the functions above we only edit the functions $v^{t,0}_\varepsilon$ in $B_R(p)$ and keep them in $W^{1,2}(N)$. By the above arguments, as the paths $l \in [-1,1] \to \check{h}^l_\varepsilon$ and $l \in [-1,1] \to \hat{h}^l_\varepsilon$ are continuous in $W^{1,2}(B_R(p))$ and so we have that the paths $t \in [-t_0, t_0] \to \check{g}^{t,l}_\varepsilon$, $l \in [-1,1] \to \check{g}^{t,l}_\varepsilon$, $t \in [-t_0,t_0] \to \hat{g}^{t,l}_\varepsilon$ and $l \in [-1,1] \to \hat{g}^{t,l}_\varepsilon$ are continuous in $W^{1,2}(N)$.

Note that then as $\check{h}^1_\varepsilon = \hat{h}^1_\varepsilon = \min\{g_\varepsilon, v_\varepsilon\}$ we have for each $t \in [-t_0, t_0]$ that $\check{g}^{t,1}_\varepsilon = \hat{g}^{t,-1}_\varepsilon = \min\{g_\varepsilon, v_\varepsilon\}$ in $B_R(p)$. We thus define, for $s \in [-2,2]$, the functions

$$g^{t,s}_\varepsilon = \begin{cases} \check{g}^{t,s+1}, & \text{if } s \in [-2,0] \\ \hat{g}^{t,s-1}_\varepsilon, & \text{if } s \in [0,2], \end{cases}$$

so that $g^{t,0}_\varepsilon = \check{g}^{t,1}_\varepsilon = \hat{g}^{t,-1}_\varepsilon = \min\{g_\varepsilon, v_\varepsilon\}$, $g^{t,-2}_\varepsilon = \check{g}^{0,-1}_\varepsilon = v^{t,0}_\varepsilon$ and $g^{0,2}_\varepsilon = \hat{g}^{0,1}_\varepsilon = g_\varepsilon$ in $B_R(p)$. By the continuity of the paths mentioned above we have that the paths, $s \in [-2,2] \to g^{t,s}_\varepsilon$ and $t \in [-t_0, t_0] \to g^{t,s}_\varepsilon$, are continuous in $W^{1,2}(N)$.

We now show that we may bound the energy of all the functions $g^{t,s}_\varepsilon$ above by $\mathcal{F}_{\varepsilon,\lambda}(v^{t,0}_\varepsilon)$ plus errors depending only on the geometry of the ball $B_R(p)$. As $\{g_\varepsilon \neq v_\varepsilon\} \subset B_{\frac{R}{2}}(p)$ we note that $g^{t,s}_\varepsilon = v^{t,0}_\varepsilon$

on $N \setminus B_{\frac{R}{2}}(p)$. We then compute that, as $\check{h}_\varepsilon^{s+1}$ is always equal to one of $g_\varepsilon, v_\varepsilon$ or p_ε^{s+1} in $B_{\frac{R}{2}}(p)$, for $s \in [-2, 0]$ we have

$$
\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) &= \frac{1}{2\sigma}\int_N e_\varepsilon(g_\varepsilon^{t,s}) - \frac{\lambda}{2}\int_N g_\varepsilon^{t,s} = \frac{1}{2\sigma}\int_{N\setminus B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(\check{h}_\varepsilon^{s+1}) \\
&\quad - \frac{\lambda}{2}\int_{N\setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\
&= \frac{1}{2\sigma}\int_{N\setminus B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)\cap\{\check{h}_\varepsilon^{s+1}=v_\varepsilon\}} e_\varepsilon(v_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)\cap\{\check{h}_\varepsilon^{s+1}=g_\varepsilon\}} e_\varepsilon(g_\varepsilon) \\
&\quad + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)\cap\{\check{h}_\varepsilon^{s+1}=p_\varepsilon^{s+1}\}} e_\varepsilon(p_\varepsilon^{s+1}) - \frac{\lambda}{2}\int_{N\setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\
&\leq \frac{1}{2\sigma}\int_{N\setminus B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s+1}) \\
&\quad - \frac{\lambda}{2}\int_{N\setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\
&= \mathcal{E}_\varepsilon(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s+1}) - \frac{\lambda}{2}\int_{N\setminus B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0} - \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} \check{h}_\varepsilon^{s+1} \\
&= \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s+1}) + \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} (v_\varepsilon^{t,0} - \check{h}_\varepsilon^{s+1}),
\end{aligned}
$$

