ISOPERIMETRY BY STRETCHING

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Abstract

We construct isoperimetric regions from separating hypersurfaces in closed manifolds. This yields isoperimetric boundaries exhibiting a wide variety of topological types and singular sets.

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1 Introduction

Isoperimetric boundaries minimise area for a fixed enclosed volume, with sharp regularity theory ensuring smoothness away from a closed singular set of codimension seven. While in closed Riemannian manifolds their existence is guaranteed for every proportion of enclosed volume, relatively few explicit examples are known; we refer to the survey in [EM13, Appendix H] and references therein. Here, we show that any smooth, closed, connected, separating hypersurface can be realised as an isoperimetric boundary for some Riemannian metric on a closed manifold (see Theorem 2). This follows as an application of a 'stretching' procedure for such hypersurfaces that are a priori assumed to be locally uniquely area-minimising, enabling a local-to-global transfer of isoperimetry when the ambient manifold is made sufficiently long (see Theorem 1).

As our technique requires local minimisation, and crucially not smoothness, of the underlying hypersurface, it can be applied to verify a claim made in [Mor16] and exhibit many singular isoperimetric boundaries in Riemannian spheres (see Subsection 3.2). Moreover, with the construction of singular minimal hypersurfaces in [Sma00], we show that a wide variety of higher dimensional singular sets can arise in isoperimetric boundaries (see Theorem 3); this includes any finite collection of smooth, closed, oriented manifolds of appropriate codimension. To the best of our knowledge, the only singular examples of isoperimetric boundaries prior to this work were those of [Niu24]. In that work, the singular sets consisted of two isolated points and, due to a topological restriction, were modelled on symmetric Simons cones (namely cones over products of spheres of the same dimension). In contrast, our construction allows for singular sets modelled on any cylindrical tangent cone, provided that the cone factor is strictly stable and strictly minimising (see Remark 6).

1.1 Strategy

We now briefly outline the construction that yields Theorem 1. Suppose that we are given a smooth, closed, connected, separating hypersurface, Σ, which is locally uniquely area-minimising in a closed Riemannian manifold, (M, g). The bounded geometry of (M, g) ensures that any isoperimetric region enclosing the same volume as Σ has uniform bounds on both the diameter and number of the connected components of its boundary. By elongating portions of the tubular neighbourhood of Σ appropriately, we obtain a family of Riemannian metrics for which the above-mentioned uniform bounds persist for any isoperimetric region with the same enclosed volume as Σ. When these manifolds are sufficiently elongated, we are able to utilise the uniform bounds on the boundary components, the unique local area-minimisation of Σ, and an elementary volume comparison argument to ensure that Σ must be the unique isoperimetric region for its enclosed volume. To carry the same construction through in the singular setting, one needs to ensure some topological control on the neighbourhood in which Σ is uniquely minimising (see Corollary 1). As a by-product of the proof of Theorem 1, the construction above also produces isoperimetric regions from volume-constrained minimisers that are, a priori, assumed to be sufficiently close in area to unique local area-minimisers (see Corollary 2).

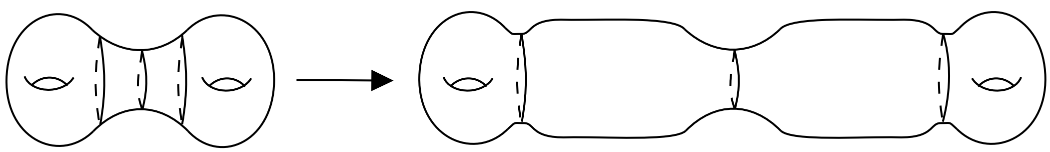

Figure 1: In each of the two graphics above, the three circles depict Σ between the boundaries of the tubular neighbourhood in which it is uniquely area-minimising. Along the stretching procedure, the 'caps' outside of this tubular neighbourhood are unchanged.

We now proceed as follows: Section 2 records the stretching of Theorem 1 and the above mentioned Corollaries 1 and 2. Section 3 contains the applications of Theorem 1, with Subsection 3.1 devoted to the smooth setting and Subsection 3.2 to the construction of singular isoperimetric boundaries.

1.2 Notation

We now collect some notation and definitions that will be used throughout this work:

- Throughout, we let M denote a smooth, closed (i.e. compact with empty boundary) manifold of dimension $n + 1 \geq 2$. Without loss of generality we will assume that M is connected. We say that a set, $S \subset M$, is **separating** if there are disjoint, connected, open sets, $E, F \subset M$, with $S = \partial E = \partial F$ and $M = E \cup S \cup F$. By a **hypersurface**, $\Sigma \subset M$, we will mean an embedded manifold of dimension n, which is smooth away from a (possibly empty) closed singular set of Hausdorff dimension at most $n - 7$.