where in the final line we have added and subtracted the term $\frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} v_\varepsilon^{t,0}$. Similarly for $s \in [0, 2]$ we have

$$
\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^{s-1}) + \frac{\lambda}{2}\int_{B_{\frac{R}{2}}(p)} (v_\varepsilon^{t,0} - \hat{h}_\varepsilon^{s-1}).
$$

Thus as for all $l \in [-1, 1]$ we have $|\check{h}_\varepsilon^l| \leq 1$, $|\hat{h}_\varepsilon^l| \leq 1$ and $|v_\varepsilon^{t,0}| \leq 1$ we see that for each $t \in [-t_0, t_0]$ and $s \in [-2, 2]$

$$
\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) + \sup_{l\in(-1,1)}\left(\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p_\varepsilon^l)\right) + \lambda \mathrm{Vol}(B_{\frac{R}{2}}(p)). \tag{55}
$$

Similarly, we note that as $v_\varepsilon = g_\varepsilon$ on $N \setminus B_{\frac{R}{2}}(p)$, $|g_\varepsilon|, |v_\varepsilon| \leq 1$ on N and $\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)$ by construction of g_ε we have that

$$
\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) \leq \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon) + \lambda \mathrm{Vol}(B_{\frac{R}{2}}(p)).
$$

and so by similar calculations to Subsection 3.3 and (25) we see that

$$
\begin{aligned}
\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(v_\varepsilon) &\leq (1+\beta\varepsilon^2)\operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon, 2\varepsilon\Lambda_\varepsilon]} \mathcal{H}^n(\{d_{\overline{M}}^{\pm} = a\} \cap B_R(p)) \\
&\to \mathcal{H}^n(M \cap B_R(p)) \text{ as } \varepsilon \to 0.
\end{aligned}
$$

By (28) we conclude that, for sufficiently small $\varepsilon > 0$, we have

$$
\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(g_\varepsilon) \leq C_p R^n + \lambda \mathrm{Vol}(B_{\frac{R}{2}}(p)) + \frac{\eta}{8}. \tag{56}
$$

We now compute upper $\mathcal{E}_\varepsilon$ bounds on the term in (55) involving the functions p^l_ε in terms of the geometry of N. Applying the co-area formula, slicing with $d^{\pm}_{P_l}$, and using (25) we see that for $l \in (-1,1)$ we have

$$\begin{aligned}\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p^l_\varepsilon) &= \frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(d^{\pm}_{P_l})) = \frac{1}{2\sigma}\int_{\mathbb{R}}\int_{\{d^{\pm}_{P_l}=a\}\cap B_{\frac{R}{2}}(p)} e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a))d\mathcal{H}^n\,da\\ &= \frac{1}{2\sigma}\int_{\mathbb{R}} \mathcal{H}^n(\{d^{\pm}_{P_l}=a\}\cap B_{\frac{R}{2}}(p))e_\varepsilon(\overline{\mathbb{H}}^\varepsilon(a))\,da\\ &\leq (1+\beta\varepsilon^2)\operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon,2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\{d^{\pm}_{P_l}=a\}\cap B_{\frac{R}{2}}(p)).\end{aligned}$$

We now bound $\operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon,2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\{d^{\pm}_{P_l}=a\}\cap B_{\frac{R}{2}}(p))$ from above independently of ε. Recall that $P_l = \psi(\Pi_l)$ where ψ is a smooth 2-bi-Lipschitz map from $\overline{B}^{\mathbb{R}^{n+1}}_R(0)$ to $\overline{B}_R(p)$ and $\Pi_l = \left\{y \in \overline{B}^{\mathbb{R}^{n+1}}_R \,\middle|\, y=(y_1,\ldots,y_n,lR)\right\}$. For $l\in(-1,1)$ we have that Π_l is a smooth embedded n-dimensional sub-manifold of $\overline{B}^{\mathbb{R}^{n+1}}_R$ and hence its image, P_l, in N is a smooth embedded n-dimensional sub-manifold of $\overline{B}_R(p)\subset N$.

We define the tubular hypersurface at distance s from P_l to be the set

$$P_l(a) = \{x\in N \mid \text{ there exists a geodesic of length } a \text{ meeting } P_l \text{ orthogonally}\}.$$

Here we choose $\varepsilon>0$ possibly smaller to ensure that $2\varepsilon\Lambda_\varepsilon < \frac{R}{2}$, so that $\{|d^{\pm}_{P_l}|=a\}\cap B_{\frac{R}{2}}(p)\subset P_l(a)$ for all $l\in(-1,1)$ such that $B_{\frac{R}{2}}(p)\cap P_l\neq\emptyset$. We then apply [Gra04, Lemma 3.12/8.2] to compute that, for each $l\in(-1,1)$ we have

$$\mathcal{H}^n(P_l(s)) \leq \int_{P_l}\int_{S^0}\theta_u(q,s)du\,d\mathcal{H}^n(q).$$

In the above we are adopting the notation that S^0 is the 0 dimensional unit sphere, and for $u\in S^0$ we define $\theta_u(q,a)$ to be the Jacobian of the exponential map $\exp_q$ at the point $\exp_q(au)$. Note then that for each $q\in P_l$ we have that $\theta_u(q,0)=1$ and $\theta_u(q,a)\to 1$ as $a\to 0$. For all $\varepsilon>0$ sufficiently small we ensure that $\theta_u(q,a)\leq 2$ for any $q\in\overline{B}_R(p)$; this may be seen by noticing that as the exponential map is smooth, its Jacobian varies continuously in each of its variables and hence its maximum, for a fixed $a\in\mathbb{R}$ on $\overline{B}_R(p)\times\mathbb{R}$, is achieved and converges to 1 as $a\to 0$. Thus, we have that

$$\mathcal{H}^n(P_l(a)) \leq 4\mathcal{H}^n(P_l).$$

Note that as ψ is 2-bi-Lipschitz we may apply [EG15, Theorem 2.8] to see that

$$\mathcal{H}^n(P_l)\leq 2^n\mathcal{H}^n(\Pi_l)\leq 2^nR^n\omega_n,$$

where ω_n is the volume of the n-dimensional unit ball. Hence, for all $l\in[-1,1]$ and $\varepsilon>0$ sufficiently small we have that

$$\mathcal{H}^n(P_l(s))\leq 2^{n+2}R^n\omega_n.$$

We thus conclude that as $\{|d^{\pm}_{P_l}|=a\}\cap B_{\frac{R}{2}}(p)\subset P_l(a)$ for all $l\in(-1,1)$ such that $B_{\frac{R}{2}}(p)\cap P_l\neq\emptyset$ we have

$$\operatorname{ess\,sup}_{a\in[-2\varepsilon\Lambda_\varepsilon,2\varepsilon\Lambda_\varepsilon]}\mathcal{H}^n(\{d^{\pm}_{P_l}=a\}\cap B_{\frac{R}{2}}(p))\leq 2^{n+2}R^n\omega_n,$$

and hence

$$\frac{1}{2\sigma}\int_{B_{\frac{R}{2}}(p)} e_\varepsilon(p^l_\varepsilon)\leq (1+\beta\varepsilon^2)2^{n+2}R^n\omega_n. \tag{57}$$

Choosing $\varepsilon > 0$ again possibly smaller we ensure that

$$2^{n+2}R^n\omega_n\beta\varepsilon^2 < \frac{\eta}{8}.$$

Recall that, given some $\eta > 0$, we chose $R > 0$ sufficiently small to ensure that

$$2^{n+2}R^n\omega_n + C_pR^n + 2\lambda\mathrm{Vol}(B_{\frac{R}{2}}(p)) < \frac{\eta}{4}.$$

Combining the above two bounds with (55), (56) and (57) we conclude that for each $t \in [-t_0, t_0]$ and $s \in [-2, 2]$ we have

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,s}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + \frac{\eta}{2},$$

for $\varepsilon > 0$ sufficiently small.