- Given a smooth Riemannian metric, g, on M we denote dist_g for the distance function on M, $B^g_r(p)$ the open geodesic ball in M of radius $r > 0$ centred at $p \in M$, $\mathrm{Vol}_g(E)$ for the volume of a measurable set $E \subset M$, and $\int_E u \, dV_g$ for the integral of a measurable function u, all with respect to the metric g. When working in $\mathbb{R}^{n+1}$ we will write g_{eucl} for the Euclidean metric. We may omit metric dependence from our notation when working in Euclidean space or when it is clear from context.

- A measurable set $E \subset M$ is a **Caccioppoli set** if the indicator function of E is of bounded variation, or equivalently if it has finite **perimeter** with respect to the metric g so that

$$\mathrm{Per}_g(E) = \sup\left\{ \int_E \mathrm{div}_g X \, dV_g \,\middle|\, X \in \Gamma(TM), g(X,X) \leq 1 \right\} < \infty,$$

where div_g is the divergence with respect to the metric g and $\Gamma(TM)$ is the set of vector fields on M; we write $\mathrm{Per}_g(E;U)$ for the perimeter relative to a measurable set $U \subset M$ by restricting the integral above to U. We write $\mathcal{C}(S)$ for the set of Caccioppoli sets in a set S, and note that $\mathcal{C}(S)$ is independent of the choice of metric on S. We say that $\Omega \in \mathcal{C}(M)$ is an **isoperimetric region** of enclosed volume $t \in \mathbb{R}$ if

$$\mathrm{Per}_g(\Omega) = \inf_{E \in \mathcal{C}(M)} \{\mathrm{Per}_g(E) \mid \mathrm{Vol}_g(E) = t\}.$$

The existence of isoperimetric regions of a given enclosed volume in M is then guaranteed by the direct method of the calculus of variations (e.g. see [Mor16, Section 13.2]). Moreover, we say that $\Omega \in \mathcal{C}(M)$ is a **volume-constrained minimiser** in an open set $U \subset M$ if

$$\mathrm{Per}_g(\Omega; U) = \inf_{E \in \mathcal{C}(M)} \{\mathrm{Per}_g(E;U) \mid \Omega \Delta E \subset\subset U \text{ and } \mathrm{Vol}_g(\Omega \cap U) = \mathrm{Vol}_g(E \cap U)\}.$$

- Given an isoperimetric region Ω as above, we denote by $\Sigma = \partial\Omega$ the closed, two-sided, embedded hypersurface of constant mean curvature (defined on its regular part) with closed **singular set**, $\mathrm{Sing}(\Sigma)$, away from which Σ is smooth; by [GMT83] and [Mor03], $\mathrm{Sing}(\Sigma)$ is of Hausdorff dimension at most $n-7$. We write $|\Sigma|_g$ for the area of Σ and denote by ν_g the outward pointing unit normal to Σ with respect to the metric g.

- Suppose that $\mathbf{C} \subset \mathbb{R}^{n+1}$ is a **regular** minimal hypercone, so that $\mathrm{Sing}(\mathbf{C}) \subset \{0\}$. We then denote the smooth, closed, manifold $\Sigma = \mathbf{C} \cap \mathbb{S}^n \subset \mathbb{R}^{n+1}$ as its **link**. By parameterising $\mathbf{C}$ in radial coordinates, $(r, \omega) \in (0,\infty) \times \Sigma$, we decompose (as in [Sim68]) the Jacobi operator, $L_{\mathbf{C}}$, of $\mathbf{C}$ as

$$L_{\mathbf{C}} = \partial_r^2 + \left(\frac{n-1}{r}\right)\partial_r + \frac{1}{r^2}(\Delta_\Sigma + |\mathrm{II}_\Sigma|^2),$$

where II_Σ is the second fundamental form of Σ in $\mathbb{S}^n$. We let $\mu_1 < \mu_2 \leq \mu_3 \leq \ldots \nearrow +\infty$ be the eigenvalues of $-(\Delta_\Sigma + |\mathrm{II}_\Sigma|^2)$, and $\varphi_1, \varphi_2, \ldots$ be the corresponding $L^2(\Sigma)$-orthonormal eigenfunctions where $\varphi_1 > 0$. By [Sim68], $\mathbf{C}$ is **stable** if and only if $\mu_1 \geq -\frac{(n-2)^2}{4}$ and, we say (as in [CHS84]) that $\mathbf{C}$ is **strictly stable** if $\mu_1 > -\frac{(n-2)^2}{4}$. Moreover, we say (as in [HS85]) that a minimal hypercone, $\mathbf{C}$, as above is **strictly minimising** if there is a $\Theta > 0$ such that for each $\varepsilon > 0$ we have

$$|\mathbf{C} \cap B_1(0)| \leq |S| - \Theta\varepsilon^n$$

whenever S is an integer rectifiable current with $\mathrm{Spt}(S) \subset \mathbb{R}^{n+1} \setminus B_\varepsilon(0)$ and $\partial S = \partial(C \cap B_1(0))$.