As $\{g_\varepsilon \neq v_\varepsilon\} \subset B_{\frac{R}{2}}(p)$ we have $g_\varepsilon^{t,s} = v_\varepsilon^{t,0}$ on $N \setminus B_{\frac{R}{2}}(p)$, so using the fact that in $B_R(p)$ we have $v_\varepsilon^{t,0} = v_\varepsilon$ and $g_\varepsilon^{t,2} = g_\varepsilon$ we thus note that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) + (\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)).$$

Using this, if the functions v_ε are such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$ for some $\tau > 0$, we therefore conclude that

$$\mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon^{t,2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{t,0}) - \tau,$$

as desired. □

6 Proof of Theorems 1, 2 & 3

We recall our setup. Let (N, g) be a smooth compact Riemannian manifold of dimension 3 or higher with positive Ricci curvature. We consider $M \subset N$ a closed embedded hypersurface of constant mean curvature $\lambda \in \mathbb{R}$, smooth away from a closed singular set of Hausdorff dimension at most $n - 7$, as produced by the one-parameter Allen–Cahn min-max in [BW20a], with constant prescribing function λ.

6.1 Proof of Theorem 3

Proof of Theorem 3. For each isolated singularity $p \in \mathrm{Sing}(M)$ we may choose, positive r_0, r and R, as in Subsection 5.1.1 and Proposition 3 in order to define the various paths constructed in Subsections 5.1.2, 5.2 and 5.3. Assuming that for $g_\varepsilon \in \mathcal{A}_{\varepsilon,\frac{R}{2}}(p)$, defined as in Lemma 2 (setting $\rho = \frac{R}{2}$ and $q = p$), there exists a $\tau > 0$ such that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$, for all $\varepsilon > 0$ sufficiently small, then using the constructions in Section 5 as mentioned above we define the following nine paths:

- First, a path from $+1$ to the constant b_ε,

  $$s \in [1, b_\varepsilon] \to s,$$

  through constant functions which, by the construction of b_ε through negative gradient flow of $\mathcal{F}_{\varepsilon,\lambda}$ in Subsection 1.2, has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(1)$.

- Second, a path from $+1$ to $v_\varepsilon^{-t_0} = v_\varepsilon^{-t_0,1}$,

  $$t \in [-2d(N), -t_0] \to v_\varepsilon^t,$$

  which by Lemma 5 has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(v^{-t_0,1}) \leq \mathcal{F}_\lambda(E) - \eta + E(\varepsilon)$. This path varies continuously by the reasoning in Subsection 5.2.

- Third, a path from $v_\varepsilon^{-t_0,1}$ to $v_\varepsilon^{-t_0,0} = g_\varepsilon^{-t_0,-2}$,

$$s \in [0,1] \to v_\varepsilon^{-t_0,s},$$

  which by Lemma 4 has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta$. This path varies continuously by (53).

- Fourth, a path from $g_\varepsilon^{-t_0,-2}$ to $g_\varepsilon^{-t_0,2}$,

$$s \in [-2,2] \to g_\varepsilon^{-t_0,s},$$

  which by Proposition 3 and Lemma 4 has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\eta}{2}$. This path varies continuously by Proposition 3.

- Fifth, a path from $g_\varepsilon^{-t_0,2}$ to $g_\varepsilon^{t_0,2}$,

$$t \in [-t_0, t_0] \to g_\varepsilon^{t,2},$$

  which by Proposition 3 and Lemma 4 has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \tau + E(\varepsilon)$. This path varies continuously by (54) and Proposition 3.

- Sixth, a path from $g_\varepsilon^{t_0,-2} = v_\varepsilon^{t_0,0}$ to $g_\varepsilon^{t_0,2}$,

$$s \in [-2,2] \to g_\varepsilon^{t_0,s},$$

  which by by Proposition 3 and Lemma 4 has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\eta}{2}$. This path varies continuously by Proposition 3.

- Seventh, a path from $v_\varepsilon^{t_0,0}$ to $v_\varepsilon^{t_0,1} = v_\varepsilon^{t_0}$,

$$s \in [0,1] \to v_\varepsilon^{t_0,s}$$

  which by Lemma 4 has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta$. This path varies continuously by (53).

- Eighth, a path from $v_\varepsilon^{t_0}$ to -1,

$$t \in [t_0, 2d(N)] \to v_\varepsilon^t$$

  which by Lemma 5 has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \eta + E(\varepsilon)$. This path varies continuously by the reasoning in Subsection 5.2.

- Ninth, a path from -1 to the constant a_ε,

$$s \in [-1, a_\varepsilon] \to s,$$

  through constant functions which, by the construction of a_ε through negative gradient flow of $\mathcal{F}_{\varepsilon,\lambda}$ in Subsection 1.2, has $\mathcal{F}_{\varepsilon,\lambda}$ energy along the path $\leq \mathcal{F}_{\varepsilon,\lambda}(-1)$.

Consider the above paths in the following order: first (reversed), second, third (reversed), fourth, fifth, sixth (reversed), seventh, eighth and ninth; this is the path depicted in Figure 2 in Subsection 1.3. In the order just given, the endpoint of each partial path matches the starting point of the next, therefore their composition in the same order provides a continuous path in $W^{1,2}(N)$, for all $\varepsilon > 0$ sufficiently small, from the constant a_ε to the constant b_ε with

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \min\left\{\frac{\eta}{2}, \tau\right\} + E(\varepsilon),$$

by Subsection 1.2, Lemma 4, Lemma 5 and Proposition 3. By (2) and the fact that $E(\varepsilon) \to 0$ by Lemma 4, by choosing $\varepsilon > 0$ sufficiently small we ensure that we have

$$\mathcal{F}_{\varepsilon,\lambda} \text{ energy along the path} \leq \mathcal{F}_\lambda(E) - \min\left\{\frac{\eta}{4}, \frac{\tau}{2}\right\}.$$

Note that as (1) holds, by (2) and the path provided in Lemma 5, we ensure $\mathcal{F}_{\varepsilon,\lambda}(u_{\varepsilon_j}) \to \mathcal{F}_\lambda(E)$ as $\varepsilon_j \to 0$. Thus, as the above path is admissible in the min-max construction of [BW20a], we contradict the assumption that $\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau$ for some $\tau > 0$. We therefore conclude that for any such M as produced by the Allen–Cahn min-max procedure in Ricci positive curvature must be such that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq \mathcal{F}_{\varepsilon,\lambda}(g_\varepsilon) + \tau_\varepsilon \text{ for some sequence } \tau_\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

With the above in mind and applying Lemma 3, by setting $r_1 = \frac{R}{4}$ and $r_2 = \frac{R}{2}$, we have that E satisfies

$$\mathcal{F}_\lambda(E) = \inf_{G \in \mathcal{C}(N)} \{\mathcal{F}_\lambda(G) \mid G \setminus B_{\frac{R}{4}}(p) = E \setminus B_{\frac{R}{4}}(p)\}.$$

In particular, by Remark 13, we note that every tangent cone at an isolated singularity of M is thus area-minimising. Applying the above reasoning for each isolated singularity of M then concludes the proof of Theorem 3. □

6.2 Proof of Theorems 1 & 2

Proof of Theorem 2. To prove Theorem 2 we exploit the results of Section 2, using the fact that the proof of Theorem 3 ensures M is one-sided minimising in the sense of Definition 1 by Remark 7. Because the set of isolated singular points of M with regular tangent cones is discrete, but not necessarily closed when $n \geq 8$, it suffices to index the isolated singularities with regular tangent cones and make a small change to the metric around each point so that the sum of the resulting perturbations is arbitrarily small.