2 Stretching

Our main result is the following:

Theorem 1. *If $\Sigma \subset M$ is a smooth, closed, connected, separating hypersurface which is uniquely homologically area-minimising in a tubular neighbourhood of Σ in (M,g), then there is a Riemannian metric, h, on M such that Σ bounds an isoperimetric region in (M,h).*

Proof. We let Ω denote one of the two open sets bounded by Σ, and $V \subset M$ denote a tubular neighbourhood of Σ in which it is uniquely homologically area-minimising. By taking a smaller neighbourhood of Σ if necessary we may suppose that V has smooth boundary, $\partial V = \Gamma^+ \cup \Gamma^-$, where each of the boundary components, $\Gamma^\pm$, are smooth, connected, and arise as the images of Σ under the exponential map (i.e. level sets of the signed distance function to Σ), chosen so that $\Gamma^- \subset \Omega$ and $\Gamma^+ \subset M \setminus \Omega$. In particular, we then ensure that Σ generates the codimension-one homology of V (with $\mathbb{Z}$ coefficients if Σ is oriented, and $\mathbb{Z}_2$ coefficients otherwise).

We now alter the metric g by stretching in the normal direction to Σ. For sufficiently small $\varepsilon > 0$ and for each $\Gamma = \Gamma^\pm$, we fix a collar, $\Phi : \Gamma \times [0,\varepsilon] \to V$, given by the exponential map on Γ (with the unit normal on Γ pointing into V) so that $\Phi(\Gamma \times [0,\varepsilon]) \cap \Sigma = \emptyset$. In this collar, the metric g then has the form

$$\Phi^* g = h_t + dt^2,$$

where $\{h_t\}_{t\in[0,\varepsilon]}$ denotes a smoothly varying family of metrics on Γ. Now, we choose a smooth cutoff function, $\eta : [0,\varepsilon] \to [0,1]$, which satisfies

$$\begin{cases} \eta(t) = 0 \text{ for } t \in \left[0, \frac{\varepsilon}{4}\right] \cup \left[\frac{3\varepsilon}{4}, \varepsilon\right], \\ \eta(t) = 1 \text{ for } t \in \left[\frac{\varepsilon}{3}, \frac{2\varepsilon}{3}\right]. \end{cases}$$

Next, we choose a smooth metric, h_Γ, on Γ such that $h_\Gamma \geq \max_{t\in[0,\varepsilon]} h_t$, and then define on $\Gamma \times [0,\varepsilon]$ the metric by setting

$$\Phi^* \tilde{g} = \Big((1-\eta(t))\, h_t(x) + \eta(t)\, h_\Gamma(x)\Big) + dt^2.$$

Note then that $\Phi^*\tilde{g} = \Phi^* g$ for $t \in [0,\varepsilon/4] \cup [3\varepsilon/4,\varepsilon]$, so that the resulting metric, $\tilde{g}$, is unchanged in a neighbourhood of $\Gamma^\pm$ and Σ. Moreover, we ensure that $\tilde{g}$ is cylindrical on $\Phi(\Gamma \times [\varepsilon/3, 2\varepsilon/3])$ and such that $\tilde{g} \geq g$. Considering the two surfaces, $\Phi(\Gamma \times \{\frac{\varepsilon}{3}\})$ and $\Phi(\Gamma \times \{\frac{2\varepsilon}{3}\})$, we now alter $\tilde{g}$ and for each $R > 1$ construct metrics, g_R, so that $\mathrm{dist}_{g_R}(\Phi(\Gamma \times \{\frac{\varepsilon}{3}\}), (\Phi(\Gamma \times \{\frac{2\varepsilon}{3}\})) = R$. Denoting $\ell = \mathrm{dist}_{\tilde{g}}\left(\Phi\left(\Gamma \times \{\frac{\varepsilon}{3}\}\right),\ \Phi\left(\Gamma \times \{\frac{2\varepsilon}{3}\}\right)\right) = \frac{\varepsilon}{3}$, for each $R > 1$ we consider the stretching function

$$\rho_R(t) = 1 + \left((R/\ell)^2 - 1\right)\eta(t),$$

so that $\rho_R = 1$ on $[0,\varepsilon/4] \cup [3\varepsilon/4,\varepsilon]$, and $\rho_R = (R/\ell)^2$ on $[\varepsilon/3,\ 2\varepsilon/3]$. We then define the metrics g_R for each $R > 1$ by setting:

$$\begin{cases} g_R = \tilde{g} & \text{on } M \setminus \Phi(\Gamma \times [0,\varepsilon]), \\ \Phi^* g_R = \tilde{g}_t + \rho_R dt^2 & \text{on } \Gamma \times [0,\varepsilon]. \end{cases}$$