With the above in hand, we now make an arbitrarily small change to the metric at each isolated singular point, $p \in \mathrm{Sing}(M)$, with regular tangent cone. By applying Proposition 1 in a ball, $B_\rho(p)$, for some $\rho > 0$ sufficiently small, there exists both:

- A metric, $\tilde{g}$, arbitrarily close to g in the $C^{k,\alpha}$ norm for each $k \geq 1$ and $\alpha \in (0,1)$, agreeing with g on $N \setminus B_\rho(p)$.
- A closed embedded hypersurface, $\widetilde{M}$, of constant mean curvature λ, which is smooth in $B_\rho(p)$, and agrees with M on $N \setminus B_\rho(p)$.

In this manner we are able to locally smooth M up to an arbitrarily small perturbation of the metric g. Thus we have shown that for each $k \geq 1$ and $\alpha \in (0,1)$ there exists a dense set of metrics, $\mathcal{G}_k \subset \mathrm{Met}^{k,\alpha}_{\mathrm{Ric}_g > 0}$, such that for each $h \in \mathcal{G}_k$, (N, h) admits a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set of Hausdorff dimension at most $n - 7$, containing no singularities with regular tangent cones. By [Whi15, Theorem 2.10] there thus exists a dense set $\mathcal{G}$ of the smooth metrics with Ricci positive curvature such that for each $h \in \mathcal{G}$, (N, h) admits a closed embedded hypersurface of constant mean curvature λ, smooth away from a closed singular set of Hausdorff dimension at most $n-7$, containing no singularities with regular tangent cones, concluding the proof of Theorem 2. □

Proof of Theorem 1. In dimension 8 all singularities are isolated with regular tangent cone. Hence, for each $g \in \mathcal{G}$, as produced in the proof of Theorem 2, there exists a smooth hypersurface of constant mean curvature. The fact that $\mathcal{G}$ as above is then open in dimension 8 follows from the results of [Whi91, Section 7]. □

References

[Alb94] Giovanni S. Alberti. On the structure of singular sets of convex functions. *Calculus of Variations and Partial Differential Equations*, 2:17–27, 1994.

[Bel20] Costante Bellettini. Multiplicity-1 minmax minimal hypersurfaces in manifolds with positive Ricci curvature. https://arxiv.org/abs/2004.10112, *(to appear in) Communications on Pure and Applied Mathematics*, 2020.

[BW20a] Costante Bellettini and Neshan Wickramasekera. The inhomogeneous Allen–Cahn equation and the existence of prescribed-mean-curvature hypersurfaces. https://arxiv.org/abs/2010.05847, 2020.

[BW20b] Costante Bellettini and Neshan Wickramasekera. Stable cmc integral varifolds of codimension 1: regularity and compactness. https://arxiv.org/abs/1802.00377, 2020.

[BW20c] Costante Bellettini and Neshan Wickramasekera. Stable prescribed-mean-curvature integral varifolds of codimension 1: regularity and compactness. https://arxiv.org/abs/1902.09669, 2020.

[BW22] Costante Bellettini and Myles Workman. Embeddedness of min-max cmc hypersurfaces in manifolds with positive Ricci curvature. https://arxiv.org/abs/2212.09605, 2022.

[CCMS20] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic initial data. https://arxiv.org/abs/2003.14344, 2020.

[CCMS22] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic low-entropy initial data. https://arxiv.org/abs/2102.11978, 2022.

[CLS22] Otis Chodosh, Yevgeny Liokumovich, and Luca Spolaor. Singular behavior and generic regularity of min-max minimal hypersurfaces. *Ars Inveniendi Analytica*, 2022.

[CMS23a] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Generic regularity for minimizing hypersurfaces in dimensions 9 and 10. https://arxiv.org/abs/2302.02253, 2023.

[CMS23b] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Improved generic regularity of codimension-1 minimizing integral currents. https://arxiv.org/abs/2306.13191, 2023.

[EG15] Lawrence C. Evans and Ronald F. Gariepy. Measure theory and fine properties of functions, revised edition. *Chapman and Hall/CRC*, 2015.

[FROS20] Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra. Generic regularity of free boundaries for the obstacle problem. *Publications mathématiques de l'IHÉS*, 132(1):181–292, 2020.