Thus, under this construction we ensure that $\mathrm{dist}_{g_R}(\Phi(\Gamma \times \{\frac{\varepsilon}{3}\}), (\Phi(\Gamma \times \{\frac{2\varepsilon}{3}\})) = R$ and g_R remains cylindrical on $\Phi(\Gamma \times [\varepsilon/3, 2\varepsilon/3])$ for each $R > 1$ (c.f. the right-hand graphic in Figure 1 for a depiction of (M, g_R)). Let us denote by

$$T = \Phi(\Gamma^+ \times [\varepsilon/3, 2\varepsilon/3]) \cup \Phi(\Gamma^- \times [\varepsilon/3, 2\varepsilon/3]) \subset V$$

the cylindrical part of (M, g_R). By the above construction, we ensure the following properties:

1. $\Sigma \subset V$ is the unique homological minimiser in (V, g_R) with $\mathrm{Per}_{g_R}(\Omega) = \mathrm{Per}_g(\Omega)$ for all $R > 1$.

2. $\mathrm{Vol}_{g_R}(\Omega), \mathrm{Vol}_{g_R}(M \setminus \Omega), \mathrm{Vol}_{g_R}(T) \to \infty$ as $R \to \infty$.

3. $\mathrm{Vol}_{g_R}(M \setminus T) = C$ for a constant $C > 0$ independent of $R > 1$.

We now show that, when $R > 1$ is made sufficiently large, Ω is an isoperimetric region for its enclosed volume in (M, g_R). First, noting that for each $R > 1$ the metrics g_R have the same uniform bounds on sectional curvature and their injectivity radius, by [Niu24, Lemma 4.3] (see also [MNP25, Lemma 2]) we ensure a uniform bound on the (constant) mean curvature of the smooth portions of any isoperimetric boundary enclosing the same volume as Ω with respect to the metric g_R. Therefore, as in the proof of [Niu24, Lemma 4.4], the monotonicity formula guarantees that there is some uniform $\delta > 0$, independent of $R > 1$, so that each connected boundary component has area at least δ. Using this, we ensure uniform upper bounds, $D, L > 0$, independent of $R > 1$ on the extrinsic diameter of each boundary component and the number of connected boundary components respectively (by direct comparison with the area of Σ). We now argue that, for $R > 1$ sufficiently large, there is only one boundary component for each such isoperimetric region, and that it must coincide with Σ.

Suppose now that E is an isoperimetric region with the same volume as Ω in (M, g_R) for some $R > 1$; from the above discussion that there are at most L connected boundary components and each of these components has extrinsic diameter at most D. Note that if some connected boundary component, Ξ, of E both lies inside of V and is homologous to Σ in V, then since Σ is uniquely homologically area-minimising in V, we ensure that $|\Xi|_{g_R} = |\Sigma|_{g_R}$ and thus $\partial E = \Sigma$ (else we would have $\mathrm{Per}_{g_R}(E) > \mathrm{Per}_{g_R}(\Omega) + \delta$ by the uniform lower area bound on each boundary component). We thus can argue that some boundary component of E both lies inside of V and is homologous to Σ in V in order to conclude.

If E has no boundary components lying inside of V which are homologous to Σ in V, then each boundary component of E must either intersect $M \setminus V$ or be null-homologous in V (since Σ generates the codimension-one homology of V). In the former case, where a boundary component of E intersects $M \setminus V$, by the uniform diameter bound we see that all components lie in the set of points at most distance D from $M \setminus V \subset M \setminus T$, which in turn has enclosed volume at most a constant, $C > 0$, independent of $R > 1$ from property 3 of the construction above (more precisely, each has at most enclosed volume at most $\mathrm{Vol}_{g_R}(M \setminus T) + C$ for a constant $C > 0$ independent of $R > 1$). In the latter case, where each boundary component is a boundary in V, the uniform diameter bound ensures that each boundary component lies in a 'tubular' region of the form $\Phi(\Gamma \times [s, t])$ for $\Gamma = \Gamma^{\pm}$ and $s < t$ with $\mathrm{dist}_{g_R}(\Phi(\Gamma \times \{s\}), \Phi(\Gamma \times \{t\})) \leq D$ (c.f. [Niu24, Proof of (4) in Lemma 4.4]), hence it encloses at most a fixed constant volume, $C > 0$ (potentially larger), independent of $R > 1$ and depending only on the geometry of Γ. Combining both cases, since there are at most L connected boundary components for E, we have