[Gra04] Alfred Gray. Tubes. *Birkhäuser*, 2004.

[Gua18] Marco A. M. Guaraco. Min–max for phase transitions and the existence of embedded minimal hypersurfaces. *Journal of Differential Geometry*, 108(1), 2018.

[HS85] Robert Hardt and Leon Simon. Area minimizing hypersurfaces with isolated singularities. *Journal für die reine und angewandte Mathematik*, 362:102–129, 1985.

[HT00] John E. Hutchinson and Yoshihiro Tonegawa. Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. *Calculus of Variations and Partial Differential Equations*, 10(1):49–84, 2000.

[KS89] Robert V. Kohn and Peter Sternberg. Local minimisers and singular perturbations. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 111(1-2):69–84, 1989.

[Law91] Gary R. Lawlor. A sufficient criterion for a cone to be area-minimizing. *Memoirs of the American Mathematical Society*, 91:0–0, 1991.

[Les23] Konstantinos Leskas. PhD thesis. *Submitted*, 2023.

[Liu19] Zhenhua Liu. Stationary one-sided area-minimizing hypersurfaces with isolated singularities. https://arxiv.org/abs/1904.02289, 2019.

[LW21] Yangyang Li and Zhihan Wang. Generic regularity of minimal hypersurfaces in dimension 8. https://arxiv.org/abs/2012.05401, 2021.

[LW22] Yangyang Li and Zhihan Wang. Minimal hypersurfaces for generic metrics in dimension 8. https://arxiv.org/abs/2205.01047, 2022.

[Mag12] Francesco Maggi. Sets of finite perimeter and geometric variational problems: An introduction to geometric measure theory. *Cambridge University Press*, 2012.

[MM02] Carlo Mantegazza and Andrea Carlo Mennucci. Hamilton–Jacobi equations and distance functions on Riemannian manifolds. *Applied Mathematics and Optimization*, 47(1):1–25, 2002.

[Moo06] John Douglas Moore. Bumpy metrics and closed parametrized minimal surfaces in Riemannian manifolds. *Transactions of the American Mathematical Society*, 358(12):5193–5257, 2006.

[Moo17] John Douglas Moore. Introduction to global analysis: Minimal surfaces in Riemannian manifolds. *American Mathematical Soc.*, 2017.

[RT07] Matthias Röger and Yoshihiro Tonegawa. Convergence of phase-field approximations to the Gibbs–Thomson law. *Calculus of Variations and Partial Differential Equations*, 32(1):111–136, 2007.

[Sak96] Takashi Sakai. Riemannian geometry. *American Mathematical Society*, 1996.

[Sim68] James Simons. Minimal varieties in Riemannian manifolds. *Annals of Mathematics*, 88:62, 1968.

[Sim83] Leon Simon. Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. *The Annals of Mathematics*, 118(3):525, 1983.

[Sim84] Leon Simon. Lectures on geometric measure theory. *Proceedings of the Centre for Mathematical Analysis 3, Canberra, VII+272*, 1984.

[Sma93] Nathan Smale. Generic regularity of homologically area minimizing hypersurfaces in eight dimensional manifolds. *Communications in Analysis and Geometry*, 1:217–228, 1993.

[Whi85] Brian White. Generic regularity of unoriented two-dimensional area minimizing surfaces. *The Annals of Mathematics*, 121(3):595, 1985.

[Whi91] Brian White. The space of minimal submanifolds for varying Riemannian metrics. *Indiana University Mathematics Journal*, 40:161–200, 1991.

[Whi15] Brian White. On the bumpy metrics theorem for minimal submanifolds. *American Journal of Mathematics*, 139(4):1149 – 1155, 2015.

[Whi19] Brian White. Generic transversality of minimal submanifolds and generic regularity of two-dimensional area-minimizing integral currents. https://arxiv.org/abs/1901.05148, 2019.

[Wic14] Neshan Wickramasekera. A general regularity theory for stable codimension 1 integral varifolds. *Annals of Mathematics*, 179(3):843–1007, 2014.

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