$$\min\{\mathrm{Vol}_{g_R}(E), \mathrm{Vol}_{g_R}(M \setminus E)\} \leq \mathrm{Vol}_{g_R}(M \setminus T) + CL < \widetilde{C},$$

where $\widetilde{C} > 0$ is independent of $R > 1$. However, this bound contradicts property 2 of the construction above for sufficiently large $R > 1$, since both $\mathrm{Vol}_{g_R}(E) = \mathrm{Vol}_{g_R}(\Omega)$ and $\mathrm{Vol}_{g_R}(M \setminus E) = \mathrm{Vol}_{g_R}(M \setminus \Omega)$. Thus, we see that E has at least, and hence at most, one boundary component in V which is also homologous to Σ in V. As mentioned, this ensures that $\partial E = \Sigma$, and thus Σ bounds an isoperimetric region of enclosed volume $\mathrm{Vol}_{g_R}(\Omega)$ as desired. □

The proof of Theorem 1 relied on smoothness of the separating hypersurface only to ensure some topological control on the neighbourhood in which it is uniquely homologically minimising. Provided we have this control, we can carry out the same construction in the singular setting:

Corollary 1. *If $\Sigma \subset M$ is a closed, connected, separating hypersurface which is uniquely homologically area-minimising in a tubular neighbourhood, V, of Σ in (M, g), with V satisfying the following topological assumptions:*

- *V is a cobordism between smooth connected $\Gamma^\pm \subset \Omega^\pm$ with $M \setminus \Sigma = \Omega^+ \cup \Omega^-$, where $\Omega^\pm$ are disjoint, Σ separates V, and ∂V separates M into three connected components.*
- *Σ generates the codimension-one homology of V (with $\mathbb{Z}$ coefficients if Σ is oriented, and $\mathbb{Z}_2$ coefficients otherwise).*

Then, there is a Riemannian metric, h, on M such that Σ bounds an isoperimetric region in (M, h).

Proof. After choosing a neighbourhood, V, of Σ satisfying the topological assumptions, one can proceed to argue identically as in the proof of Theorem 1 to conclude. □

Remark 1. *As utilised in Subsection 3.2 below, the examples of singular minimal hypersurfaces constructed in [Sma00] have neighbourhoods satisfying the above topological assumptions of Corollary 1.*

The method of proof of Theorem 1 also allows for constructions of isoperimetric regions from volume-constrained minimisers that are, a priori, close in area to a unique local area-minimiser:

Corollary 2. *If $\Sigma \subset M$ is a locally homologically area-minimising hypersurface, as in the statement of Theorem 1, and $\widetilde{\Sigma} \subset V$ is a smooth, closed, connected, separating hypersurface,bounding a unique volume-constrained minimiser in a tubular neighbourhood, $V \subset M$, of Σ in (M, g), with $\widetilde{\Sigma}$ both homologous and sufficiently close in area to Σ, then there is a Riemannian metric, h, on M such that $\widetilde{\Sigma}$ bounds an isoperimetric region in (M, h).*

Proof. Denoting $\widetilde{\Omega} \subset M$ an open set bounded by $\widetilde{\Sigma}$, one can follow the construction in the proof of Theorem 1 verbatim to ensure that, for $R > 1$ sufficiently large, any isoperimetric region, $E \subset M$, of enclosed volume $\mathrm{Vol}_{g_R}(\widetilde{\Omega})$ in (M, g_R) has at least one connected component homologous to Σ in a neighbourhood V. Since $\mathrm{Per}_{g_R}(E) \geq \mathrm{Per}_{g_R}(\Omega)$, where $\partial\Omega = \Sigma$, we see that provided $\mathrm{Per}_{g_R}(\widetilde{\Omega}) < \mathrm{Per}_{g_R}(\Omega) + \delta$ (where $\delta > 0$ is uniform lower area bound as in the proof of Theorem 1), E has at most one boundary component; else $\mathrm{Per}_{g_R}(\widetilde{\Omega}) < \mathrm{Per}_{g_R}(E)$. The volume-constrained minimisation of $\widetilde{\Sigma}$ in V ensures that $\mathrm{Per}_{g_R}(\widetilde{\Omega}) = \mathrm{Per}_{g_R}(E)$, and the uniqueness implies that $\partial E = \widetilde{\Sigma}$ as desired. □

Remark 2. *With the same assumptions and reasoning as for Corollary 1, Corollary 2 can, in principle, be applied to volume-constrained minimisers that possess non-empty singular sets.*

3 Applications

We record some direct applications of Theorem 1, showing that the space of isoperimetric boundaries in M is rich both in terms of topological complexity and singular behaviour.

3.1 Topological types

We show that the space of isoperimetric boundaries is topologically diverse:

Theorem 2. *Given a smooth, closed, connected, separating hypersurface, $\Sigma \subset M$, there is a Riemannian metric, g, on M such that Σ bounds an isoperimetric region in (M, g).*

Proof. Starting with an arbitrary Riemannian metric, g, the idea is to find a suitable metric perturbation which ensures Σ is uniquely homologically area-minimising in a tubular neighbourhood, then directly apply Theorem 1 to obtain the desired conclusion.

For a smooth function, f, on M, if we define $h = e^{2f}g$ then by [Bes87, 1.159/(1.163)] the mean curvature, H_h, of Σ with respect to h satisfies

$$H_h = e^{-f}\left(H_g - \frac{\partial f}{\partial \nu}\right),$$

and the Ricci curvature, Ric_h, of M with respect to h satisfies

$$\mathrm{Ric}_h = \mathrm{Ric}_g - (n-1)(\nabla_g^2 f - df \otimes df) - (\Delta_g f + (n-1)|\nabla_g f|^2)g.$$

Choosing a smooth function, f_1, on M with $\frac{\partial f_1}{\partial \nu} = H_g$, we ensure that Σ is a minimal hypersurface in (M, h). We then choose another smooth function, f_2, on M so that both f_2 and $\nabla^h f_2$ vanish on Σ, ensuring that ν_h is also a unit normal with respect to the metric $\tilde{h} = e^{2f_2}h$, $\mathrm{II}_{\Sigma,h} = \mathrm{II}_{\Sigma,\tilde{h}}$ (equality of the second fundamental forms of Σ with respect to each metric), and $\Delta_h f_2 = \nabla_h^2 f_2(\nu,\nu)$ on Σ. We thus compute that

$$\mathrm{Ric}_{\tilde{h}}(\nu_h, \nu_h) = \mathrm{Ric}_h(\nu_h, \nu_h) - n\nabla_h^2 f_2(\nu_h, \nu_h),$$

and so choosing $\nabla_h^2 f_2(\nu_h, \nu_h)$ sufficiently large (in particular so that $\mathrm{Ric}_{\tilde{h}}(\nu_h, \nu_h) < -|\mathrm{II}_{\Sigma,h}|^2$ on Σ), we ensure that Σ is a strictly stable minimal hypersurface in $(M, \tilde{h})$ in the sense that for each non-zero $\phi \in C_c^\infty(\Sigma)$ we have

$$\int_\Sigma |\nabla^{\tilde{h}}\phi|^2 - (|\mathrm{II}_{\Sigma,\tilde{h}}|^2 + \mathrm{Ric}_{\tilde{h}}(\nu_h, \nu_h))\phi^2 \, dV_{\tilde{h}} > 0.$$

Concretely, one can choose f_2 so that it agrees with a large multiple of the square of the distance function to Σ in a tubular neighbourhood. By the results of [Whi94], the strict stability of Σ ensures its unique homological area-minimisation in a tubular neighbourhood. Applying Theorem 1 in this tubular neighbourhood, we find a Riemannian metric on M for which Σ bounds an isoperimetric region as desired. □

Remark 3. *While above we prescribe the ambient closed manifold and ensure isoperimetry for any smooth, closed, connected, separating hypersurface, the same technique shows that one can instead prescribe any null-cobordant, smooth, closed, connected manifold, Σ, and construct an ambient closed manifold (formed by doubling the manifold bounded by Σ) along with a Riemannian metric on it so that Σ bounds an isoperimetric region.*

Remark 4. *From the proof of Theorem 2, the resulting isoperimetric boundaries, Σ, are strictly stable and locally uniquely homologically area-minimising. One can instead produce isoperimetric boundaries with small non-zero constant mean curvature by applying Corollary 2. More precisely, the strict stability of Σ ensures the invertibility of its Jacobi operator and hence, for small constant mean curvature values, one can find strictly stable CMC graphs, $\widetilde{\Sigma}$, over Σ. The results of [Whi94] (see also [Gro96]) ensure that such $\widetilde{\Sigma}$ are locally uniquely volume-constrained minimising and hence, as the graphs will have area close to that Σ for small values of the mean curvature, one can apply Corollary 2 to ensure that $\widetilde{\Sigma}$ bounds an isoperimetric region for some Riemannian metric on M.*

3.2 Singular sets

Our initial motivation for this work stemmed from a claim made in [Mor16], concerning the possibility of constructing singular isoperimetric regions in Riemannian spheres, which we now describe. If we consider a strictly stable, strictly minimising, minimal hypercone, $\mathbf{C} \subset \mathbb{R}^{n+1}$, with $\mathrm{Sing}(\mathbf{C}) = \{0\}$ (thus necessarily $n+1 \geq 8$), we see that $\Sigma = (\mathbf{C} \times \mathbb{R}) \cap \mathbb{S}^{n+1}$ is a minimal hypersurface in

$(\mathbb{S}^{n+1}, g_{\text{round}})$ with two isolated singularities (antipodal to one another), where g_{round} is the metric on $\mathbb{S}^{n+1}$ induced from g_{eucl}. The results of [Sma99, Section 3] show that, after successive conformal metric perturbations, Σ is in fact uniquely homologically area-minimising in a neighbourhood with respect to some Riemannian metric on $\mathbb{S}^{n+1}$. It was then claimed in [Mor16, Section 13.2] that "if you make the metric huge on the rest of the sphere, it should be globally area-minimising for its volume" which we interpreted to mean that Σ can be shown to bound an isoperimetric region for some choice of Riemannian metric on $\mathbb{S}^{n+1}$. The discussion on [Sma99, Pages 168-169] shows that the hypotheses of our Corollary 1 are satisfied, and hence its application directly verifies the above claim.

In fact, we can use our Corollary 1 to produce isoperimetric boundaries with the singular sets exhibited in [Sma00]. There, a variety of singular minimal hypersurfaces were constructed, and then shown to be locally area-minimising, in closed Riemannian manifolds. We first introduce the class of manifolds from which the singular sets will be formed:

Definition 1. *For $n \geq 7$, we denote by $\mathcal{F}(n+1)$ the collection of smooth, closed, connected, orientable manifolds, Γ, of dimension at most $n-7$ such that there exists a smooth embedding $\Phi : \overline{B}_1^{m+1}(0) \times \Gamma \to \mathbb{R}^{n+1}$, where $m + \dim(\Gamma) + 1 = n + 1$ and $\overline{B}_1^{m+1}(0)$ denotes the closed unit ball in $\mathbb{R}^{m+1}$.*

Remark 5. *As noted in [Sma00], there are many examples of manifolds in the class $\mathcal{F}(n+1)$. For instance, if Γ is embedded into $\mathbb{R}^{n+1}$ with trivial normal bundle, then $\Gamma \in \mathcal{F}(n+1)$. In particular, if Γ is embedded in $\mathbb{R}^{\dim(\Gamma)+1}$ with $\dim(\Gamma) \leq n-7$, then Γ is also in $\mathcal{F}(n+1)$.*

With the above notation, we construct singular isoperimetric boundaries as follows:

Theorem 3. *Suppose that M is an orientable, closed manifold of dimension $n+1 \geq 8$ and $L \geq 1$, then for any collection of Riemannian manifolds, $\{(\Gamma_i, h_i)\}_{i=1}^L$, with $\Gamma_i \in \mathcal{F}(n+1)$, and any collection, $\{\mathbf{C}_i\}_{i=1}^L$, of strictly stable, strictly minimising, regular minimal hypercones, $\mathbf{C}_i \subset \mathbb{R}^{n_i+1}$, where we denote $n_i = n - \dim(\Gamma_i)$, there is a Riemannian metric, g, on M, and an isoperimetric region, Ω, in (M, g) with boundary Σ such that*

$$\operatorname{Sing}(\Sigma) = \bigcup_{i=1}^{L} \Lambda_i,$$

where Λ_i is an embedded copy of Γ_i for $i = 1, \dots, L$. In particular, in a neighbourhood of Λ_i for each $i = 1, \dots, L$, (M, g) is locally isometric to $B_\sigma^{n_i+1}(0) \times \Gamma_i$ endowed with the product metric $g_{\text{eucl}} \times h_i$, with Σ in this region isometric to $(\mathbf{C}_i \cap B_\sigma^{n_i+1}(0)) \times \Gamma_i$, for some $\sigma > 0$ sufficiently small.

Proof. First, we pick a smooth, closed, connected, separating hypersurface, $S \subset M$; for example, one can choose S to be the boundary of a small geodesic ball for some Riemannian metric on M. Because M is orientable, S is also orientable. We now describe the procedure used in the proof of [Sma00, Theorem 1] in order to construct singular minimal hypersurfaces from S. Consider a tubular neighbourhood, $\mathcal{E}$, which S separates into disjoint connected components, $\mathcal{E}^{\pm}$, then choose points, $p_1, \dots, p_L \in S$, and pairwise disjoint balls, B_i, each diffeomorphic to a Euclidean ball of dimension $n+1$ centred at each point, p_i, so that $B_i \subset\subset \mathcal{E}$. Then, for each $i = 1, \dots, L$, the ball B_i is separated by S into two disjoint connected components, $B_i^{\pm}$.

For each $i = 1, \dots, L$, let $\widehat{\mathbf{C}}_i \subset B_1^{n_i+1}(0)$ be a capped-off version of the minimising stable hypercone $\mathbf{C}_i$ as constructed in [Sma00, Section 2, Proposition]; for the construction, one intersects the cone with the ball and then smooths out the intersection to produce a smooth, closed hypersurface in the unit ball. By the hypothesis on Γ_i and since each B_i is diffeomorphic to a Euclidean ball of dimension $n+1$, for each $i = 1, \dots, L$ we choose an embedding

$$\Phi_i : B_1^{n_i+1}(0) \times \Gamma_i \hookrightarrow B_i^+,$$

and define

$$\Sigma_i = \Phi_i(\widehat{\mathbf{C}}_i \times \Gamma_i) \subset B_i^+.$$

Next, for each $i = 1, \dots, L$ we choose two n-disks, $D_i \subset \Sigma_i, D_i' \subset S \cap B_i$, which by removing D_i and D_i', allow us to connect S and Σ_i inside B_i^+ by an n-dimensional handle (a hypersurface diffeomorphic to $S^{n-1} \times I$) whose boundary components are glued to ∂D_i and $\partial D_i'$. We then finally let Σ be the resulting smooth, closed, connected hypersurface

$$\Sigma \ = \ S \# \Sigma_1 \# \cdots \# \Sigma_L,$$

where # denotes the connected sum, which can be done in such a way to ensure that Σ is embedded away from $\mathrm{Sing}(\Sigma)$, is orientable, and separates M. For each $i = 1, \dots, L$, on the images, $\Phi_i(B_1^{n_i+1}(0) \times \Sigma_i)$, we define the pull-back metric

$$g_i = (\Phi_i^{-1})^*(g_{eucl} \oplus h_i).$$

Then, by using a partition of unity, we can choose a Riemannian metric, g_0, on M such that it agrees with g_i on $\Phi_i\big(B_{1/2}^{n_i+1}(0) \times \Sigma_i\big)$ for each $i = 1, \dots, L$. The hypersurface $\Sigma \subset (M, g_0)$ then satisfies each of the assumptions of [Sma00, Lemma 1], and hence we can apply it to find a Riemannian metric, h, on M that agrees with g_0 in a neighbourhood of $\mathrm{Sing}(\Sigma)$ and ensures that Σ is uniquely homologically minimising in a tubular neighbourhood.

As noted on [Sma00, Pages 2323-2324] (see also the discussion on [Sma99, Pages 168-169]), we can find a tubular neighbourhood of Σ in which the assumptions of Corollary 1 are satisfied by virtue the above construction. We can thus apply Corollary 1 in order to conclude that Σ bounds an isoperimetric region for some Riemannian metric on M as desired. □

Remark 6. *As mentioned in Section 1, due to a topological restriction, the singular examples of isoperimetric boundaries constructed in [Niu24] required the prescribed regular minimal hypercones,* $\mathbf{C}_i$*, on which the singularities were modelled to be symmetric Simons cones, namely those with link given by the products of spheres,* $\mathbb{S}^p(r) \times \mathbb{S}^p(r) \subset \mathbb{S}^{2p+1}$ *for* $p \geq 3$ *and some* $r > 0$*. Our construction does not need this constraint, and ensures the singularities have unique cylindrical tangent cones, of the form* $\mathbf{C} \times \mathbb{R}^k$*, for any choice,* $\mathbf{C} \subset \mathbb{R}^{n-k+1}$*, of strictly stable, strictly minimising regular minimal hypercone; this directly addresses a question left open by the constructions in [Niu24].*

Remark 7. *The results of [MNP25] show that isoperimetric boundaries are generically (in various settings) smooth and nondegenerate in closed manifolds of dimension eight. In particular, [MNP25, Theorem 1] ensures that the above examples of Riemannian metric and enclosed volume pairs on a given closed eight-manifold which admit singular isoperimetric regions (having only finitely many isolated singularities in this case) are necessarily contained in a Baire meagre set of such pairs. Moreover, the existence of singular isoperimetric boundaries in any closed manifold of dimension eight, as guaranteed by Theorem 3, retroactively shows that the results of [MNP25] are sharp.*

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